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Équations aux dérivées partielles/*Partial Differential Equations*

On one-homogeneous solutions to elliptic systems in two dimensions

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Abstract In this Note we consider a class of nonlinear second order elliptic systems in divergence form and two independent variables. We prove that all Lipschitz continuous one-homogeneous weak solutions are linear. ***To cite this article: D. Phillips, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 39–42.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur les solutions un-homogènes des systèmes elliptiques en dimension deux

Résumé Dans cette Note, nous considérons une classe de systèmes d'équations elliptiques non linéaires du second ordre sous forme divergence à deux variables indépendantes. Nous prouvons que toutes les solutions faibles un-homogènes et Lipschitz continues sont linéaires. ***Pour citer cet article : D. Phillips, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 39–42.***
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Let $M^{N\times 2}$ denote the space of real $N\times 2$ matrices and let $\mathbf{A}\in C^1(M^{N\times 2};M^{N\times 2})$,

$$\mathbf{A}=\left[A_i^\alpha\right],\quad 1\leqslant\alpha\leqslant N,\ 1\leqslant i\leqslant 2.$$

We consider the following elliptic system in divergence form

$$\partial_{x^i}A_i^\alpha(D\mathcal{U})=0\quad\text{for }1\leqslant\alpha\leqslant N,\tag{1}$$

where $\mathcal{U}:\mathbf{x}\in\mathbb{R}^2\to\mathbb{R}^N$. To say that the system is *elliptic* means that $\mathbf{A}$ satisfies the condition

$$\partial_{P_j^\beta}A_i^\alpha(\mathbf{P})\zeta^i\zeta^j\eta_\alpha\eta_\beta>0\quad\text{for all }\mathbf{P}\in M^{N\times 2},\ \boldsymbol{\zeta}\in\mathbb{R}^2,\ \boldsymbol{\eta}\in\mathbb{R}^N,\ \boldsymbol{\zeta}\neq 0,\ \boldsymbol{\eta}\neq 0.\tag{2}$$

We consider weak solutions that are *one-homogeneous*, that is,

$$\mathcal{U}(t\mathbf{x})=t\mathcal{U}(\mathbf{x})\quad\text{for all }t\geqslant 0,\mathbf{x}\in\mathbb{R}^2.$$

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Expressing $\mathbf{x}$ in polar coordinates we see that $\mathcal{U}$ is Lipschitz continuous and one-homogeneous if and only if

$$\mathcal{U} = r\mathbf{g}(\theta) \quad \text{where } \mathbf{g} \in \mathrm{C}^{0,1}\big(\mathbb{S}^1; \mathbb{R}^N\big).$$

We prove the following.

THEOREM. – *Let $\mathcal{U}$ be a Lipschitz continuous one-homogeneous weak solution to the system* (1). *Then $\mathcal{U}$ is linear, i.e., $\mathcal{U} = r\mathbf{g}(\theta)$ where $\mathbf{g} = \mathbf{c}\cos\theta + \mathbf{d}\sin\theta$ for some $\mathbf{c}$ and $\mathbf{d} \in \mathbb{R}^N$.*

Our calculation is motivated by the question of regularity for weak solutions to elliptic variational problems in two independent variables. Assuming the more stringent hypothesis of *strong ellipticity,*

$$\partial_{P_j^\beta} A_i^\alpha(\mathbf{P})\pi_\alpha^i \pi_\beta^j > 0 \quad \text{for all } \mathbf{P} \in M^{N\times 2},\ \boldsymbol{\pi} \in M^{2\times N},\ \boldsymbol{\pi} \neq 0.$$

C.B. Morrey proved that Lipschitz continuous weak solutions to (1) are continuously differentiable, see [2]. It is not known however if there exist nondifferentiable weak solutions to (1), (2). Our result is that such a singular solution could not be one-homogeneous.

For the case of n independent variables, $n \geqslant 3$, examples of nondifferentiable weak solutions to strongly elliptic systems do exist. See [1,3], and [4]. All of these examples are various one-homogeneous functions.

