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Classes of time-dependent measures and the behavior of Feynman–Kac propagators

Archil Gulisashvili

Department of Mathematics, Ohio University, Athens, OH 45701, USA

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Abstract The Feynman–Kac propagators, corresponding to the forward and backward heat equations with a time-dependent measure as a potential are studied. Various generalizations of the Kato class of potentials are considered. Under appropriate conditions on the potential, the solvability and uniqueness theorems for the heat equation with a potential are obtained, and the mapping properties of the Feynman–Kac propagators are discussed. ***To cite this article: A. Gulisashvili, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 445–449.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur les propagateurs de Feynman–Kac pour l'équation de la chaleur avec un potentiel qui est une mesure dépendant du temps

Résumé On étudie les propagateurs de Feynman–Kac, correspondant à l'équation de la chaleur dans le sens direct et inverse avec un potentiel qui est une mesure dépendant du temps. Nous considérons diverses généralisations de la classe de Kato de potentiels. Sous des conditions appropriées sur le potentiel nous obtenons les théorèmes de résolubilité et d'unicité pour l'équation de la chaleur avec un potentiel, et nous étudions les propriétés des transformations définies par les propagateurs. ***Pour citer cet article : A. Gulisashvili, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 445–449.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit $\mu=\{\mu(t): 0\leqslant t\leqslant T\}$ une famille de mesures sur $\mathbb{R}^n$. On dit que μ appartient à la classe $\mathcal{P}_{n,T}$, si

$$\lim_{t\to 0+}\sup_{h:t\leqslant h\leqslant T}\sup_{x\in\mathbb{R}^n}\int_0^t \mathrm{e}^{\frac{s}{2}\Delta}\big|\mu(h-s)\big|\,\mathrm{d}s=0,$$

où $\mathrm{e}^{\frac{s}{2}\Delta}$ désigne le semi-groupe de la chaleur. Pour $\mu\in\mathcal{P}_{n,T}$, le propagateur de Feynman–Kac U_μ est défini par

$$U_\mu(t,\tau)f(x)=E_x f(B_{t-\tau})\exp\big\{-C_\mu(t-\tau,t)\big\}$$

E-mail address: guli@math.ohiou.edu (A. Gulisashvili).

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pour $0 \leqslant \tau \leqslant t \leqslant T$, oú E_x est l'espérance dans l'espace de Wiener, B_t le mouvement brownien, $f \in \mathrm{L}^p$, et C_μ la fonctionnelle additive de mouvement brownien, correspondant à la famille μ.

On notera BC l'espace des fonctions bornées et continuées sur $\mathbb{R}^n$, BUC l'espace des fonctions bornées et uniformément continuées, et C_∞ l'espace des fonctions qui tendent vers zéro à infini. Nous montrons que pour $\mu \in \mathcal{P}_{n,T}$: (1) U_μ est un propagateur sur L^p pour $1 < p \leqslant \infty$. (2) La solution au sens des distributions du problème de la valeur initiale,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u - \mu(t)u, \\ u(\tau) = f, \end{cases}$$

est donnée par $u(t) = U_\mu(t,\tau)f$. (3) $U_\mu(t,\tau) : BUC \to BUC$ et $U_\mu(t,\tau) : \mathrm{C}_\infty \to \mathrm{C}_\infty$ pour $0 \leqslant \tau \leqslant t \leqslant T$. (4) $U_\mu(t,\tau) : \mathrm{L}^p \to BC$ pour $1 < p \leqslant \infty$ et $0 \leqslant \tau < t \leqslant T$. (5) $U_\mu(t,\tau) : \mathrm{L}^p \to \mathrm{L}^q$ pour $1 < p \leqslant q \leqslant \infty$ et $0 \leqslant \tau < t \leqslant T$.

