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Équations aux dérivées partielles/*Partial Differential Equations*

What is a solution to the Navier–Stokes equations?

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Abstract The definition of a solution to the Navier–Stokes equations varies according to authors, but the link between those different definitions is not always explicit. In this Note, we intend to prove that six of the most common definitions are equivalent under a physically reasonable assumption. We then indicate a few consequences of this result. ***To cite this article: S. Dubois, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 27–32.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Qu'est-ce qu'une solution des équations de Navier–Stokes ?

Résumé La définition d'une solution des équations de Navier–Stokes varie avec les auteurs mais le lien entre ces différentes définitions n'est pas toujours explicite. Dans cette Note, on se propose de montrer que six des définitions les plus courantes sont équivalentes sous une hypothèse physiquement raisonnable. On indique ensuite quelques conséquences de ce résultat. ***Pour citer cet article : S. Dubois, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 27–32.***
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Pour chacune des six formulations différentes des équations de Navier–Stokes : (NSD1), (NS$\mathbb{P}$), (NSI), (NSD2), (NSW) et (NSWW), on donne la notion de solution associée. Soit $T \in]0, +\infty]$.

La première forme est (NSD1). On dira que $v \in \mathcal{D}'(]0, T[\times\mathbb{R}^n)$ est une *solution faible des équations de Navier–Stokes*, ou plus simplement que v est une solution de (NSD1), sur $]0, T[$ si $v \in \mathrm{L}^2_{\mathrm{loc}}(]0, T[\times\mathbb{R}^n)$ et si il existe $p \in \mathcal{D}'(]0, T[\times\mathbb{R}^n)$ tel que (v, p) satisfait (NSD1).

En faisant les hypothèses qui permettent de projeter l'équation sur les champs de vecteur à divergence nulle, on obtient la deuxième forme : (NS$\mathbb{P}$) où $\mathbb{P}$ désigne le projecteur de Leray. On dira que $v \in \mathcal{D}'(]0, T[\times\mathbb{R}^n)$ est une *solution faible des équations projetées de Navier–Stokes*, ou plus simplement que v est une solution de (NS$\mathbb{P}$), sur $]0, T[$ si $\mathbb{P}\nabla \cdot (v \otimes v)$ peut être défini dans $\mathcal{D}'(]0, T[\times\mathbb{R}^n)$ et si v satisfait (NS$\mathbb{P}$).

Une troisième forme, très utilisée depuis son introduction par Fujita et Kato dans [5], est la formulation intégrale, laquelle s'obtient formellement à partir de (NS$\mathbb{P}$) par la formule de Duhamel. Dans (NSI) et dans la suite, $v_0 \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ et $(\mathrm{e}^{t\Delta})_{t\geqslant 0}$ désigne le semi-groupe de la chaleur sur $\mathbb{R}^n$. On dit que

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$v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ est une *solution mild des équations de Navier–Stokes issue de* v_0, ou plus simplement que v est une solution de (NSI), sur $]0,T[$ si l'intégrale peut être définie dans $\mathcal{D}'(]0,T[\times\mathbb{R}^n)$ et si v satisfait (NSI). On sait d'après [6] que ces trois notions de solution sont équivalentes sous l'hypothèse $v \in \bigcap_{0<\tau<T} \mathrm{L}^2(]0,\tau[,E_2)$, où E_2 est défini dans la version anglaise.

La quatrième formulation (NSD2) a été introduite par Foias dans [7]. Cette fois, la donnée initiale apparaît dans l'équation. On dira que $v \in \mathcal{D}'(]-\infty,T[\times\mathbb{R}^n)$ est une *solution faible des équations de Navier–Stokes issue de* v_0, ou plus simplement que v est une solution de (NSD2), sur $]0,T[$ si $v \in \mathrm{L}^2_{\mathrm{loc}}(]-\infty,T[\times\mathbb{R}^n)$, $v=0$ presque partout dans $]-\infty,0[\times\mathbb{R}^n$ et s'il existe $p \in \mathcal{D}'(]-\infty,T[\times\mathbb{R}^n)$ tel que (v,p) satisfait (NSD2).