Proof. – Set

$$\mathbf{e}_r = (\cos\theta, \sin\theta)^t \quad \text{and} \quad \mathbf{e}_\theta = (-\sin\theta, \cos\theta)^t.$$

Consider a weak solution $\mathcal{U}$ where

$$\mathcal{U} = r\mathbf{g}(\theta), \quad \mathbf{g} \in \mathrm{C}^{0,1}\big(\mathbb{S}^1; \mathbb{R}^N\big).$$

Then

$$D\mathcal{U} = \mathbf{g} \otimes \mathbf{e}_r + \mathbf{g}' \otimes \mathbf{e}_\theta.$$

We see $\partial_r D\mathcal{U} = 0$ for $r > 0$. Now (1) is equivalent to

$$\partial_r\big(r\mathbf{A}(D\mathcal{U})\mathbf{e}_r\big) + \partial_\theta\big(\mathbf{A}(D\mathcal{U})\mathbf{e}_\theta\big) = 0.$$

Since $\partial_r D\mathcal{U} = 0$ this reduces to the weak ordinary differential equation

$$\mathbf{A}\mathbf{e}_r + (\mathbf{A}\mathbf{e}_\theta)' = 0 \quad \text{on } \mathbb{S}^1, \tag{3}$$

where $\mathbf{A} = \mathbf{A}(\mathbf{g} \otimes \mathbf{e}_r + \mathbf{g}' \otimes \mathbf{e}_\theta)$. This implies that $\mathbf{A}\mathbf{e}_\theta$ has a continuous representative. Since $\mathbf{A}(P)$ and $\mathbf{g}(\theta)$ are continuous, we see that if $\mathbf{g}'$ does not have a continuous representative then there exist $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^N$, $\mathbf{a} \neq \mathbf{b}$, and θ_0 so that for $\theta = \theta_0$ we have

$$\begin{aligned} \mathbf{A}_1 &\equiv \mathbf{A}\big(\mathbf{g}(\theta_0) \otimes \mathbf{e}_r(\theta_0) + \mathbf{a} \otimes \mathbf{e}_\theta(\theta_0)\big)\mathbf{e}_\theta(\theta_0) \\ &= \mathbf{A}\big(\mathbf{g}(\theta_0) \otimes \mathbf{e}_r(\theta_0) + \mathbf{b} \otimes \mathbf{e}_\theta(\theta_0)\big)\mathbf{e}_\theta(\theta_0) \equiv \mathbf{A}_2. \end{aligned}$$

Using (2) this leads to

$$\begin{aligned} 0 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{A}_1 - \mathbf{A}_2) \\ &= \int_0^1 \partial_{P_j^\beta} A_i^\alpha\big(L(t)\big)(\mathbf{a} - \mathbf{b})_\alpha(\mathbf{a} - \mathbf{b})_\beta(\mathbf{e}_\theta)^i(\mathbf{e}_\theta)^j\,\mathrm{d}t > 0, \end{aligned}$$

where

$$\mathbf{L}(t) = \mathbf{g}(\theta_0) \otimes \mathbf{e}_r(\theta_0) + \big(\mathbf{b} + t(\mathbf{a} - \mathbf{b})\big) \otimes \mathbf{e}_\theta(\theta_0).$$

Thus we have $\mathbf{g} \in \mathrm{C}^1$.

We now prove that $\mathbf{g}''$ exists. From (3) we get

$$\boldsymbol{\mathcal{A}}(\theta) = \mathbf{A}\big(\mathbf{g}(\theta) \otimes \mathbf{e}_r(\theta) + \mathbf{g}'(\theta) \otimes \mathbf{e}_\theta(\theta)\big)\mathbf{e}_\theta(\theta) \in \mathrm{C}^1.$$

Fix $\tilde{\theta}$. Set

$$\tilde{\boldsymbol{\mathcal{A}}}(\theta) = \mathbf{A}\big(\mathbf{g}(\theta) \otimes \mathbf{e}_r(\theta) + \mathbf{g}'(\tilde{\theta}) \otimes \mathbf{e}_\theta(\theta)\big)\mathbf{e}_\theta(\theta).$$

