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Théorie des nombres/*Number Theory*

Some transcendental functions over function fields with positive characteristic

Jia-Yan Yao

Department of Mathematics, Nonlinear Science Center, Wuhan University, Wuhan 430072, People's Republic of China

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Abstract In this work we shall define two families of functions over function fields with positive characteristic and show that such a function is transcendental if and only if its generating sequence is not ultimately zero. As a result, the Carlitz exponential and the Carlitz logarithm are transcendental functions. Our proof is elementary in the sense that we only use a theorem due to H. Sharif and C. Woodcock, and to T. Harase which generalizes the theorem of Christol about automatic sequences. ***To cite this article: J.-Y. Yao, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 939–943.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Certaines fonctions transcendantes sur des corps de fonctions de caractéristique positive

Résumé Dans ce travail, nous allons définir deux familles de fonctions sur des corps de fonctions de caractéristique positive et montrer qu'une telle fonction est transcendante si et seulement si sa suite génératrice n'est pas ultimement nulle. Comme conséquence, l'exponentielle de Carlitz et le logarithme de Carlitz sont des fonctions transcendantes. Notre preuve est élémentaire dans le sens que nous allons utiliser seulement un théorème dû à H. Sharif et C. Woodcock, ainsi qu'à T. Harase qui généralise le théorème de Christol pour les suites automatiques. ***Pour citer cet article : J.-Y. Yao, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 939–943.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit p ($p \geqslant 2$) un nombre premier. Soit $q = p^s$ avec $s \in \mathbb{N}_*$ où $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$ et $\mathbb{N} = \{0, 1, 2, \ldots\}$ est l'ensemble des entiers naturels. Nous désignons par $\mathbb{F}_q$ le corps fini composé de q éléments et par $\mathbb{F}_q[T]$ l'anneau intègre des polynômes à coefficients dans $\mathbb{F}_q$. Notons $\mathbb{F}_q(T)$ le corps de fractions de $\mathbb{F}_q[T]$. Pour tous les $P, Q \in \mathbb{F}_q[T]$ avec $Q \neq 0$, définissons $v_\infty(P/Q) = -(\deg P - \deg Q)$. Alors v_∞ est une valuation discrète sur $\mathbb{F}_q(T)$.

Pour tout entier $j \in \mathbb{N}_*$, nous définissons

$$[j] := T^{q^j} - T, \quad L_j := \prod_{i=1}^{j} [i] \quad \text{et} \quad D_j := \prod_{l=0}^{j-1} \left(T^{q^j} - T^{q^l}\right).$$

E-mail address: yaojiaya@public.wh.hb.cn (J.-Y. Yao).

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Par convention, nous posons $L_0 = 1$ et $D_0 = 1$. Il est clair que pour tout $j \in \mathbb{N}$, nous avons $v_\infty(D_j) = -jq^j$ et $v_T(L_j) = j$, où v_T est la valuation T-adique, i.e., pour tout $Q \in \mathbb{F}_q[T]$, $v_T(Q)$ est le plus grand entier k tel que T^k divise Q dans $\mathbb{F}_q[T]$. Les polynômes $[j]$, L_j et D_j sont fondamentaux pour l'arithmétique de $\mathbb{F}_q[T]$ et le lecteur pourra consulter par exemple [10, p. 46] pour leurs diverses propriétés.

Soit $K := \mathbb{F}_q(T)$ et soit t une indéterminée sur K. Notons $K[t]$ l'anneau intègre des polynômes en t à coefficients dans K. Le corps de fractions de $K[t]$ est noté $K(t)$. Soit ω_t la valuation t-adique sur $K[t]$, i.e., pour tout $P \in K[t]$, $\omega_t(P)$ est le plus grand entier k tel que t^k divise P dans $K[t]$. Étendons ω_t à $K(t)$ et désignons par $K((t))$ le complété topologique de $K(t)$ pour ω_t. Alors pour tout $f \in K((t))$, nous avons $f = \sum_{m=k}^{+\infty} a(m)t^m$ avec $\omega_t(f) = k \in \mathbb{Z}$, où $a(k) \neq 0$ et $a(m) \in K$ pour tout $m \in \mathbb{Z}$ $(m \geqslant k)$.

