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Probabilités/*Probability Theory*
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On the volume of the intersection of a sphere with random half spaces

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Abstract We find an asymptotic expression of the volume of the intersection of the N dimensional sphere with $p = \alpha N$ random half spaces when α is less than a critical value. This expression coincides with the one found by Gardner [3] using replica calculations. We get also the same value for α_c. Our proof is rigorous and based on the cavity method. The required decay of correlations is obtained by means of a geometrical argument which holds for general Hamiltonians. ***To cite this article: M. Shcherbina, B. Tirozzi, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 803–806.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur le volume de l'intersection d'une boule avec des demi espaces aléatoires

Résumé Nous trouvons une expression asymptotique du volume de l'intersection d'une boule à N dimensions avec $p = \alpha N$ demi espaces aléatoires quand α ne depasse pas la valeur critique α_c. Cette expression est la même que celle trouvée par Gardner [3] en utilisant un calcul de repliques. Nous trouvons aussi la mème valeur de α_c. Notre démonstration est rigoureuse et basée sur la methode de la cavité. La nécessaire décroissance des corrélations est obtenue en utilisant un argument géométrique qui est vrai pour des hamiltoniens généraux. ***Pour citer cet article : M. Shcherbina, B. Tirozzi, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 803–806.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

For very large integer N consider the N-dimensional sphere S_N of radius $N^{1/2}$ centered in the origin and $p = \alpha N$ independent random half spaces Π_μ $(\mu = 1, \ldots, p)$. Let $\Pi_\mu = \{\boldsymbol{J} \in \mathbf{R}^N : N^{-1/2}(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J}) \geqslant k\}$, where $\boldsymbol{\xi}^{(\mu)}$ are i.i.d. random vectors with i.i.d. Bernulli components $\xi_j^{(\mu)}$ and k is the distance from Π_μ to the origin. The problem is to find the maximum value of α such that the volume of the intersection of S_N with $\bigcap \Pi_\mu$ is not "too small" (i.e., of order $\mathrm{e}^{-N\,\mathrm{const}}$). More precisely, we study the "typical" behaviour as

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$N \to \infty$ of

$$\Theta_{N,p}(k) = \sigma_N^{-1} \int_{S_N} \mathrm{d}\boldsymbol{J} \prod_{\mu=1}^{p} \theta\big(N^{-1/2}\big(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J}\big) - k\big), \tag{1}$$

where σ_N is the volume of S_N and $\theta(x)$ as usually is the function, assuming $+1$ in the positive semi axis and 0 in the negative.

This geometrical question is motivated also by the neural networks theory. It corresponds to the question how many patterns can be stored by the network of N spins, or more precisely, what is the volume of the interaction couplings J_{ij} for which p independent patterns of N independent ± 1 bits can be retrieved by the neural dynamics in the limit $N,\ p \to \infty,\ p/N \to \alpha$. The answer was conjectured by Gardner [3] by means of nonrigorous replica calculations. She found the critical value $\alpha_c(k) \equiv \big(\frac{1}{\sqrt{2\pi}} \int_{-k}^{\infty} (u+k)^2 \,\mathrm{e}^{-u^2/2} \,\mathrm{d}u\big)^{-1}$ and shows that for $\alpha \geqslant \alpha_c$ the volume of the intersection decays as $N \to \infty$ faster than e^{-LN} with any positive L. Our main goal is to prove rigorously the results of [3].

To formulate our main theorem we should remark that since $\Theta_{N,p}(k)$ can be zero with nonzero probability (e.g., if for some $\mu \neq \nu$, $\boldsymbol{\xi}^{(\mu)} = -\boldsymbol{\xi}^{(\nu)}$), we cannot, as usually in statistical mechanics, just study $\log \Theta_{N,p}(k)$. To avoid this difficulty, we take some large enough M and replace the log function by the function $\log_{(MN)}$, defined as $\log_{(MN)} X = \log \max\{X, \mathrm{e}^{-MN}\}$.

THEOREM 1.1. – *For any $\alpha \leqslant \alpha_c(k)$ $N^{-1} \log_{(MN)} \Theta_{N,p}(k)$ is self-averaging in the limit $N, p \to \infty$, $p/N \to \alpha$ (i.e., its variance tends to* 0 *in this limit) and for M large enough there exists*

$$\lim_{N,p\to\infty} E\big\{N^{-1} \log_{(MN)} \Theta_{N,p}(k)\big\} = \min_{0 \leqslant q \leqslant 1} \left[\alpha E\left\{\log \mathrm{H}\left(\frac{u\sqrt{q}+k}{\sqrt{1-q}}\right)\right\} + \frac{1}{2}\frac{q}{1-q} + \frac{1}{2}\log(1-q)\right],$$

where $\mathrm{H}(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \mathrm{e}^{-t^2/2} \,\mathrm{d}t$, *$u$ is a Gaussian random variable with zero mean variance* 1 *and $E\{\cdot\}$ is the averaging with respect to u.*

For $\alpha > \alpha_c(k)$, $E\{N^{-1} \log_{(MN)} \Theta_{N,p}(k)\} \to -\infty$, as $N \to \infty$ and then $M \to \infty$.