On introduit encore deux formulations où la pression est éliminée par un bon choix de fonctions tests. La cinquième formulation (NSW) correspond aux solutions turbulentes de Leray dans [12]. Une formulation voisine étant utilisée dans [10]. On dira que $v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ est une *très faible solution des équations de Navier–Stokes issue de* v_0, ou plus simplement que v est une solution de (NSW), sur $]0,T[$ si $v \in \mathrm{L}^2_{\mathrm{loc}}(]0,T[\times\mathbb{R}^n)$ et si v satisfait (NSW).

La formulation (NSWW) est utilisée par Amann dans [1] ou encore par Lions dans [13]. On dit que $v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ est une *très très faible solution des équations de Navier–Stokes issue de* v_0, ou plus simplement que v est une solution de (NSWW), sur $]0,T[$ si $v \in \mathrm{L}^2_{\mathrm{loc}}(]0,T[\times\mathbb{R}^n)$ et si v satisfait (NSWW).

Le théorème suivant permet de définir la trace d'une solution sur $t=0$.

THÉORÈME 1. – *Soit* $T>0$ *et* $v \in \bigcap_{0<\tau<T} \mathrm{L}^2(]0,\tau[,E_2)$. *On a équivalence entre* :

- v *est une solution de* (NSD1),
- v *est une solution de* (NS$\mathbb{P}$),
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *tel que* v *est une solution de* (NSI) *issue de* v_0,
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *tel que* v*, prolongé par* 0*, est une solution de* (NSD2) *issue de* v_0,
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *tel que* v *est une solution de* (NSW) *issue de* v_0,
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *tel que* v *est une solution de* (NSWW) *issue de* v_0.

De plus, on observe que dans ce cas, $v_0 \in \mathcal{S}'(\mathbb{R}^n)$ *est unique,* $v \in \mathcal{C}([0,T[,\mathcal{S}'(\mathbb{R}^n))$ *et pour tout* $t \in [0,T[$, $v(t) \overset{\mathcal{S}'}{=} \mathrm{e}^{t\Delta}v_0 - \int_0^t \mathbb{P}\nabla\,\mathrm{e}^{(t-s)\Delta}(v\otimes v)(s)\,\mathrm{d}s$. *On peut aussi choisir* p *tel que* p *soit défini presque partout en temps et* $\nabla p = \sum_{j,k=1}^3 \nabla R_j R_k(v_j v_k) \in \mathrm{L}^1(]0,T[,\mathcal{S}')$.

Idée de la démonstration. – Les propriétes de v_0, v et p sont des conséquences de la preuve de l'équivalence des trois premiers points (*voir* [6]). Il suffit alors de montrer que (NSI) $\Rightarrow$ (NSD2), (NSD2) $\Rightarrow$ (NSW) et (NSWW) $\Rightarrow$ (NS$\mathbb{P}$). Pour montrer que (NSI) $\Rightarrow$ (NSD2), on prolonge v, p, $t \mapsto \mathrm{e}^{t\Delta}v_0$, $t \mapsto \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla\cdot(v\otimes v)(s)\,\mathrm{d}s$ par 0 et on note de même les extensions. On montre alors que $\partial_t \mathrm{e}^{t\Delta}v_0 - \Delta\,\mathrm{e}^{t\Delta}v_0 = v_0 \otimes \delta_0$ dans $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^n)$ et $\partial_t(\int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla\cdot(v\otimes v)(s)\,\mathrm{d}s) = \Delta(\int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla\cdot(v\otimes v)(s)\,\mathrm{d}s) - \mathbb{P}\nabla\cdot(v\otimes v)$ dans $\mathcal{D}'(]-\infty,T[\times\mathbb{R}^n)$ en utilisant des fonctions test de la forme $\theta\varphi$ avec $\theta \in \mathcal{D}(\mathbb{R})$ et $\mathcal{D}(]-\infty,T[)$, respectivement, et $\varphi \in \mathcal{D}(\mathbb{R}^n)$. L'expression de ∇p permet de montrer que $\mathbb{P}\nabla\cdot(v\otimes v) = \nabla(v\otimes v) + \nabla p$, ce qui permet de conclure. L'implication (NSD2) $\Rightarrow$ (NSW) s'obtient en adaptant la preuve d'un résultat de Serrin (*voir* [15], p. 79). Enfin, pour montrer que (NSWW) $\Rightarrow$ (NS$\mathbb{P}$) il suffit de poser $F = \partial_t v - \Delta v + \nabla\cdot(v\otimes v)$ et de reprendre la preuve de (NSD1) $\Rightarrow$ (NS$\mathbb{P}$) dans [6] en remplaçant $-\nabla p$ par F. □

