C. R. Acad. Sci. Paris, Ser. I 334 (2002) 689–692

Probabilités/*Probability Theory*
(Statistique/*Statistics*)

Extreme value attractors for star unimodal copulas

Ioan Cuculescu [a], Radu Theodorescu [b]

[a] Facultatea de matematică, Universitatea Bucureşti, str. Academiei 14, RO-70109 Bucureşti, Romania
[b] Département de mathématiques et de statistique, Université Laval, Québec, QC, Canada G1K 7P4

Received 5 December 2001; accepted after revision 18 February 2002

Note presented by Paul Deheuvels.

Abstract We determine maximum attractors for copulas star (or 2-) unimodal (about a point $(a, b) \in \mathbf{R}^2$). If $(a, b) \neq (1, 1)$ these attractors form a two-parameter family of copulas extending that of Cuadras–Augé, whereas if $(a, b) = (1, 1)$ they cover all maximum value copulas. We also examine the relationship between unimodality and Archimax copulas. ***To cite this article: I. Cuculescu, R. Theodorescu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 689–692.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Attracteurs de valeurs extrêmes pour les copules 2-unimodales

Résumé Nous déterminons les attracteurs des valeurs maximales pour les copules 2-unimodales (par rapport à (a, b)). Si $(a, b) \neq (1, 1)$ ces attracteurs forment une famille de copules à deux paramètres généralisant celle de Cuadras–Augé alors que si $(a, b) = (1, 1)$ elles couvrent toutes les copules de valeurs maximales. Nous examinons aussi la relation entre l'unimodalité et les copules Archimax. ***Pour citer cet article : I. Cuculescu, R. Theodorescu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 689–692.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

An important property of a distribution is unimodality. It is then natural to ask whether copulas are unimodal. This question has been answered for central convex, block, and star unimodality in Cuculescu and Theodorescu [3]. As a follow-up we examine in this Note the maximum domain of attraction for star unimodal copulas.

The Note is organized as follows. Section 2 has an auxiliary character; here we indicate several definitions, notations, and results to be used throughout this note. In Section 3 we show that the maximum domain of attraction to which a copula that is star unimodal about $(a, b) \neq (1, 1)$ belongs is an element of a two-parameter family of copulas extending that of Cuadras–Augé. When $(a, b) = (1, 1)$ the set of all possible attractors changes dramatically and covers all maximum value copulas. As a consequence of the results in Section 3 we examine in Section 4 the relationship between star unimodality and the Archimax copulas of Capéraà, Fougères, and Genest [2]; we show that many of them are not star unimodal.

This Note summarizes the results obtained in Cuculescu and Theodorescu [4]; details and proofs will be published elsewhere.

E-mail addresses: icucul@pro.math.unibuc.ro (I. Cuculescu); radutheo@mat.ulaval.ca (R. Theodorescu).

S1631-073X(02)02322-1/FLA

2. Prelude

We shall use the term probability measure or distribution at our convenience; m is Lebesgue measure, $\otimes$ stands for measure product, 1_A for the indicator function of A, $\overline{\mu}$ for the 'survival' function of μ, and $f\mu$ for the measure $\int_{\cdot} f\,\mathrm{d}\mu$.

Let $I = [0,1]$. It was Sklar [9] who in 1959 coined the term *copula* for a distribution on I^2 whose margins are uniform. The notations M, W, and Π stand for the copulas $\min\{u,v\}$, $\max\{u+v-1,0\}$, and uv respectively. For details on copulas we shall refer the reader to the recent book by Nelsen [7].