1. Introduction

In this Note we study the behavior of Feynman–Kac propagators for the heat equation with a time-dependent measure as a potential and for the corresponding backward heat equation. We obtain generalizations of various known results concerning the Kato class of potentials and Schrödinger semigroups (*see* [1,11], *see* also [2,13] where the Kato class of time-independent measures is considered). Nonautonomous parabolic equations were also studied in [6–9].

Let $\mu = \{\mu(t) : 0 \leqslant t \leqslant T\}$ be a family of signed Borel measures on $\mathbb{R}^n$. We assume that the variation $|\mu(t)|$ of the measure $\mu(t)$ is locally bounded for all $t \in [0, T]$.

DEFINITION 1. – A two-parametric family $\{U(t,\tau) : 0 \leqslant \tau \leqslant t \leqslant T\}$ of bounded linear operators on the space L^p with $1 \leqslant p < \infty$ is called a propagator for the heat equation,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \mu(t)u, \tag{1}$$

provided (a) $U(t,\tau) = U(t,\lambda)U(\lambda,\tau)$ for $0 \leqslant \tau \leqslant \lambda \leqslant t \leqslant T$; (b) $U(\tau,\tau) = I$ for $0 \leqslant \tau \leqslant T$, where I stands for the identity operator on L^p; (c) For every $f \in \mathrm{L}^p$, the L^p-valued function $(t,\tau) \to U(t,\tau)f$ is continuous for $0 \leqslant \tau \leqslant t \leqslant T$; (d) For every τ with $0 \leqslant \tau < T$, the function $u(t) = U(t,\tau)f$ where $\tau \leqslant t \leqslant T$, is a solution in the $D'((\tau,T) \times \mathbb{R}^n)$-sense to the following initial value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u - \mu(t)u, \\ u(\tau) = f. \end{cases} \tag{2}$$

For $p = \infty$, we requre the weak* continuity in L^∞ in part 3 of Definition 1.

DEFINITION 2. – A two-parametric family $\{Y(t,\tau) : 0 \leqslant \tau \leqslant t \leqslant T\}$ of bounded linear operators on the space L^p with $1 \leqslant p < \infty$ is called a backward propagator for the backward heat equation,

$$\frac{\partial w}{\partial \tau} = -\frac{1}{2}\Delta w + \mu(\tau)w, \tag{3}$$

if the family $U(t,\tau) = Y(T-\tau, T-t)$ is a propagator for Eq. (1) with $\nu(t) = \mu(T-t)$ instead of $\mu(t)$.

It is easy to see that the function $w(\tau) = Y(t,\tau) f$ is a solution in the $D'((0,t)\times\mathbb{R}^n)$-sense to the following terminal value problem:

$$\begin{cases} \dfrac{\partial w}{\partial \tau} = -\dfrac{1}{2}\Delta w + \mu(\tau) w, \\ w(t) = f. \end{cases} \tag{4}$$

Under appropriate restrictions on the potential μ, the family of linear operators, given by

$$U_\mu(t,\tau) f(x) = E_x f(B_{t-\tau}) \exp\bigl\{-C_\mu(t-\tau,t)\bigr\}, \quad 0 \leqslant \tau \leqslant t \leqslant T, \tag{5}$$

is a propagator for Eq. (1), and the family

$$Y_\mu(t,\tau) f(x) = E_x f(B_{t-\tau}) \exp\bigl\{-A_\mu(t-\tau,\tau)\bigr\}, \quad 0 \leqslant \tau \leqslant t \leqslant T, \tag{6}$$

is a backward propagator for Eq. (3) (*see* Section 3). In (5) and (6), E_x is the expectation in the Wiener space, B_t stands for a standard Brownian motion in $\mathbb{R}^n$, and C_μ and A_μ are additive functionals of the Brownian motion which will be defined in Section 2. We call the family U_μ (the family Y_μ) the Feynman–Kac propagator for Eq. (1) (the backward Feynman–Kac propagator for Eq. (3)), respectively.