We give six different formulations of the Navier–Stokes equations and the associated notion of a solution. Let $T \in]0,+\infty]$. The first form is the most physical one:

$$\text{(NSD1)}\qquad \begin{cases} \partial_t v - \Delta v + \nabla\cdot(v\otimes v) + \nabla p = 0, \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \mathcal{D}'\big(]0,T[\times\mathbb{R}^n\big).^{1}$$

We will say that $v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ is a *weak solution of the Navier–Stokes equations*, or more simply that v is a solution of (NSD1), on $]0,T[$ if $v \in \mathrm{L}^2_{\mathrm{loc}}(]0,T[\times\mathbb{R}^n)$ and if there exists $p \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ such that (v,p) satisfy (NSD1).

By doing the assumptions which allow to project the equation onto divergence free vector fields, we obtain the formulation

$$\text{(NS}\mathbb{P}\text{)}\qquad \begin{cases} \partial_t v - \Delta v + \mathbb{P}\nabla\cdot(v\otimes v) = 0, \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \mathcal{D}'\big(]0,T[\times\mathbb{R}^n\big),$$

where $\mathbb{P}$ stands for the Leray projector. We will say that $v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ is a *weak solution of the projected Navier–Stokes equations*, or more simply that v is a solution of (NS$\mathbb{P}$), on $]0,T[$ if $\mathbb{P}\nabla\cdot(v\otimes v)$ can be defined in $\mathcal{D}'(]0,T[\times\mathbb{R}^n)$ and if v satisfies (NS$\mathbb{P}$). A third form which is often used since its introduction by Fujita and Kato in [5], is the integral formulation, obtained formally from (NS$\mathbb{P}$) by applying the Duhamel formula:

$$\text{(NSI)}\qquad v = \mathrm{e}^{t\Delta}v_0 - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla\cdot\big(v(s)\otimes v(s)\big)\,\mathrm{d}s \quad \text{in } \mathcal{D}'\big(]0,T[\times\mathbb{R}^n\big),$$

where here and subsequently $v_0 \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ and $(\mathrm{e}^{t\Delta})_{t\geqslant 0}$ stands for the heat semi-group on $\mathbb{R}^n$. We say that $v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ is a *mild solution of the Navier–Stokes equations arising from* v_0, or more simply that v is a solution of (NSI), on $]0,T[$ if the integral can be defined in $\mathcal{D}'(]0,T[\times\mathbb{R}^n)$ and if v satisfies (NSI).

It is proved in [6] that those three notions of solution are equivalent if $v \in \bigcap_{0<\tau<T}\mathrm{L}^2(]0,\tau[,E_2)$ with $E_2 = \overline{\mathcal{S}}^X$ (the closure of the Schwarz class in X) where $X = \mathrm{L}^2_{\mathrm{uloc}} = \{f \in \mathcal{S}'(\mathbb{R}^3) \mid \|f\|_{\mathrm{L}^2_{\mathrm{uloc}}} < +\infty\}$ with $\|f\|^2_{L^2_{\mathrm{uloc}}} = \sup_{x\in\mathbb{R}^n}\int_{|y-x|<1}|f(y)|^2\,\mathrm{d}y$.