We remark here that the self-averaging of $N^{-1} \log \Theta_{N,p}(k)$ was proven also in [16].

2. Method and results

It can be easily seen, that the Gardner problem (1) is very similar to problems of statistical mechanics, where the integrals with respect to N variables in the limit $N \to \infty$ are studied. But due to technical reasons it is not convenient to study directly the model (1) with θ functions. That is why we use a common trick: substitute the θ-functions appearing in the expression of the partition function (1) by some smooth functions which depend on the small parameter ε and tend, as $\varepsilon \to 0$, to the θ-functions. We choose for this purpose $\mathrm{H}(-x\varepsilon^{-1/2})$ with $\mathrm{H}(x)$ defined in Theorem 1.1 but the particular form of this function is not important for us. The most important fact is that its logarithm should be a convex function. To substitute in (1) the integration over S_N by the integration over the whole $\mathbf{R}^N$ we use another well known trick in statistical mechanics. We add to the Hamiltonian a term depending on the additional free parameter z. At the end of our considerations we can choose this parameter in order to provide the condition that for large N only a small neighborhood of S_N gives the main contribution to our integral. Thus, we consider the Hamiltonian of the form

$$\mathcal{H}_{N,p}(\boldsymbol{J}, k, z, \varepsilon) \equiv -\sum_{\mu=1}^{p} \log \mathrm{H}\left(\frac{k - (\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J})N^{-1/2}}{\sqrt{\varepsilon}}\right) + \frac{z}{2}(\boldsymbol{J}, \boldsymbol{J}) + h(\boldsymbol{h}, \boldsymbol{J}). \tag{2}$$

Where the last term $h(\boldsymbol{h}, \boldsymbol{J})$ is the scalar product of the variables $\boldsymbol{J}$ with some vector $\boldsymbol{h}$ with independent random components introduced for getting the self averaging of the order parameters of the theory (*see* below) [5,8,9]. The free energy and the Gibbs average for this Hamiltonian are

$$Z_{N,p}(k,z,\varepsilon)=\sigma_N^{-1}\int \mathrm{d}\boldsymbol{J}\,\mathrm{e}^{-\mathcal{H}_{N,p}(\boldsymbol{J})},\quad \langle\cdot\rangle=\int(\cdot)\,\mathrm{d}\boldsymbol{J}\,\mathrm{e}^{-\mathcal{H}_{N,p}(\boldsymbol{J})},\quad f_{N,p}(k,z,\varepsilon)\equiv\frac{1}{N}\log Z_{N,p}(k,z,\varepsilon).$$

Now we have the typical problem of statistical mechanics which we solve by a method usually called the cavity method. The idea of the cavity method is to choose one variable, e.g., J_N and to try to express $\langle J_N\rangle$ through the Gibbs average of the others J_i, and then, using the symmetry of the Hamiltonian, write the self-consistent equations for the so-called order parameters of the problem $q\equiv\frac{1}{N}\sum\langle J_i\rangle^2$ and $R\equiv\frac{1}{N}\sum\langle J_i^2\rangle$. This procedure allows us to reduce the problem to a finite number of nonlinear equations. The rigorous version of the cavity method was proposed in [8] and developed in [9–12]. The key problem of the application of the cavity method is the proof of the vanishing of the correlation functions $\langle J_iJ_j\rangle-\langle J_i\rangle\langle J_j\rangle$ as $N\to\infty$. We derived this property from a geometrical statement (*see* Theorem 2.1 below). We consider a general convex Hamiltonian and the Gibbs measure generated by it. Then the Gibbs average of any linear combination of $(\boldsymbol{J},\mathbf{e})$ ($|\mathbf{e}|=1$) can be expressed in terms of the two-dimensional integral with respect to the energy U (the value of the Hamiltonian) and $c=(\boldsymbol{J},\mathbf{e})N^{-1/2}$. The additional function, which appears under this change of variables is the "partial entropy", given by the logarithm of the volume of the intersections of the level surfaces of the Hamiltonian with the hyper planes $(\boldsymbol{J},\mathbf{e})=cN^{1/2}$. We study these intersections using a theorem of classical geometry known since the nineteenth century as the Brunn–Minkowski theorem [6]. From this theorem we obtain that the "partial entropy" is a concave function of (U,c). Thus we can apply the Laplace method to evaluate the Gibbs averages. So we obtain the vanishing of correlation functions, which allows us to find the expression for the free energy. A similar idea was used in [1] where the results of [2] (also based on the Brunn–Minkowski theorem) have been used. We would like to remark that, differently from [1], we cannot just use the results of [2], because they are true for $\mathbf{R}^N$ while the most nontrivial part of our proof (i.e., the limiting transition $\varepsilon\to 0$) is based on similar results for the intersections of p random half spaces.