2. Classes of time-dependent measures

In the following definitions of classes of time-dependent measures on $[0,T]\times\mathbb{R}^n$ or $[0,\infty)\times\mathbb{R}^n$ the symbol $\mathrm{e}^{\frac{t}{2}\Delta}$ denotes the heat semigroup on $\mathbb{R}^n$.

(a) $\mu \in \widehat{\mathcal{P}}_{n,T}$ if $\sup_{h:0\leqslant h\leqslant T} \sup_{x\in\mathbb{R}^n} \int_0^h \mathrm{e}^{\frac{s}{2}\Delta}|\mu(h-s)|(x)\,\mathrm{d}s < \infty$,
(b) $\mu \in \widehat{\mathcal{P}}^*_{n,T}$ if $\sup_{h:0\leqslant h\leqslant T} \sup_{x\in\mathbb{R}^n} \int_0^{T-h} \mathrm{e}^{\frac{s}{2}\Delta}|\mu(s+h)|(x)\,\mathrm{d}s < \infty$,
(c) $\mu \in \widehat{\mathcal{P}}_{n,\infty}$ if $\sup_{h:h\geqslant 0} \sup_{x\in\mathbb{R}^n} \int_0^h \mathrm{e}^{-s}\,\mathrm{e}^{\frac{s}{2}\Delta}|\mu(h-s)|(x)\,\mathrm{d}s < \infty$,
(d) $\mu \in \widehat{\mathcal{P}}^*_{n,\infty}$ if $\sup_{h:h\geqslant 0} \sup_{x\in\mathbb{R}^n} \int_0^\infty \mathrm{e}^{-s}\,\mathrm{e}^{\frac{s}{2}\Delta}|\mu(s+h)|(x)\,\mathrm{d}s < \infty$,
(e) $\mu \in \mathcal{P}_{\mathrm{n},\infty}$ if $\mu \in \widehat{\mathcal{P}}_{\mathrm{n},\infty}$ and $\lim_{\mathrm{t}\to 0+} \sup_{\mathrm{h:h}\geqslant \mathrm{t}} \sup_{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}} \int_0^{\mathrm{t}} \mathrm{e}^{\frac{\mathrm{s}}{2}\Delta}|\mu(\mathrm{h}-\mathrm{s})|(\mathrm{x})\,\mathrm{ds} = 0$,
(f) $\mu \in \mathcal{P}_{\mathrm{n,T}}$ if $\mu \in \widehat{\mathcal{P}}_{\mathrm{n,T}}$ and $\lim_{\mathrm{t}\to 0+} \sup_{\mathrm{h:t}\leqslant\mathrm{h}\leqslant\mathrm{T}} \sup_{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}} \int_0^{\mathrm{t}} \mathrm{e}^{\frac{\mathrm{s}}{2}\Delta}|\mu(\mathrm{h}-\mathrm{s})|\,\mathrm{ds} = 0$,
(g) $\mu \in \mathcal{P}^*_{\mathrm{n},\infty}$ if $\mu \in \widehat{\mathcal{P}}^*_{\mathrm{n},\infty}$ and $\lim_{\mathrm{t}\to 0+} \sup_{\mathrm{h:h}\geqslant 0} \sup_{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}} \int_0^{\mathrm{t}} \mathrm{e}^{\frac{\mathrm{s}}{2}\Delta}|\mu(\mathrm{s}+\mathrm{h})|(\mathrm{x})\,\mathrm{ds} = 0$,
(h) $\mu \in \mathcal{P}^*_{\mathrm{n,T}}$ if $\mu \in \widehat{\mathcal{P}}^*_{\mathrm{n,T}}$ and $\lim_{\mathrm{t}\to 0+} \sup_{\mathrm{h}:0\leqslant\mathrm{h}\leqslant\mathrm{T-t}} \sup_{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}} \int_0^{\mathrm{t}} \mathrm{e}^{\frac{\mathrm{s}}{2}\Delta}|\mu(\mathrm{s}+\mathrm{h})|(\mathrm{x})\,\mathrm{ds} = 0$.