The fourth formulation has been introduced by Foias in [7]. Here, the initial data appears in the PDE:

$$\text{(NSD2)}\qquad \begin{cases} \partial_t v - \Delta v + \nabla\cdot(v\otimes v) + \nabla p = v_0\otimes\delta_0, \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \mathcal{D}'\big(]-\infty,T[\times\mathbb{R}^n\big).$$

We will say that $v \in \mathcal{D}'(]-\infty,T[\times\mathbb{R}^n)$ is a *weak solution of the Navier–Stokes equations arising from* v_0, or more simply that v is a solution of (NSD2), on $]0,T[$ if $v \in \mathrm{L}^2_{\mathrm{loc}}(]-\infty,T[\times\mathbb{R}^n)$, $v=0$ almost everywhere in $]-\infty,0[\times\mathbb{R}^n$ and if there exists $p \in \mathcal{D}'(]-\infty,T[\times\mathbb{R}^n)$ such that (v,p) satisfy (NSD2). In the last formulations the pressure is killed thanks to a good choice of test functions. The fifth one corresponds to Leray's turbulent solutions in [12]:

$$\text{(NSW)}\qquad \begin{cases} (v(t),\varphi(t)) = (v_0,\varphi(0)) + \int_0^t(v,\Delta\varphi+\partial_t\varphi)\,\mathrm{d}s - \int_0^t(\nabla\cdot(v\otimes v),\varphi)\,\mathrm{d}s, \\ \quad\text{for almost every } t\in[0,T[\text{ and } \forall\varphi\in\mathcal{C}^\infty([0,t],\mathcal{D}_\sigma(\mathbb{R}^n)), \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{D}'(]0,T[\times\mathbb{R}^n). \end{cases}$$

The index σ stands for the divergence free condition. A similar formulation is used by Kozono and Taniuchi in [10]. We will say that $v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ is a *very weak solution of the Navier–Stokes equations arising from* v_0, or more simply that v is a solution of (NSW), on $]0,T[$ if $v \in \mathrm{L}^2_{\mathrm{loc}}(]0,T[\times\mathbb{R}^n)$ and if v satisfies (NSW). The sixth formulation is used by Amann in [1] or also by Lions in [13]:

$$\text{(NSWW)}\qquad \begin{cases} \int_0^T(\partial_t v - \Delta v + \nabla\cdot(v\otimes v),\varphi)\,\mathrm{d}s = (v_0,\varphi(0)), \quad \forall\varphi\in\mathcal{C}^\infty_{\mathrm{comp}}([0,T[,\mathcal{D}_\sigma(\mathbb{R}^n)), \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{D}'(]0,T[\times\mathbb{R}^n). \end{cases}$$

We say that $v \in \mathcal{D}'(]0,T[\times\mathbb{R}^n)$ is a *very very weak solution of the Navier–Stokes equations arising from* v_0, or more simply that v is a solution of (NSWW), on $]0,T[$ if $v \in \mathrm{L}^2_{\mathrm{loc}}(]0,T[\times\mathbb{R}^n)$ and if v satisfies (NSWW).

Theorem 1 allows to define the trace of a solution to the Navier–Stokes equations on $t=0$.

THEOREM 1. – *Let* $T>0$ *and* $v \in \bigcap_{0<\tau<T}\mathrm{L}^2(]0,\tau[,E_2)$. *The following propositions are equivalent*:
- v *is a solution of* (NSD1),

- *v is a solution of* (NS$\mathbb{P}$),
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *such that v is a solution of* (NSI) *arising from v_0,*
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *such that v, extended by* 0*, is a solution of* (NSD2) *arising from v_0,*
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *such that v is a solution of* (NSW) *arising from v_0,*
- $\exists v_0 \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div} v_0 = 0$ *such that v is a solution of* (NSWW) *arising from v_0.*

