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# Complete classification of homoclinic cycles in $\mathbb{R}^4$ in the case of a symmetry group $G \subset \mathrm{SO}(4)$

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**Abstract** Some homoclinic cycles in $\mathbb{R}^4$ with symmetry groups contained in SO(4) have already appeared in the literature. These cycles have 2, 3, 6, 8, 12, or 24 equilibria. In this Note we show that this classification is complete using a result in diophantine trigonometric equations with rational angles. ***To cite this article: N. Sottocornola, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 859–864.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

**Classification complète des cycles homoclines de $\mathbb{R}^4$ dans le cas d'un groupe de symétrie $G \subset \mathrm{SO}(4)$**

**Résumé** Des cycles homoclines avec groupes de symétrie contenus dans SO(4) sont déjà apparus dans la littérature. Ces cycles ont 2, 3, 6, 8, 12 ou 24 points d'équilibre. Dans cette Note, on montre que cette classification est complète en utilisant un résultat sur les équations diophantiennes à angles rationnels. ***Pour citer cet article : N. Sottocornola, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 859–864.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## *Version française abrégée*

Les cycles homoclines robustes (CHR) apparaissent dans l'étude des champs de vecteurs équivariants. Dans le cas d'un groupe de symétrie $G$ fini ils ont été classés, sur la base de l'action de $G$, en cycles de type A, B et C (*cf.* [2]). La question de savoir quelles sont les valeurs possibles pour $n$, le nombre de points d'équilibre du cycle, a été étudiée dans [10] pour les cycle de type B ($n = 2$, 3 ou 6) et C ($n = 4$ ou 8) et dans [11] pour les cycles de type A avec groupes de symétrie non contenus dans SO(4) ($n$ pair ou $n = 3$). Dans cette note on complète la classification dans le cas $G \subset \mathrm{SO}(4)$.

On peut imaginer un CHR comme une suite finie de plans $P_i$ invariants par rapport au flot, qui sont, chacun, l'image du précédent par l'action d'un élément $\gamma \in G : \gamma(P_i) = P_{i+1}$. Chaque plan contient deux points d'équilibre connectés entre eux par une ligne intégrale. Pour classer les cycles on se sert de la notion d'*angle de structure* du cycle. Le premier angle de structure est noté $t$ et représente l'angle entre deux

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points d'équilibre consécutifs sur le plan invariant (*angle de connexion*). Pour définir le deuxième angle de structure on considère la suite d'hyperplans $Q_i$, sommes vectorielles de deux plans invariants consécutifs : $Q_i = P_{i-1} + P_i$. L'angle entre deux hyperplans consécutifs est noté $s$ (*angle « tilt » des hyperplans*). Si les angles de structure sont des multiples de $\pi/2$ on dit que le cycle est *simple*. La liste complète des cycles simple est donnée dans les Tableaus 1 et 2.

Pour limiter la recherche de cycles non simples on utilise l'homomorphisme bien connu de $\mathcal{Q} \times \mathcal{Q}$ sur SO(4), où $\mathcal{Q}$ est le corps des quaternions unitaires. Ceci permet d'obtenir les conditions (12) et (13) sur les angles de structure. À l'aide d'un résultat sur les solutions rationnelles d'une équation trigonométrique (*cf.* [9]) on conclut que les angles de structure doivent être des multiples de $\pi/4$. On retrouve donc les cycles à 12 ($t = \pi/4$, $s = 3\pi/4$) et 24 points d'équilibre ($t = s = \pi/4$) déjà connus (*cf.* [11]). La classification est complète.