Soit $\gamma \in \mathbb{N}_*$ et soit $u = (u(m))_{m\geqslant 0}$ une suite à valeurs dans $\mathbb{F}_q$. Nous posons

$$\psi_u^{(\gamma)} = \sum_{j=0}^{+\infty} \frac{u(j)}{D_j^\gamma} t^{q^j} \quad \text{et} \quad \lambda_u^{(\gamma)} = \sum_{i=0}^{+\infty} \frac{u(i)}{L_i^\gamma} t^{q^i}.$$

Exemple. – D'abord soit $u = ((-1)^j)_{j\geqslant 0}$. Alors $\psi_u^{(1)} = \sum_{j=0}^{+\infty}(-1)^j(t^{q^j}/D_j)$ est l'exponentielle de Carlitz. Ensuite soit v la suite constante 1. Cette fois nous obtenons le logarithme de Carlitz $\lambda_v^{(1)} = \sum_{k=0}^{+\infty}(t^{q^k}/L_k)$. Ces deux fonctions, introduites par L. Carlitz, ont marqué le début de la théorie du module de Carlitz (*voir* par exemple [7] et [10]).

Le but principal de cet article est de montrer les deux résultats suivants :

THÉORÈME 1. – *La fonction $\psi_u^{(\gamma)}$ est transcendante sur le corps $K(t)$ si et seulement si la suite u n'est pas ultimement nulle. En particulier, la fonction exponentielle de Carlitz est transcendante sur $K(t)$.*

THÉORÈME 2. – *La fonction $\lambda_u^{(\gamma)}$ est transcendante sur le corps $K(t)$ si et seulement si la suite u n'est pas ultimement nulle. En particulier, la fonction logarithme de Carlitz est transcendante sur $K(t)$.*

1. Introduction

In 1990, Allouche gave an elementary proof (*see* [4]) of a theorem of Wade about the transcendence of the formal power series Π (*cf.* [15]). This marks the inception of applications of automata theory in the study of the theory of Carlitz module and it follows many interesting and important results (see, for example, [5,12,14], and their bibliographies). Inspired by [4], we would like to give an automata-style proof of the following theorem of Wade [15]: the value of the Carlitz exponential (resp. logarithm) function at a nonzero algebraic argument is transcendental. Unfortunately for the moment it seems that such a result is quite beyond the grasp of automata theory, which demands a good knowledge about the coefficients of the formal power series in discussion. Nevertheless we can still notice that the theorem of Wade quoted above implies immediately the weaker result that the Carlitz exponential and the Carlitz logarithm are transcendental functions. This remark leads us to consider and prove the two theorems presented in this note, much weaker than our original purpose.

Before stating our principal results, we shall give below some notations and definitions.

Let p $(p \geqslant 2)$ be a prime number and let $q = p^s$ with $s \in \mathbb{N}_*$ where $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Let $\mathbb{F}_q$ be the finite field with q elements and let $\mathbb{F}_q[T]$ be the integral domain of polynomials with coefficients in $\mathbb{F}_q$. We denote by $\mathbb{F}_q(T)$ the fraction field of $\mathbb{F}_q[T]$. For all $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$, define $v_\infty(P/Q) = -(\deg P - \deg Q)$. Then v_∞ is a discrete valuation over $\mathbb{F}_q(T)$.

For every integer $j \in \mathbb{N}_*$, we define

$$[j] := T^{q^j} - T, \quad L_j := \prod_{i=1}^{j} [i] \quad \text{and} \quad D_j := \prod_{l=0}^{j-1} \left(T^{q^j} - T^{q^l}\right).$$

By convention we put $L_0 = 1$ and $D_0 = 1$. Then for every $j \in \mathbb{N}$, we have $v_\infty(D_j) = -jq^j$ and $v_T(L_j) = j$, where v_T is the T-adic valuation over $\mathbb{F}_q(T)$ such that for all $Q \in \mathbb{F}_q[T]$, $v_T(Q)$ is the greatest integer k such that T^k divides Q in the integral domain $\mathbb{F}_q[T]$. The polynomials $[j]$, L_j and D_j are fundamental for the arithmetic of $\mathbb{F}_q[T]$ and the reader can consult for example [10, p. 46] for their various properties.

Let $K := \mathbb{F}_q(T)$ and let t be an indeterminate over K. Let $K[t]$ be the integral domain of polynomials in t with coefficients in K. The fraction field of $K[t]$ is denoted by $K(t)$. Let ω_t be the t-adic valuation over $K[t]$, i.e., for all $P \in K[t]$, $\omega_t(P)$ is the greatest integer k such that t^k divides P in $K[t]$. Extend ω_t over $K(t)$ and denote by $K((t))$ the topological completion of $K(t)$ for ω_t. Then for every $f \in K((t))$, we have $f = \sum_{m=k}^{+\infty} a(m)t^m$ with $\omega_t(f) = k \in \mathbb{Z}$, where $a(k) \neq 0$ and $a(m) \in K$ for all $m \in \mathbb{Z}$ $(m \geqslant k)$.