Since $\mathbf{A}$ and $\mathbf{g}$ are C^1 if follows that $\tilde{\boldsymbol{\mathcal{A}}}(\theta) \in \mathrm{C}^1$ as well. We write

$$\frac{\boldsymbol{\mathcal{A}}(\theta) - \tilde{A}(\theta)}{\theta - \tilde{\theta}} = \frac{\boldsymbol{\mathcal{A}}(\theta) - \boldsymbol{\mathcal{A}}(\tilde{\theta})}{\theta - \tilde{\theta}} - \frac{\tilde{\boldsymbol{\mathcal{A}}}(\theta) - \boldsymbol{\mathcal{A}}(\tilde{\theta})}{\theta - \tilde{\theta}}. \tag{4}$$

Since $\boldsymbol{\mathcal{A}}(\theta) \in \mathrm{C}^1$ the first term on the right side has a limit as $\theta \to \tilde{\theta}$. Furthermore since $\boldsymbol{\mathcal{A}}(\tilde{\theta}) = \tilde{\boldsymbol{\mathcal{A}}}(\tilde{\theta})$ the second term on the right also has a limit. Next we see

$$\frac{\boldsymbol{\mathcal{A}}(\theta) - \tilde{\boldsymbol{\mathcal{A}}}(\theta)}{\theta - \tilde{\theta}} = \mathbf{N} \cdot \left[\frac{\mathbf{g}'(\theta) - \mathbf{g}'(\tilde{\theta})}{\theta - \tilde{\theta}}\right], \tag{5}$$

where

$$N^{\alpha\beta}(\theta) = \left(\int_0^1 \partial_{P_j^\beta} A_i^\alpha\big(\mathbf{L}(t)\big)\,\mathrm{d}t\right)\big(\mathbf{e}_\theta(\theta)\big)^i\big(\mathbf{e}_\theta(\theta)\big)^j$$

with

$$\mathbf{L}(t) = \mathbf{g}(\theta) \otimes \mathbf{e}_r(\theta) + \big(\mathbf{g}'(\tilde{\theta}) + t\big(\mathbf{g}'(\theta) - \mathbf{g}'(\tilde{\theta})\big)\big) \otimes \mathbf{e}_\theta(\theta).$$

Note that $\mathbf{N}(\theta)$ is continuous and that

$$N^{\alpha\beta}(\tilde{\theta}) = \partial_{P_j^\beta} A_i^\alpha\big(\mathbf{g}(\tilde{\theta}) \otimes \mathbf{e}_r(\tilde{\theta}) + \mathbf{g}'(\tilde{\theta}) \otimes \mathbf{e}_\theta(\tilde{\theta})\big)\big(\mathbf{e}_\theta(\tilde{\theta})\big)^i\big(\mathbf{e}_\theta(\tilde{\theta})\big)^j.$$

It follows from (2) that $\mathbf{N}^{-1}(\tilde{\theta})$ exists. From this, (4), and (5) we see that $\mathbf{g}''(\tilde{\theta})$ exists.

Carrying out the differentiation in (3) gives

$$\partial_{P_j^\beta} A_i^\alpha(\mathbf{g} \otimes \mathbf{e}_r + \mathbf{g}' \otimes \mathbf{e}_\theta)(\mathbf{e}_\theta)^i(\mathbf{e}_\theta)^j(\mathbf{g} + \mathbf{g}'')_\beta = 0$$

for $1 \leqslant \alpha \leqslant N$. From (2) we see $\mathbf{g} + \mathbf{g}'' = 0$ on $\mathbb{S}^1$. This implies that $\mathbf{g} = \mathbf{c}\cos\theta + \mathbf{d}\sin\theta$ for some $\mathbf{c}$ and $\mathbf{d} \in \mathbb{R}^N$. $\square$

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