## 1. Introduction

Homoclinic cycles naturally appear in the study of bifurcations in symmetric systems. The presence of symmetries is necessary for these cycles to be robust, i.e., stable with respect to perturbations compatible with the symmetries of the system. For more details and general results on homoclinic cycles see [3] and [6]; the problem of the (asymptotic) stability of the cycles is studied in [7]. In this Note we focus our attention on the case where the symmetry group $G$ is finite, and its action on $\mathbb{R}^4$ is linear and faithful. We can suppose, without loss of generality, that $G \subset O(4)$. These cycles has been classified into type A, B, and C depending of the action of $G$ (*see* [2]). A problem arises concerning the possible number $n$ of equilibria for a homoclinic cycle. The answer to this problem has been given in [10] for type B and C cycles and in [11] for type A cycles with symmetry group not included in SO(4). For type B cycles the number of equilibria can only be 2, 3, or 6; for type C, $n$ is 4 or 8 and for type A cycles with $G \not\subset \mathrm{SO}(4) n$ is even or $n = 3$. In this Note we investigate the case of type A cycles with $G \subset \mathrm{SO}(4)$.

## 2. Rotations and quaternions

Let $\mathbb{Q}$ be the noncommutative division algebra of unitary quaternions. In the following we identify the element $(x, y, z, w)$ of $\mathbb{R}^4$ with the quaternion $x + y\vec{i} + z\vec{j} + w\vec{k}$, which we will frequently write as $(x, \vec{v})$, with $\vec{v}$ a 3-dimensional vector. The operations defined in the algebra of quaternions are:

$$(x_1, \vec{v}) + (x_2, \vec{w}) = (x_1 + x_2, \vec{v} + \vec{w}), \tag{1}$$

$$(x_1, \vec{v})(x_2, \vec{w}) = (x_1 x_2 - \vec{v} \cdot \vec{w}, x_1 \vec{w} + x_2 \vec{v} + \vec{v} \wedge \vec{w}), \tag{2}$$

where $\cdot$ and $\wedge$ denote the usual scalar and cross products (*see* [4], § 14).

The elements of $\mathbb{Q}$ can be written in the form:

$$\mathbf{q} = (\cos\theta, \sin\theta \vec{v}), \quad \theta \in S^1,\ \vec{v} \in \mathbb{R}^3,\ \|\vec{v}\| = 1. \tag{3}$$

*Remark* 1. – If the unitary quaternion

$$\mathbf{q} = (\cos\theta, \sin\theta \vec{v}) \tag{4}$$

is periodic ($\mathbf{q}^n = 1$ for a given integer $n > 0$), the angle $\theta$ is rational.

It is a well-known result that we have an homomorphism $\Psi$ from $\mathbb{Q} \times \mathbb{Q}$ on SO(4) whose kernel is $\{1, -1\}$. The action of $(\mathbf{p}, \mathbf{q}) \in \mathbb{Q} \times \mathbb{Q}$ on the quaternion $\mathbf{x}$ is

$$\mathbf{x} \longrightarrow \mathbf{p}\,\mathbf{x}\,\mathbf{q}^{-1}. \tag{5}$$

## 3. The homoclinic cycles

The following definition can be found in [2]: let $X$ be a vector field on $\mathbb{R}^n$, equivariant with respect to the linear action of a finite group $G$. Let $\xi \neq 0$ be a hyperbolic saddle point with a one-dimensional unstable manifold $W^u(\xi)$ and suppose that $W^u(\xi) \subset P = \mathrm{Fix}(K)$, where $P$ is a two-dimensional fixed-point subspace corresponding to an isotropy subgroup $K \subset G$.

DEFINITION 3.1. – If there is an element $\gamma \in G$ (*twist*) such that $\gamma\xi$ is a sink in $P$ and there is a saddle-sink connection in $P$ connecting $\xi$ to $\gamma\xi$, then the collection of points $\gamma^j\xi$, $j \in \mathbb{N}$, together with their unstable manifolds, forms what we call a *robust homoclinic cycle*.