Moreover, we observe that in that case, $v_0 \in \mathcal{S}'(\mathbb{R}^n)$ is unique, $v \in \mathcal{C}([0,T[,\mathcal{S}'(\mathbb{R}^n))$ and for all $t \in [0,T[$, $v(t) \overset{\mathcal{S}'}{=} \mathrm{e}^{t\Delta}v_0 - \int_0^t \mathbb{P}\nabla\,\mathrm{e}^{(t-s)\Delta}(v\otimes v)(s)\,\mathrm{d}s$. We may choose p such that p is defined almost everywhere in time and $\nabla p = \sum_{j,k=1}^3 \nabla R_j R_k(v_j v_k) \in \mathrm{L}^1(]0,T[,\mathcal{S}')$.

Proof. – The properties for v_0, v and p are consequences of the proof of the equivalence between the first three points (*see* [6]). Clearly, we have the following diagram (where the single arrows are the missing implications):

$$\begin{array}{ccccc}
(\mathrm{NSI}) & \Longleftrightarrow & (\mathrm{NSD1}) & \Longleftrightarrow & (\mathrm{NS}\mathbb{P}) \\
 & \searrow & \Uparrow & & \uparrow \\
 & & (\mathrm{NSD2}) & \Rightarrow & (\mathrm{NSWW}) \\
 & & & \searrow & \Uparrow \\
 & & & & (\mathrm{NSW})
\end{array}$$

Step 1: We indicate how to prove that (NSI) $\Rightarrow$ (NSD2). A complete proof is given in [4]. Let $v \in \bigcap_{0<\tau<T} \mathrm{L}^2(]0,\tau[,E_2)$ be a solution of (NSI). We extend v, p, $t \mapsto \mathrm{e}^{t\Delta}v_0$, $t \mapsto \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla\cdot(v\otimes v)(s)\,\mathrm{d}s$ by 0 when $t<0$. In this step, we use those extensions and denote them in the same way. If $w : t \mapsto \mathrm{e}^{t\Delta}v_0$, then we have $\partial_t w - \Delta w = v_0 \otimes \delta_0$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^n)$. This is classical but the argument is useful later. We consider test functions $\theta\varphi$ where $\theta \in \mathcal{D}(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We then have by dominated convergence,

$$(\partial_t w - \Delta w, \theta\varphi) = -\lim_{\varepsilon\to 0}\int_\varepsilon^{+\infty} \big(v_0, \partial_t\big(\theta\,\mathrm{e}^{t\Delta}\varphi\big)\big)\,\mathrm{d}t.$$

Let $\varepsilon > 0$. We now introduce $\eta \in \mathcal{D}(\mathbb{R}^{+*})$ such that $\eta = 1$ on $[\varepsilon/2, A]$ where A is chosen such that $\operatorname{supp}\theta \subset]-\infty, A]$. We then have:

$$\begin{aligned}
-\int_\varepsilon^{+\infty} \big(v_0, \partial_t\big(\theta\,\mathrm{e}^{t\Delta}\varphi\big)\big)\,\mathrm{d}t &= -\int_\varepsilon^{+\infty} \big(v_0, \partial_t\big(\theta\eta\,\mathrm{e}^{t\Delta}\varphi\big)\big)\,\mathrm{d}t \\
&= -\big(v_0\otimes \mathbf{1}_{\{t\geqslant\varepsilon\}}, \partial_t\big(\theta\eta\,\mathrm{e}^{t\Delta}\varphi\big)\big) = \big(\partial_t\big(v_0\otimes\mathbf{1}_{\{t\geqslant\varepsilon\}}\big), \theta\eta\,\mathrm{e}^{t\Delta}\varphi\big) \\
&= \big(v_0\otimes\delta_\varepsilon, \theta\eta\,\mathrm{e}^{t\Delta}\varphi\big) = \big(v_0, \mathrm{e}^{\varepsilon\Delta}\varphi\big)\theta(\varepsilon)\eta(\varepsilon) \\
&= \big(\mathrm{e}^{\varepsilon\Delta}v_0, \varphi\big)\theta(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (v_0,\varphi)\theta(0) = (v_0\otimes\delta_0, \theta\varphi).
\end{aligned}$$