A copula C^* is said to be the *maximum attractor* of copula C (or C belongs to the *maximum domain of attraction* of C^*) if we have (Galambos [6, Theorem 5.2.3, p. 294])

$$\lim_{n\to\infty} C^n\left(x^{1/n}, y^{1/n}\right) = C^*(x,y), \quad x,y \in I. \tag{1}$$

Here (1) is equivalent to

$$\lim_{n\to\infty} n\left(1 - C\left(x^{1/n}, y^{1/n}\right)\right) = -\log C^*(x,y), \quad x,y \in I. \tag{2}$$

Since $1 - C(x,y) = (1-x) + (1-y) - \overline{C}(x,y)$, $x,y \in I$, where $\overline{C}(x,y) = C((x,1] \times (y,1]))$ is the joint *survival function* of C, (2) is equivalent to

$$\lim_{n\to\infty} n\overline{C}\left(x^{1/n}, y^{1/n}\right) = -\log(xy) + \log C^*(x,y), \quad x,y \in I.$$

Only the behavior of $\overline{C}$ near the point $(1,1)$ is important when deciding whether C belongs or not to the maximum domain of attraction of C^*. Similarly copula C belongs to the *minimum domain of attraction* of C_* if and only if C_* is the maximum attractor of the *survival copula* $\widehat{C}(x,y) = \overline{C}(1-x, 1-y) = x + y - 1 + C(1-x, 1-y)$, $x,y \in I$. Therefore any assertion concerning the minimum domain of attraction is equivalent to one concerning the maximum domain of attraction by changing C to $\widehat{C}$.

Since the work of Pickands [8] (see also Tawn [10]) it is known that C^*, also called *extreme value copula*, can be expressed in the form

$$C^*(x,y) = C_A(x,y) = \exp\left\{\log(xy) A\left(\log(x)/\log(xy)\right)\right\}, \quad x,y \in I, \tag{3}$$

in terms of a convex *dependence function* A defined on I in such a way that $\max\{t, 1-t\} \leqslant A(t) \leqslant 1$ for all $t \in I$. The bounds 1 and $\max\{t, 1-t\}$ correspond to copulas Π and M respectively. A dependence function A which will occur in the sequel is

$$A_{\theta_1,\theta_2}(t) = \max\left\{1 - \theta_1(1-t), 1 - \theta_2 t\right\}, \quad \theta_1, \theta_2 \in (0,1], \tag{4}$$

and $A_{0,0} = 1$. Such an A_{θ_1,θ_2} leads to the copula

$$C_{\theta_1,\theta_2}(x,y) = \begin{cases} xy^{1-\theta_1} & \text{for } x^{1/\theta_1} \leqslant y^{1/\theta_2}, \\ x^{1-\theta_2} y & \text{for } y^{1/\theta_2} \leqslant x^{1/\theta_1}, \end{cases}$$

and $C_{0,0} = \Pi$. This two-parameter family of copulas is an extension of the one-parameter Cuadras–Augé (Nelsen [7, p. 12 and 17]) family of copulas

$$C_\theta(x,y) = \begin{cases} xy^{1-\theta} & \text{for } x \leqslant y, \\ x^{1-\theta} y & \text{for } y \leqslant x, \end{cases}$$

where $\theta \in I$.

In what follows we shall be concerned with *star unimodality* (*n-unimodality*) (Dharmadhikari and Joagdev [5, p. 38], Bertin, Cuculescu, and Theodorescu [1, p. 72]): a distribution C is said to be *star unimodal* about $x \in \mathbf{R}^n$ if it belongs to the closed convex hull of the set of all uniform distributions on sets which are star-shaped about x.

3. Asymptotics of extremes

Let copula C be star unimodal about $(a,b) \neq (1,1)$.

To cite this article: I. Cuculescu, R. Theodorescu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 689–692

PROPOSITION 3.1. – *Let copula C be star unimodal about $(a,b) \neq (1,1)$. Then C belongs to the maximum domain of attraction of copula C_{θ_1,θ_2} with $\theta_1 = 2c_{11}/(1-b)$, $\theta_2 = 2c_{11}/(1-a)$ for $a,b<1$ and $\theta_1=\theta_2=0$ for $a=1$ or $b=1$; the quantity c_{11} is defined in* (6).