We choose an orthonormal basis $\mathcal{B}' = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$ of $\mathbb{R}^4$ in the following way: let $P_1$ be an invariant plane and $\{\mathbf{e_1}, \mathbf{e_2}\}$ an orthonormal basis of $P_1$ such that $\mathbf{e_2} = P_1 \cap \gamma(P_1)$. Let $\mathbf{e_3}$ be a vector orthogonal to $P_1$ and such that $P_2 = \gamma(P_1) = \langle \mathbf{e_2}, \mathbf{e_3}\rangle$. Finally, we complete the basis $\mathcal{B}'$ with $\mathbf{e_4}$ to obtain a direct orthonormal basis. The action of $\gamma$ on the planes used to construct $\mathcal{B}'$ is as follows:

$$P_1 = \langle \mathbf{e_1}, \mathbf{e_2}\rangle \longrightarrow P_2 = \langle \mathbf{e_2}, \mathbf{e_3}\rangle \longrightarrow P_3 = \big\langle \cos(t)\,\mathbf{e_2} + \sin(t)\,\mathbf{e_3}, \cos(s)\,\mathbf{e_1} + \sin(s)\,\mathbf{e_4}\big\rangle$$

for some $s$ and $t$. If $A$ is the matrix representing $\gamma$ in the basis $\mathcal{B}'$ we can replace some $\mathbf{e_j}$ with $-\mathbf{e_j}$ to obtain a direct basis $\mathcal{B}$ in which

$$A = \begin{pmatrix} 0 & 0 & \cos(s) & -\sin(s) \\ \sin(t) & \cos(t) & 0 & 0 \\ -\cos(t) & \sin(t) & 0 & 0 \\ 0 & 0 & \sin(s) & \cos(s) \end{pmatrix}. \tag{6}$$

It is easy to show that $\det(A) = 1$. The equilibria are $\xi_{j+1} = A^j\xi_1$ where $\xi_1 = \mathbf{e_2}$. The matrix fixing the first invariant plane $P_1 = \langle \mathbf{e_1}, \mathbf{e_2}\rangle$ is $S = \mathrm{diag}(1, 1, -1, -1)$ and $G = \langle A, S\rangle$.

DEFINITION 3.2. – The angle $t$ between two consecutive equilibria $\xi_i$ and $\xi_{i+1} = A\,\xi_i$ in the flow-invariant plane $P_{i+1}$ is called *connecting angle*. The angle $s$ between the hyperplanes $Q_i = P_{i-1} + P_i$ and $Q_{i+1} = A\,Q_i$ is called *hyperplanes tilt angle*. We will refer to $s$ and $t$ as the *structure angles* of the cycle $X$. The cycles where at least one of the structure angles is a multiple of $\pi/2$ are called *simple cycles*.

LEMMA 3.3. – (a) *If* $P_1 = \langle \mathbf{e_1}, \mathbf{e_2}\rangle = \mathrm{Fix}(K_1)$ *then* $\{I, S\} \subseteq K_1$ *where* $S = \mathrm{diag}(1, 1, -1, -1)$.
(b) *If* $H_1$ *is the isotropy subgroup of* $\xi_1 = (0, 1, 0, 0)$,

$$\big\{I, S, \mathrm{diag}(-1, 1, 1, -1), \mathrm{diag}(-1, 1, -1, 1)\big\} \subseteq H_1.$$

*Proof.* – It is a direct consequence of the hypothesis $\dim(W^u(\xi)) = 1$ for each equilibrium point $\xi$. □

Finally we need the following

DEFINITION 3.4. – The cycle $X$ is of *type A* if $Q = P + \gamma P$ is not a fixed-point subspace; of *type B* if $Q$ is a fixed-point subspace and $X \subset Q$; and of *type C* if $Q$ is a fixed-point subspace and $X \not\subset Q$.

The following lemma will be useful in determining the possible values of $n$.

LEMMA 3.5. – *In the case of a non-simple cycle the structure angles satisfy the inequalities*

$$0 \leqslant t \leqslant \pi/2, \qquad 0 \leqslant s \leqslant \pi. \tag{7}$$

*Proof.* – By changing the sign of $s$ we obtain a symmetric cycle with respect to the hyperplane $\langle \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\rangle$.