We now put $w : t \mapsto \mathbb{P}\nabla\cdot(v\otimes v)(t) \in \mathrm{L}^1(]0,T[,\mathcal{S}')$. We consider test functions $\theta\varphi$ where $\theta \in \mathcal{D}(]-\infty,T[)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We have:

$$\left(\partial_t \int_0^t \mathrm{e}^{(t-s)\Delta} w(s)\,\mathrm{d}s, \theta\varphi\right) = -\int_0^T\!\!\int_0^t \big(w(s), \partial_t\big(\theta\,\mathrm{e}^{(t-s)\Delta}\varphi\big) - \theta\Delta\,\mathrm{e}^{(t-s)\Delta}\varphi\big)\,\mathrm{d}s\,\mathrm{d}t.$$

For the first term, we use the same limiting argument as above for $\mathrm{e}^{t\Delta}v_0$:

$$-\int_0^T\!\!\int_0^t \big(w(s), \partial_t\big(\theta\,\mathrm{e}^{(t-s)\Delta}\varphi\big)\big)\,\mathrm{d}s\,\mathrm{d}t = \int_0^T \big(w(s), \theta(s)\varphi\big)\,\mathrm{d}s = (w, \theta\varphi).$$

For the second one, we easily obtain:

$$-\int_0^T\!\!\int_0^t \big(w(s), \theta\Delta\,\mathrm{e}^{(t-s)\Delta}\varphi\big)\,\mathrm{d}s\,\mathrm{d}t = \left(\Delta\int_0^t \mathrm{e}^{(t-s)\Delta}w(s)\,\mathrm{d}s, \theta\varphi\right).$$

We then have $\partial_t v - \Delta v + \mathbb{P}\nabla(v \otimes v) = v_0 \otimes \delta_0$ in $\mathcal{D}'(]-\infty, T[\times\mathbb{R}^3)$. We then observe that since $\nabla p = \sum_{j,k=1}^{3} \nabla R_j R_k(v_j v_k)$, simple calculations give $\mathbb{P}\nabla \cdot (v \otimes v) = \nabla \cdot (v \otimes v) + \nabla p$, which allows us to conclude.

Step 2: We now prove that (NSD2) $\Rightarrow$ (NSW). To do this, we adapt the proof of a result due to Serrin (*see* [15], p. 79). Let $v \in \bigcap_{0<\tau<T} \mathrm{L}^2(]0,\tau[, E_2)$ be a solution of (NSD2). Let $t \in [0, T[$. Let $\varphi \in \mathcal{C}^\infty([0,t], \mathcal{D}_\sigma)$ and continue φ into an element of $\mathcal{D}(]-\infty, T[\times\mathbb{R}^3)$, which we denote by φ again, such that $\operatorname{div}\varphi = 0$. Let $h > 0$ and $\theta \in \mathcal{D}(\mathbb{R})$ such that $\theta = 1$ on $[0,t]$, $\theta = 0$ on $[t+h, T[$ and $-C/h \leqslant \partial_t\theta \leqslant 0$. As $\theta\varphi \in \mathcal{D}(]-\infty, T[\times\mathbb{R}^3)$, $\operatorname{div}\theta\varphi = 0$ and v is a solution of (NSD2) arising from v_0, we have:

$$\int_0^T (v, \partial_t\varphi + \Delta\varphi)\theta \,\mathrm{d}s - \int_0^T (\nabla \cdot v \otimes v, \varphi)\theta \,\mathrm{d}s = -\int_0^T (v,\varphi)\partial_t\theta \,\mathrm{d}s - \big(v_0, \varphi(0)\big).$$

For the left-hand side, we have by Lebesgue's dominated convergence theorem,

$$\int_0^T (v, \partial_t\varphi + \Delta\varphi)\theta \,\mathrm{d}s - \int_0^T (\nabla \cdot v \otimes v, \varphi)\theta \,\mathrm{d}s \underset{h\to 0}{\longrightarrow} \int_0^t (v, \partial_t\varphi + \Delta\varphi) - \int_0^t (\nabla \cdot v \otimes v, \varphi)\,\mathrm{d}s.$$