Sketch of the proof. – In view of Choquet's representation, a distribution C is star unimodal about (a,b) if and only if it is a mixture of the form

$$C = \int \sigma_{(a,b),(u,v)}\, \mathrm{d}\mu(u,v), \tag{5}$$

where the probability measure μ on $\mathbf{R}^2$ is unique, $\sigma_{(a,b),(a,b)} = \varepsilon_{(a,b)}$($\varepsilon_w$ stands for the point mass at w), $\sigma_{(a,b),(u,v)}$, for $(u,v) \neq (a,b)$, is concentrated on the segment joining (a,b) to (u,v) and has with respect to the uniform distribution a probability density function $f(u',v')$ which is proportional to the distance between (u',v') and (a,b). We now suppose that C is a copula which is star unimodal about $(a,b) \in I^2$. In Cuculescu and Theodorescu [3, Proposition 3.3] we completely determined the unique probability measure μ:

$$\mu = \sum_{\alpha,\beta\in\{0,1\}} c_{\alpha\beta}\varepsilon_{(\alpha,\beta)} + d_0^1\varepsilon_0 \otimes \left(f_0^1 m\right) + d_1^1\varepsilon_1 \otimes \left(f_1^1 m\right) + d_0^2\left(f_0^2 m\right)\otimes\varepsilon_0 + d_1^2\left(f_1^2 m\right)\otimes\varepsilon_1 + c\xi, \tag{6}$$

where $c = \sum_{\alpha,\beta\in\{0,1\}} c_{\alpha\beta}$, the remaining c's and d's are nonnegative such that

$$c_{00} + c_{01} + d_0^1 = a/2, \qquad c_{10} + c_{11} + d_1^1 = (1-a)/2,$$

$$c_{00} + c_{10} + d_0^2 = b/2, \qquad c_{01} + c_{11} + d_1^2 = (1-b)/2,$$

and f_α^i are probability density functions on I satisfying

$$\left(d_0^1 f_0^1 + d_1^1 f_1^1\right)m + c\xi_2 = \left(d_0^2 f_0^2 + d_1^2 f_1^2\right)m + c\xi_1 = 1_I m/2,$$

ξ_1 and ξ_2 being the margins of the distribution ξ. The remaining of the proof is a succession of evaluations involving survival functions.

Remark 3.2. – Copula C_{θ_1,θ_2} is not star unimodal except when $\theta_1=\theta_2=1$ (i.e., $C_{1,1} = C_{A_{1,1}} = M$) and $\theta_1=\theta_2=0$ (i.e., $C_{0,0}=\Pi$). This is the consequence of the fact that the singular part of a star unimodal (about (a,b)) copula is concentrated on a union of half-lines originating in (a,b) (Cuculescu and Theodorescu [3, Remark 4.3]).

The following result deals with the case $(a,b)=(1,1)$ which was left out in the preceding proposition. In what follows copula C is related to the measure μ by (5).

PROPOSITION 3.3. – *Let copula C be star unimodal about* $(1,1)$. *The following are equivalent*:

(I) *C belongs to the maximum domain of attraction of C_A (given by* (3)*) for some dependence function A.*

(II) *For all $x,y \in I$ there exists $\lim_{n\to\infty} n(1-\mu(x^{1/n}, y^{1/n})) = h(x,y)$. Moreover $h(x,y) = -0.5 \times \log C_A(x,y)$, $x,y \in I$.*

4. More on unimodality and Archimax copulas

The results in Section 3 allow us to infer that certain copulas are not star unimodal.