Table 1. – Type A simple cycles with symmetry group in SO(4).
*Tableau 1. – Cycles simples avec un groupe de symétrie dans* SO(4).

| $n$ | $t$ (connection) | $s$ (tilt) | $\|G\|$ |
|---|---|---|---|
| 2 | $\pi$ | 0 | 8 |
| 3 | $\pi/2$ | 0 | 24 |
| 6 | $\pi/2$ | $\pi$ | 24 |
| 8 | $\pi/2$ | $\pi/2$ | 32 |

We can of course change $t$ in $-t$ obtaining a symmetry with respect to the axis $\mathbf{e_2}$ so we can suppose $t \in [0, \pi]$. Furthermore, let $K_1$ be the subgroup of $G$ fixing the plane $P_1$. Because of Lemma 3.3 the group $N(K_1)$, acting on $P_1$, contains the symmetries with respect to all directions obtained from $\xi_1$ with a rotation by an angle equal to $t$. These directions being flow-invariant if $\pi/2 < t < \pi$ the connection in $P_1$ must go through an invariant direction which is impossible. □

## 4. The simple cycles

We start from a result from [11] which gives the fundamental property characterizing a simple cycle.

THEOREM 4.1. – *The structure angles of a simple homoclinic cycle are multiples of* $\pi/2$.

Using this theorem it is easy to find all possible type A simple cycles. The complete list is given in Table 1.

We now look for non-simple homoclinic cycles using the quaternion homomorphism to characterize the finite groups generated by $A$ and $S$.

## 5. The non-simple cycles

Consider the homomorphism $\Psi : \mathcal{Q} \times \mathcal{Q} \to \mathrm{SO}(4)$ introduced in Section 2. One of the two inverse images of $A$ is the element $((p_1, q_1, r_1, s_1), (P_1, Q_1, R_1, S_1))$ where

$$r_1 = -p_1 = \frac{\sqrt{2}}{2}\cos\left(\frac{s+t}{2}\right), \qquad q_1 = s_1 = -\frac{\sqrt{2}}{2}\sin\left(\frac{s+t}{2}\right) \tag{8}$$

and

$$R_1 = P_1 = -\frac{\sqrt{2}}{2}\cos\left(\frac{s-t}{2}\right), \qquad -Q_1 = S_1 = \frac{\sqrt{2}}{2}\sin\left(\frac{s-t}{2}\right). \tag{9}$$

In the case of $AS$ one of the inverse images is $((p_2, q_2, r_2, s_2), (P_2, Q_2, R_2, S_2))$ with

$$r_2 = -p_2 = -\frac{\sqrt{2}}{2}\sin\left(\frac{s+t}{2}\right), \qquad q_2 = s_2 = -\frac{\sqrt{2}}{2}\cos\left(\frac{s+t}{2}\right) \tag{10}$$

and

$$R_2 = P_2 = \frac{\sqrt{2}}{2}\sin\left(\frac{s-t}{2}\right), \qquad -Q_2 = S_2 = \frac{\sqrt{2}}{2}\cos\left(\frac{s-t}{2}\right). \tag{11}$$

We can now state the following lemma:

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LEMMA 5.1. – *The angles $s$ and $t$ satisfy the following conditions*:

$$\cos\left(\frac{s+t}{2}\right) = \sqrt{2}\cos(p\pi), \qquad \cos\left(\frac{s-t}{2}\right) = \sqrt{2}\cos(q\pi), \tag{12}$$

$$\sin\left(\frac{s+t}{2}\right) = \sqrt{2}\cos(r\pi), \qquad \sin\left(\frac{s-t}{2}\right) = \sqrt{2}\cos(u\pi), \tag{13}$$

*where $p$, $q$, $r$ and $u$ are rational numbers.*

*Proof.* – It is a simple consequence of Remark 1 and Eqs. (8)–(10) and (11). □

We recall the following Theorem from [9] (for a more general result *see* [8]):

THEOREM 5.2. – *All rational values of $x$ and $y$ such that $\sin\pi x \sin\pi y$ is a positive rational, normalized so that $0 < x \leqslant y \leqslant 1/2$, are given by*