It remains to show that there exists a set $N(v)$ of measure zero and depending only of v such that $\forall t \in [0, T[\setminus N(v)$, $-\int_0^T (v,\varphi)\partial_t\theta \,\mathrm{d}s \to (v(t), \varphi(t))$ when $h \to 0$. We observe that $\int_0^T \partial_t\theta \,\mathrm{d}s = \int_t^{t+h} \partial_t\theta \,\mathrm{d}s = -1$, so we can write, using Jensen's and Hölder's inequalities:

$$\begin{aligned}\left|-\int_0^T (v,\varphi)\partial_t\theta \,\mathrm{d}s - \big(v(t),\varphi(t)\big)\right| &\leqslant \frac{C}{h}\int_t^{t+h} \big|\big(v(t),\varphi(t)\big) - \big(v(s),\varphi(s)\big)\big|\,\mathrm{d}s \\ &\leqslant C\big\|\varphi(t)\big\|_N \left(\frac{1}{h}\int_t^{t+h} \big\|v(t)-v(s)\big\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}\,\mathrm{d}s\right)^{1/2} + C\|v\|_{\mathrm{L}^2(\mathrm{L}^2_{\mathrm{uloc}})}\left(\frac{1}{h^2}\int_t^{t+h}\big\|\varphi(t)-\varphi(s)\big\|^2_N\,\mathrm{d}s\right)^{1/2},\end{aligned}$$

where $N = \{f \in \mathrm{L}^2 \mid f \overset{\mathrm{L}^2}{=} \sum_{k\in\mathbb{N}} g_k,\ (g_k)_k \subset \mathrm{L}^2_{\mathrm{comp}},\ \forall k \in \mathbb{N},\ \operatorname{diam}(\operatorname{supp} g_k) \leqslant 1,\ \sum_{k\in\mathbb{N}} \|g_k\|_{\mathrm{L}^2} < +\infty\}$ is the predual of $\mathrm{L}^2_{\mathrm{uloc}}$. We have clearly $\mathcal{D} \hookrightarrow N \hookrightarrow \mathrm{L}^2_{\mathrm{comp}}$ when N is equipped with the norm $\|f\|_N = \inf\{\sum_{k\in\mathbb{N}} \|g_k\|_{\mathrm{L}^2}\}$ where the infimum is taken over all possible decompositions. Since $\varphi \in \mathcal{D}(\mathcal{D}) \hookrightarrow \mathrm{Lip}(N)$ and $v \in \bigcap_{0<\tau<T} \mathrm{L}^2(]0,\tau[, E_2)$, we conclude by applying Lebesgue's differentiation theorem to the first term.

Step 3: We prove that (NSWW) $\Rightarrow$ (NS$\mathbb{P}$). Let $v \in \bigcap_{0<\tau<T} \mathrm{L}^2(]0,\tau[, E_2)$ be a solution of (NSWW). We put $F = \partial_t v - \Delta v + \nabla \cdot (v \otimes v)$. Using the notations introduced in [6], we have $F \in \tau' \subset \mathcal{D}'(]0, T[\times\mathbb{R}^n)$ and by assumption $(F, \varphi) = 0$, for all $\varphi \in \mathcal{D}(]0, T[, \mathcal{D}_\sigma(\mathbb{R}^n))$. As in [6], we can apply the Leray projector on high frequency so as to get for all $j \in \mathbb{Z}$,

$$\mathbb{P}(\mathrm{Id} - S_j)F \overset{\tau'}{=} (\mathrm{Id} - S_j)(\partial_t v - \Delta v) + \mathbb{P}(\mathrm{Id} - S_j)\nabla \cdot (v \otimes v).$$