Let $\gamma \in \mathbb{N}_*$ and let $u = (u(m))_{m\geqslant 0}$ be a sequence with terms in $\mathbb{F}_q$. Define

$$\psi_u^{(\gamma)} = \sum_{j=0}^{+\infty} \frac{u(j)}{D_j^\gamma} t^{q^j} \quad \text{and} \quad \lambda_u^{(\gamma)} = \sum_{i=0}^{+\infty} \frac{u(i)}{L_i^\gamma} t^{q^i}.$$

Example. – First let $u = ((-1)^j)_{j\geqslant 0}$. Then $\psi_u^{(1)} = \sum_{j=0}^{+\infty} (-1)^j (t^{q^j}/D_j)$ is just the Carlitz exponential function. Second let v be the constant sequence 1. Then we get the Carlitz logarithm function $\lambda_v^{(1)} = \sum_{k=0}^{+\infty} (t^{q^k}/L_k)$. These two functions, introduced firstly by Carlitz, marked the beginning of the theory of Carlitz module (*see*, for example, [7] and [10]).

The aim of this work is to show the following two results:

THEOREM 1. – *The function $\psi_u^{(\gamma)}$ is transcendental over the field $K(t)$ if and only if the sequence u is not ultimately zero. In particular, the Carlitz exponential function is transcendental over $K(t)$.*

THEOREM 2. – *The function $\lambda_u^{(\gamma)}$ is transcendental over the field $K(t)$ if and only if the sequence u is not ultimately zero. In particular, the Carlitz logarithm function is transcendental over $K(t)$.*

Remark. – It is well known that the Carlitz exponential function and the Carlitz logarithm function are transcendental. Indeed Theorem 1 is a special case of a general theorem of Wade (*see* [17, Theorem 5]), established by a purely analytic method. However since $\lambda_u^{(\gamma)}$ is not an entire function, it seems difficult to deduce Theorem 2 directly from Theorem 5 in [17]. The interest of our present work is to offer a pure algebraic but elementary method to treat $\psi_u^{(\gamma)}$, $\lambda_u^{(\gamma)}$, and similar problems (*cf.* [15,16,18], and [19] and its bibliography).

2. Some preliminary results

We begin with a theorem independently shown by Sharif and Woodcock (*cf.* [13]) and by Harase (*cf.* [11]), which generalizes the theorem of Christol (*cf.* [8], *see* also [9] and [2]) about automatic sequences. For our need, we adopt the reformulation of Allouche (*cf.* [3]) and only present a rather special case.

THEOREM 3. – *Let $\overline{K}$ be an algebraic closure of K and let $v = (v(m))_{m\geqslant 0}$ be a sequence with terms in K. Then the formal power series $\sum_{m=0}^{+\infty} v(m)t^m$ is algebraic over $K(t)$ if and only if the $\overline{K}$-vector space generated by the set of sequences*

$$\left\{ \left(v^{1/q^a} \left(q^a m + b \right) \right)_{m\geqslant 0} \mid a, b \in \mathbb{N},\ 0 \leqslant b < q^a \right\}$$

has finite dimension over $\overline{K}$.

When the sequence $v = (v(m))_{m\geqslant 0}$ takes a very particular form, we obtain

THEOREM 4. – *Let $w = (w(m))_{m\geqslant 0}$ be a sequence with terms in K. Then the formal power series $\sum_{m=0}^{+\infty} w(m)t^{q^m}$ is algebraic over $K(t)$ if and only if the vector space generated over $\overline{K}$ by the sequences $(w^{1/q^k}(m+k))_{m\geqslant 0}$ $(k \in \mathbb{N})$ has finite dimension over $\overline{K}$.*

Remark. – In the case that the field K is finite, the reader can consult [6, p. 77] for a previous version of Theorem 4 (*see* also [1] and [21, Corollary 1.4] for another combinatorial version).

Fix $n \in \mathbb{N}$ $(n \geqslant 2)$ and let S_n be the symmetric group of degree n, i.e., S_n is the set of all permutations on the set $\Omega_n := \{1, 2, \ldots, n\}$. Let $a_1 < a_2 \cdots < a_n$ and $b_1 < b_2 \cdots < b_n$ be real numbers. For all $\sigma \in S_n$, define $g(\sigma) = \sum_{j=1}^{n} a_j b_{\sigma(j)}$. Finally for all $j \in \Omega_n$, define $\sigma_0(j) = n + 1 - j$. Then $\sigma_0 \in S_n$.

LEMMA. – *For every* $\sigma \in S_n \backslash \{\sigma_0\}$, *we have* $g(\sigma) > g(\sigma_0)$.