As far as we know, the Gardner problem is one of the first problems of spin glass theory completely solved (i.e., for all values of α and k) in a rigorous way. The explanation is that the problem (1) can be reduced to the problem with the convex Hamiltonian (2) in the convex configuration space. It is just this convexity that allows us to prove the vanishing of all correlation functions for all values of α and k, while, e.g., in the Hopfield and Sherrington–Kirkpatrick models the vanishing is valid only for small enough α or for high temperatures (*see* [7] for the physical theory and [11–14] for the respective rigorous results). Also for the Gardner–Derrida [4] model there is only a justification of the Replica Simmetry solution in a certain region of parameters (*see* [15]).

Our last step is the limiting transition $\varepsilon\to 0$, i.e., the proof that θ-functions in (1) can be replaced by $\mathrm{H}(x/\sqrt{\varepsilon})$ with a small difference when ε is small enough. It is the most difficult step from the technical point of view. It is rather straightforward to obtain that the free energy of (2) is an upper bound of $\log\Theta_N$, $p(k)$. But the estimate from below is much more complicated. The problem is that to estimate the difference between the free energies corresponding to the two Hamiltonians we, as a rule, need to have them defined in a common configuration space, or at least, we need to know some a priori bounds for some Gibbs averages. In the case of the Gardner problem we do not possess this information. That is why we need to apply our geometrical theorem not only to the model (2) (for these purposes it would be enough to apply the results of [2]) but also to some models, interpolating between (2) and (1), with a complicated random (but convex) configuration space.

At the end we formulate our analog of the result of [2], which allows to prove the vanishing of the correlation functions for a large class of the models of statistical mechanics.

Let $\{\Phi_N(\boldsymbol{J})\}_{N=1}^{\infty}$ ($\boldsymbol{J}\in\mathbf{R}^N$) be a system of convex functions which have third derivatives bounded in any compact set. Consider also a system of convex domains $\{\Gamma_N\}_{N=1}^{\infty}$ ($\Gamma_N\subset\mathbf{R}^N$) whose boundaries consist of a finite number (may be depending on N) of smooth pieces. Define the Gibbs measure and the free energy, corresponding to $\Phi_N(\boldsymbol{J})$ in Γ_N:

$$\langle\cdot\rangle_{\Phi_N} \equiv \Sigma_N^{-1}\int_{\Gamma_N} \mathrm{d}\boldsymbol{J}(\cdot)\,\mathrm{e}^{-\Phi_N(\boldsymbol{J})}, \quad \Sigma_N(\Phi_N) \equiv \int_{\Gamma_N} \mathrm{d}\boldsymbol{J}\,\mathrm{e}^{-\Phi_N(\boldsymbol{J})}, \quad f_N(\Phi_N) \equiv \frac{1}{N}\log\Sigma_N(\Phi_N).$$

Denote

$$\widetilde{\Omega}_N(U) \equiv \bigl\{\boldsymbol{J} : \Phi_N(\boldsymbol{J}) \leqslant NU\bigr\}, \quad \Omega_N(U) \equiv \widetilde{\Omega}_N(U) \cap \Gamma_N, \quad \mathcal{D}_N(U) \equiv \widetilde{\mathcal{D}}_N(U) \cap \Gamma_N,$$

where $\widetilde{\mathcal{D}}_N(U)$ is the boundary of $\widetilde{\Omega}_N(U)$. Define also $f_N^*(U) = \frac{1}{N}\log\int_{J\in\mathcal{D}_N(U)} \mathrm{d}\boldsymbol{J}\,\mathrm{e}^{-NU}$.