These classes are generalizations of the celebrated Kato class K_n (*see* [1,11]), of the Kato class of measures $\widetilde{K}_n$ (*see*, e.g., [2,3]), of the enlarged Kato class $\widehat{K}_n$ (*see* [12], *see* also [3,5]), and of the nonautonomous Kato classes (*see* [4,6–10]). It is known that the classes $\mathcal{P}_{\mathrm{n,T}}$ and $\mathcal{P}^*_{\mathrm{n,T}}$ do not coincide (*see* [4]).

For a family $\mu(t)$ with $0 \leqslant t < \infty$ put

$$M_\mu(h,r,x) = \int_0^h \mathrm{e}^{-s}\,\mathrm{e}^{\frac{s}{2}\Delta}\mu(r-s)(x)\,\mathrm{d}s, \tag{7}$$

where $0 \leqslant h < \infty$, $h \leqslant r < \infty$, and $x \in \mathbb{R}^n$. Put also

$$N_\mu(h,r,x) = \int_0^h \mathrm{e}^{-s}\,\mathrm{e}^{\frac{s}{2}\Delta}\mu(s+r)(x)\,\mathrm{d}s, \tag{8}$$

where $0 \leqslant h \leqslant \infty$, $0 \leqslant r < \infty$, and $x \in \mathbb{R}^n$. In (7) and (8) we assume that the integrals on the right-hand side make sense. It was shown in [3,5] that if a measure μ does not depend on time, then we have

$$\mu \in \widetilde{K}_n \iff (I-\Delta)^{-1}|\mu| \in BUC. \tag{9}$$

The next theorem generalizes equivalence (9) to the case of time-dependent measures.

THEOREM 1. – (i) *A family* $\mu \in \widehat{\mathcal{P}}^*_{n,\infty}$ *belongs to the class* $\mathcal{P}^*_{\mathrm{n},\infty}$ *if and only if the function* $(h,x) \to N_{|\mu|}(\infty,h,x)$ *is bounded and uniformly continuous on* $[0,\infty) \times \mathbb{R}^n$. (ii) *A family* $\mu \in \widehat{\mathcal{P}}^*_{n,T}$ *belongs to the class* $\mathcal{P}^*_{\mathrm{n,T}}$ *if and only if the function* $(h,x) \to N_{|\mu|}(T-h,h,x)$ *is bounded and uniformly continuous on* $[0,T] \times \mathbb{R}^n$. (iii) *A family* $\mu \in \widehat{\mathcal{P}}_{n,T}$ *belongs to the class* $\mathcal{P}_{\mathrm{n,T}}$ *if and only if the function* $(h,x) \to M_{|\mu|}(T-h,T-h,x)$ *is bounded and uniformly continuous on* $[0,T] \times \mathbb{R}^n$.

We refer the reader to [4] for various examples of families of measures which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ and belong to the classes $\mathcal{P}_{\mathrm{n,T}}$ and $\mathcal{P}^*_{\mathrm{n,T}}$. Similar examples of families of singular measures can also be constructed. The next lemma gives new examples in the case $n = 1$. We denote by δ_x the Dirac measure concentrated at $x \in \mathbb{R}^1$. It can be shown that the time-dependent measure $\mu(t) = t^{-1/2}\delta_x$ does not belong to the class $\mathcal{P}_{1,1}$. However, the following assertion holds:

LEMMA 1. – *Let* $\mu(t) = t^{-1/2}\delta_{\phi(t)}$ *where* $\phi(t) = t^\beta$ *with* $\beta > 0$. *Then* $\mu \in \mathcal{P}_{1,1}$ *if and only if* $\beta < 1/2$.

It is known that the Kato class of measures can be described using the Brownian motion (*see* [2]). Our next goal is to give a similar description of the classes $\mathcal{P}_{\mathrm{n,T}}$ and $\mathcal{P}^*_{\mathrm{n,T}}$. We denote by Ω the Wiener space, by $\mathcal{F}_{\mathrm{t}}$ the standard Brownian filtration, and by θ_t the family of shift operators on Ω.