Proof. – We need only note that if $\sigma \in S_n \backslash \{\sigma_0\}$ and $j, l \in \Omega_n$ such that $j < l$ and $\sigma(j) < \sigma(l)$, then we have $g(\sigma') - g(\sigma) = (b_{\sigma(j)} - b_{\sigma(l)})(a_l - a_j) < 0$ where $\sigma' \in S_n$ is defined by $\sigma'(j) = \sigma(l)$, $\sigma'(l) = \sigma(j)$ and $\sigma'(i) = \sigma(i)$ for all $i \in \Omega_n \backslash \{j, l\}$. $\square$

3. Proof of Theorem 1

The necessity is quite evident. So we need only show the sufficiency. Let $u = (u(m))_{m \geqslant 0}$ be a sequence not ultimately zero in $\mathbb{F}_q$ and let $V = \{m \in \mathbb{N} \mid u(m) \neq 0\}$. Then V is an infinite set. For every $k \in \mathbb{N}$, put

$$W_k = \left(\frac{u(m+k)}{D_{m+k}^{\gamma/q^k}} \right)_{m \geqslant 0}.$$

By Theorem 4, to show that $\psi_u^{(\gamma)}$ is transcendental over $K(t)$, it suffices to show that for all $n \in \mathbb{N}_*$ $(n \geqslant 2)$, we can find n natural numbers $c_1 < c_2 \cdots < c_n$ such that the n sequences $W_{c_1}, W_{c_2}, \ldots, W_{c_n}$ are $\overline{K}$-linearly independent.

Since the set V is infinite, we can find $d_1 < d_2 \cdots < d_n$ in V such that $d_1 \geqslant n$. For every integer $j \in \mathbb{N}$ $(1 \leqslant j \leqslant n)$, put $c(j) = d(j) - (n + 1 - j)$. Then $c(j) \in \mathbb{N}$ and $c(j) > c(j-1)$ if $j \geqslant 2$. Set $H_n = \det(W_{c(i)}(j))_{1 \leqslant i,\ j \leqslant n}$ and prove $H_n \neq 0$. Indeed we have

$$\begin{aligned} H_n &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} W_{c(i)}\bigl(\sigma(i)\bigr) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \frac{u(\sigma(i) + c(i))}{D_{\sigma(i)+c(i)}^{\gamma/q^{c(i)}}} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{n} u\bigl(\sigma(i) + c(i)\bigr) \right) \left(\prod_{i=1}^{n} D_{\sigma(i)+c(i)}^{-\gamma/q^{c(i)}} \right), \end{aligned}$$

where for all $\sigma \in S_n$, we have $\operatorname{sgn}(\sigma) = 1$ or -1 according as σ is even or odd.

For every $\sigma \in S_n$, put $G_\sigma = \prod_{i=1}^{n} D_{\sigma(i)+c(i)}^{-\gamma/q^{c(i)}}$. Then we obtain

$$\begin{aligned} v_\infty(G_\sigma) &= -\sum_{i=1}^{n} \frac{\gamma}{q^{c(i)}} v_\infty(D_{\sigma(i)+c(i)}) = \sum_{i=1}^{n} \frac{\gamma}{q^{c(i)}} \bigl(\sigma(i) + c(i)\bigr) q^{\sigma(i)+c(i)} \\ &= \gamma \sum_{i=1}^{n} \bigl(\sigma(i) + c(i)\bigr) q^{\sigma(i)} = \gamma \sum_{j=1}^{n} j q^j + \gamma \sum_{i=1}^{n} c(i) q^{\sigma(i)}. \end{aligned}$$

By our lemma, we know that for all $\sigma \in S_n \backslash \{\sigma_0\}$, we have

$$v_\infty(G_\sigma) > v_\infty(G_{\sigma_0}).$$

Remark also that for all $i \in \mathbb{N}$ $(1 \leqslant i \leqslant n)$, we have

$$u\bigl(\sigma_0(i) + c(i)\bigr) = u\bigl(n + 1 - i + c(i)\bigr) = u\bigl(d(i)\bigr) \neq 0.$$

So $v_\infty(H_n) = v_\infty(G_{\sigma_0}) < +\infty$ and $H_n \neq 0$. Thus the n sequences $W_{c_1}, W_{c_2}, \ldots, W_{c_n}$ must be $\overline{K}$-linearly independent. $\square$

4. Proof of Theorem 2

The proof is almost identical as above except that this time we shall use the valuation v_T instead of v_∞. Indeed if we compute as above the ∞-adic valuation of each nonzero summand of the determinant in discussion, we can see that they are all equal. So the ∞-adic valuation is useless for the present problem. □

5. Further study

We can also consider the series $\sum_{m=1}^{+\infty}(u(m)/[m]^\gamma)t^{q^m}$ and conjecture that it is transcendental over the function field $K(t)$ if and only if the sequence u is not ultimately zero (*see* [5,12], and [20] for related topics about this function). The necessity is evident. However until now we have not yet found a proof for the sufficiency.

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