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 1/2 | 1/3 | 1/4 | 1/6 | 1/10 | 1/12 |
| $y$ | 1/2 | 1/3 | 1/4 | 1/6 | 3/10 | 5/12 |

We can now state the theorem:

THEOREM 5.3. – *The structure angles of a type A homoclinic cycle in $\mathbb{R}^4$, with symmetry group included in* $\mathrm{SO}(4)$*, are multiples of $\pi/4$.*

*Proof.* – From Lemma 5.1 we obtain

$$\cos^2(p\pi) + \cos^2(r\pi) = \frac{1}{2}, \tag{14}$$

then

$$\sin\left(\left(p + r - \frac{1}{2}\right)\pi\right)\sin\left(\left(-p + r + \frac{1}{2}\right)\pi\right) = \frac{1}{2} \tag{15}$$

and the statement follows from Theorem 5.2. □

A consequence of Lemma 3.5 is that the only possible non-simple cycles can be obtained with
(a) $t = s = \pi/4$,
(b) $t = \pi/4$ and $s = 3\pi/4$.

It has been shown in [11] that these two cases give two homoclinic cycles with 24 and 12 equilibria, respectively. The symmetry group is the same in the two cases and its order is 192. The projection of the 24-equilibria cycle on the plane $\langle \mathbf{e}_1, \mathbf{e}_2\rangle$ is shown in the left-hand side of Fig. 1. From this projection it is not so easy to guess the topology of the original curve on the sphere $S^3$. A simple question is about this curve to be a (non-trivial) knot or not. The answer is negative, such a curve being isotopic to a circle on $S^3$ as it can be seen plotting its stereographic projection on $\mathbb{R}^3$ (*see* Fig. 1, right-hand side).

To complete the classification we have to add the two cycles in Table 2 to Table 1 in Section 4.

Table 2. – Continuatio of Table 1.
*Tableau 2. – Suite du Tableau 1.*

| $n$ | $t$ (connection) | $s$ (tilt) | $\|G\|$ |
|---|---|---|---|
| 12 | $\pi/4$ | $3\pi/4$ | 192 |
| 24 | $\pi/4$ | $\pi/4$ | 192 |

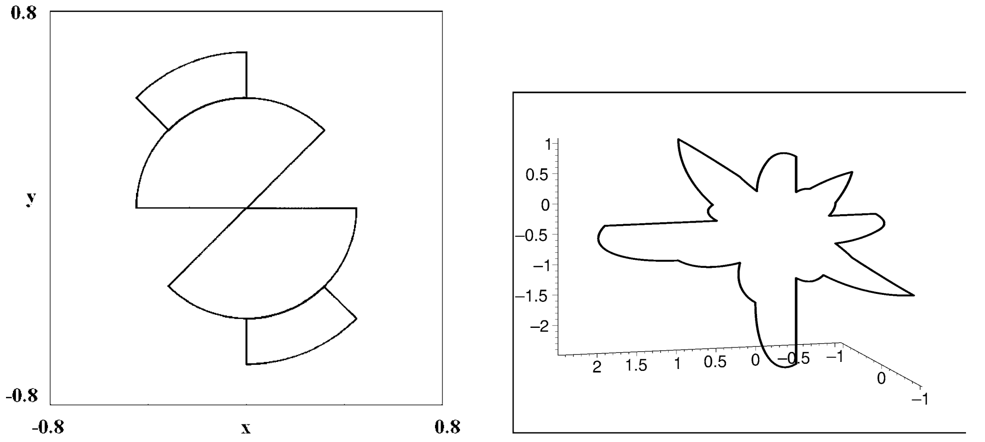


Figure 1. – The 24 equilibria homoclinic cycle. Left: the projection on the plane $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. Right: the stereographic projection on $\mathbb{R}^3$: this curve is clearly unknotted.

*Figure 1. – Cycle honocline de 24 points d'équilibre* : *gauche* : *La projection sur le plan* $\langle e_1, e_2 \rangle$, *droite* : *la projection stereographique sur* $\mathbb{R}^3$ : *cette courbe est visiblement dénovée.*

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