THEOREM 2.1. – *Let the functions $\Phi_N(\boldsymbol{J})$ satisfy the conditions*:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Phi_N(\boldsymbol{J}+t\mathbf{e})|_{t=0} \geqslant C_0 > 0, \quad \Phi_N(\boldsymbol{J}) \geqslant C_1(\boldsymbol{J},\boldsymbol{J}) - NC_2, \quad \bigl|\nabla\Phi_N(\boldsymbol{J})\bigr| \leqslant N^{1/2}C_3(U) \quad \bigl(\boldsymbol{J} \in \widetilde{\Omega}_N(U)\bigr),$$

where $\mathbf{e}$ *is an arbitrarily direction* ($|\mathbf{e}| = 1$), $C_0, C_1, C_2, C_3(U)$ *are some positive N-independent constants and $C_3(U)$ is continuous in U* ($U > U_{\min} \equiv \min_{J\in\Gamma_N} N^{-1}\Phi_N(\boldsymbol{J}) \equiv N^{-1}\Phi_N(\boldsymbol{J}^*)$).

Assume also that there exists some finite N-independent C_4 such that $f_N(\Phi_N) \geqslant -C_4$.

Then for any $U > U_{\min}$, $f_N^*(U) = \min_{z>0}\{f_N(z\Phi_N) + zU\} + \mathrm{O}(N^{-1}\log N)$, *and for any* $\mathbf{e} \in \mathbf{R}^N$ ($|\mathbf{e}| = 1$) *and any natural p*

$$\bigl\langle(\dot{\boldsymbol{J}},\mathbf{e})^p\bigr\rangle_{\Phi_N} \leqslant C(p), \quad \frac{1}{N^2}\sum_{i,j}\bigl\langle\dot{J}_i\dot{J}_j\bigr\rangle_{\Phi_N}^2 \leqslant \frac{C(2)}{N} \quad \bigl(\dot{J}_i \equiv J_i - \langle J_i\rangle_{\Phi_N}\bigr)$$

with some positive N-independent $C(p)$.

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References

[1] A. Bovier, V. Gayrard, Hopfield models as a generalized random mean field models, in: A. Bovier, P. Picco (Eds.), Mathematical Aspects of Spin Glasses and Neuronal Networks, Progr. Probab., Vol. 41, Birkhäuser, 1998, pp. 3–89.
[2] H.J. Brascamp, E.H. Lieb, On the extension of the Brunn–Minkowsky and Pekoda–Leindler theorems, includings inequalities for log concave functions, and with an application to the diffusion equation, J. Func. Anall. 22 (1976) 366–389.
[3] G. Gardner, The space of interactions in neural network models, J. Phys. A 21 (1988) 271–284.
[4] E. Gardner, B. Derrida, Optimal stage properties of neural network models, J. Phys. A 21 (1988) 257–270.
[5] S. Ghirlanda, F. Guerra, General properties of overlap probability distributions in disordered spin system, J. Phys. A 31 (1988) 9149–9155.
[6] H. Hadwiger, Vorlesungen über Inhalt, Oberlache und Isoperimetrie, Springer-Verlag, 1957.
[7] M. Mezard, G. Parisi, M.A. Virasoro, Spin Glass Theory and Beyond, World Scientific, Singapore, 1987.
[8] L. Pastur, M. Shcherbina, Absence of self-averaging of the order parameter in the Sherrington–Kirkpatrick model, J. Statist. Phys. 62 (1991) 1–26.
[9] L. Pastur, M. Shcherbina, B. Tirozzi, The replica-symmetric solution without replica trick for the hopfield model, J. Statist. Phys. 74 (5/6) (1994) 1161–1183.
[10] L. Pastur, M. Shcherbina, B. Tirozzi, On the replica symmetric equations for the hopfield model, J. Math. Phys. 40 (8) (1999) 3930–3947.
[11] M. Shcherbina, On the replica symmetric solution for the Sherrington–Kirkpatrick model, Helv. Phys. Acta 70 (1997) 772–797.
[12] M. Shcherbina, Some estimates for the critical temperature of the Sherrington–Kirkpatrick model with magnetic field, in: Mathematical Results in Statistical Mechanics, World Scientific, Singapore, 1999, pp. 455–474.
[13] M. Talagrand, Rigorous results for the Hopfield model with many patterns, Probab. Theory Related Fields 110 (1998) 176–277.
[14] M. Talagrand, Exponential inequalities and replica symmetry breaking for the Sherrington–Kirkpatrick model, Ann. Probab. 28 (2000) 1018–1068.
[15] M. Talagrand, Intersecting random half-spaces: Toward the Gardner–Derrida problem, Ann. Probab. 28 (2000) 725–758.
[16] M. Talagrand, Self averaging and the space of interactions in neural networks, Random Structures and Algorithms 14 (1988) 199–213.