Let $j \in \mathbb{Z}$, we observe that $\mathbb{P}(\mathrm{Id} - S_j)F \overset{\tau'}{=} 0$. Indeed,

$$\begin{aligned}\mathbb{P}(\mathrm{Id} - S_j)F \overset{\tau'}{=} 0 \quad &\Leftrightarrow \quad (\mathrm{Id} - S_j)F \overset{\tau'}{=} \Delta^{-1}\nabla \otimes \nabla(\mathrm{Id} - S_j)F \\ &\Leftrightarrow \quad \big(\Delta - \nabla \otimes \nabla\big)(\mathrm{Id} - S_j)F \overset{\tau'}{=} 0\end{aligned}$$

and we have $\forall\varphi \in \mathcal{D}(]0, T[\times\mathbb{R}^n)$, $\operatorname{div}(\Delta - \nabla \otimes \nabla)\varphi = 0$, so

$$\big((\Delta - \nabla \otimes \nabla)(\mathrm{Id} - S_j)F, \varphi\big) = \big(F, \big(\mathrm{Id} - \check{S}_j\big)(\Delta - \nabla \otimes \nabla)\varphi\big) = 0.$$

Finally, we have for all $j \in \mathbb{Z}$, $(\mathrm{Id} - S_j)(\partial_t v - \Delta v) + \mathbb{P}(\mathrm{Id} - S_j)\nabla \cdot (v \otimes v) \overset{\tau'}{=} 0$ and we can let j tends to $-\infty$ in the same way as in [6]. □

Remarks. – (1) Leray shows in [12] the existence of solutions for (NSW) which fulfil energy inequalities, hence they are clearly solutions of all six formulations (we have not found this explicitly in the literature). We then have solutions of (NSI) not obtained by the fixed point method.

(2) One can obtain better results by selecting the most appropriate formulation. As an example, in the case of weak-strong uniqueness, (NSD2) used in [4] permits to extend the result obtained in [8] with (NSD1).

(3) The definition of $\mathbb{P}\nabla\cdot(v\otimes v)$ and $\int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla\cdot(v(s)\otimes v(s))\,\mathrm{d}s$ requires a regularity assumption on v. It is shown in [6] that if $v\in\bigcap_{0<\tau<T}\mathrm{L}^2(]0,\tau[,E_2)$ then the integral converges in $\mathcal{C}([0,T[,Y_\infty)$ via Littlewood–Paley analysis, where Y_∞ is contained in some inhomogeneous Besov space. Other occurrence of use of Littlewood–Paley analysis to define the integral with different assumption on v is in [2].

(4) The assumption $v\in\bigcap_{0<\tau<T}\mathrm{L}^2(]0,\tau[,E_2)$ is not restrictive in numerous cases and in particular in the physical framework. For example, it is proved in [3] that Koch and Tataru's solutions with initial data in L^2 fulfil energy inequalities and so satisfy the six formulations.

Theorem 1 has an immediate and interesting corollary:

COROLLARY 1. – *Let* $T>0$ *and* $v\in\bigcap_{0<\varepsilon<\tau<T}\mathrm{L}^2(]\varepsilon,\tau[,E_2)$.
If there exists $v_0\in\mathcal{S}'(\mathbb{R}^n)$, $\operatorname{div}v_0=0$ *such that* v *is a solution of* (NSI) *arising from* v_0, *then* v *is a solution of* (NSD1) *and* (NS$\mathbb{P}$).

In particular, all solutions v of (NSI) fulfiling the condition $\sup_{0<s<t}\sqrt{s}\|v(s)\|_{\mathrm{L}^\infty}<+\infty$ for all $t\in]0,T[$ are also solutions of (NSD1) and (NS$\mathbb{P}$). This is the case of many solutions described as critical and constructed via the fixed point method. As an example, one can cite solutions constructed by Auscher and Tchamitchian [2], Koch and Tataru [9], Kozono and Yamazaki [11] and Planchon [14] (again the equivalence between formulations is not explicitly stated).

[1] We do the slight abuse of notation which consists in writing a vector field v in a scalar space X instead of X^n.

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