Let $\phi : I \to [0,\infty]$ with $\phi(1)=0$ be a continuous, convex, and strictly decreasing function and denote by $\phi^{[-1]}$ its *pseudo-inverse* given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & \text{for } 0 \leqslant t \leqslant \phi(0), \\ 0 & \text{for } \phi(0) \leqslant t \leqslant \infty. \end{cases}$$

If $\phi(0)=\infty$ then $\phi^{[-1]} = \phi^{-1}$. Further we consider a dependence function A. A copula C is *Archimax* if

$$C_{\phi,A}(u,v) = \phi^{[-1]}\left[\left(\phi(u)+\phi(v)\right)A\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right)\right], \quad u,v \in I.$$

The function ϕ is its *generator*. For $\phi(t) = \log(1/t)$ we are led to an extreme value copula (3) and for $A \equiv 1$ we obtain an *Archimedean* copula (Nelsen [7, p. 90]). Archimax copulas were introduced by Capéraà, Fougères, and Genest [2]. The name 'Archimax' was chosen to reflect the fact that the new family includes both the maximum value distributions and the Archimedean copulas. We observe that for any generator ϕ we have $C_{\phi,A_{1,1}} = M$, where $A_{1,1}$ is given by (4).

According to Cuculescu and Theodorescu [3, Propositions 6.1 and 6.2] Archimedean copulas are not star unimodal except Π and W. In addition we have:

PROPOSITION 4.1. – *No extreme value copula except* Π *and* M *is star unimodal about* $(a,b) \in I^2$*, whatever the choice of* (a,b).

Since in general neither Archimedean nor maximum value copulas are star unimodal let us look closer at Archimax copulas.

PROPOSITION 4.2. – *Suppose that* $A \neq A_{1,1}$. *If* $\phi(1-1/t)$ *is regularly varying at infinity with degree* $-r$ *for some* $r > 1$ *then copula* $C_{\phi,A}$ *is not star unimodal about any* $(a,b) \neq (1,1)$.

PROPOSITION 4.3. – *Suppose that* $A \neq A_{1,1}$. *If* $\phi(0) = \infty$ *and* $\phi(1/t)$ *is regularly varying at infinity with degree* k *for some* $k > 0$ *then copula* $C_{\phi,A}$ *is not star unimodal about any* $(a,b) \neq (0,0)$.

From Propositions 4.2 and 4.3 we deduce

COROLLARY 4.4. – *Suppose that* $A \neq A_{1,1}$. *Under the regularity conditions in Propositions* 4.2 *and* 4.3 *copula* $C_{\phi,A}$ *is not star unimodal about any* $(a,b) \in I^2$.

Acknowledgements. Work supported by the Natural Sciences and Engineering Research Council of Canada and by the Fonds pour la formation de chercheurs et l'aide à la recherche du Gouvernement du Québec. The authors also wish to thank Christian Genest for helpful comments and suggestions.

References

[1] E. Bertin, I. Cuculescu, R. Theodorescu, Unimodality of Probability Measures, Kluwer Academic, Dordrecht, 1997.
[2] P. Capéraà, A.-L. Fougères, C. Genest, Bivariate distributions with given extreme value attractor, J. Multivariate Anal. 72 (2000) 30–49.
[3] I. Cuculescu, R. Theodorescu, Are copulas unimodal?, J. Multivariate Anal. (2002) (accepted with revision).
[4] I. Cuculescu, R. Theodorescu, Unimodal copulas: maximum domain of attraction, Preprint nr. 01-02, University Laval, Department Math. Statist., 2001.
[5] S.W. Dharmadhikari, K. Joag-dev, Unimodality, Convexity, and Applications, Academic Press, New York, 1988.
[6] J. Galambos, The Asymptotic Theory of Extreme Order Statistics, 2nd ed., Krieger, Malabar, 1987.
[7] R.B. Nelsen, An Introduction to Copulas, Springer, New York, 1999.
[8] J. Pickands, Multivariate extreme value distributions, in: Proc. 43rd Session Internat. Statist. Inst., Buenos Aires, 1981, pp. 859–878.
[9] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.
[10] J.A. Tawn, Bivariate extreme value theory: Models and estimation, Biometrika 75 (1988) 397–415.