THEOREM 2. – *Let* μ *be a family of nonnegative measures such that* $\mu \in \mathcal{P}^*_{\mathrm{n},\infty}$. *Then for all* $x \in \mathbb{R}^n$ *there exists a unique* (*up to equivalence*) *family* $A_\mu(t,h)$ *of random variables on* Ω *such that* (a) *for all* $h \geqslant 0$, $A_\mu(0,h) = 0$ *a.s.*; (b) *for all* $t,h \geqslant 0$, *the random variable* $A_\mu(t,h)$ *is* $\mathcal{F}_{\mathrm{t}}$*-measurable*; (c) *for every* $h \geqslant 0$, *the function* $t \to A_\mu(t,h)$ *is non-decreasing and continuous a.s.*; (d) $A_\mu(t+\tau,h) = A_\mu(t,h) + \theta_t \circ A_\mu(\tau,h+t)$ *a.s. for all* $t,\tau,h \geqslant 0$; (e) $\int_0^t \mathrm{e}^{\frac{s}{2}\Delta}\mu(s+h)(x)\,\mathrm{d}s = E_x A_\mu(t,h)$ *for all* $t,h \geqslant 0$.

For a signed measure $\mu \in \mathcal{P}^*_{\mathrm{n},\infty}$, we put $A_\mu(t,h) = A_{\mu^+}(t,h) + A_{\mu^-}(t,h)$, where μ^+ and μ^- denote the positive and the negative part of the measure μ, respectively. If $\mu \in \mathcal{P}^*_{\mathrm{n,T}}$, then the functional $A_\mu(t,h)$ is defined for all pairs (t,h) such that $0 \leqslant t \leqslant T$ and $0 \leqslant h \leqslant T-t$. For $\mu \in \mathcal{P}_{\mathrm{n,T}}$, $0 \leqslant t \leqslant T$, and $t \leqslant h \leqslant T$, we put $C_\mu(t,h) = A_\nu(t,T-h)$ where $\nu(t) = \mu(T-t)$.

From Theorem 2 we get: $\mu \in \mathcal{P}_{\mathrm{n,T}} \Leftrightarrow \lim_{\mathrm{t}\to 0+} \sup_{\mathrm{h}:\mathrm{t}\leqslant \mathrm{h}\leqslant \mathrm{T}} \sup_{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}} E_x C_{|\mu|}(\mathrm{t},\mathrm{h}) = 0$; $\mu \in \mathcal{P}^*_{\mathrm{n,T}} \Leftrightarrow \lim_{\mathrm{t}\to 0+} \sup_{\mathrm{h}:0\leqslant \mathrm{h}\leqslant \mathrm{T}-\mathrm{t}} \sup_{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}} \mathrm{E_x A}_{|\mu|}(\mathrm{t},\mathrm{h}) = 0$.

3. Existence and uniqueness of the Feynman–Kac propagators

In this section we gather our main results concerning the families of operators U_μ and Y_μ, defined by (5) and (6). We denote by *BC* the space of bounded continuous functions on $\mathbb{R}^n$, and by *BUC* and C_∞ the space of bounded uniformly continuous functions and the space of continuous functions which tend to zero at infinity, respectively. For Banach spaces A and B, the symbol $L(A,B)$ will stand for the space of all bounded linear operators from A into B.

THEOREM 3. – *Let* $\mu \in \mathcal{P}_{\mathrm{n,T}}$ *and* $1 < p \leqslant \infty$. *Then the following assertions hold*: (a) *the family* U_μ *is a propagator for Eq.* (1) *on the space* L^p; (b) $U_\mu(t,\tau) \in L(BUC,BUC)$ *for all* $0 \leqslant \tau \leqslant t \leqslant T$; (c) $U_\mu(t,\tau) \in L(\mathrm{C}_\infty,\mathrm{C}_\infty)$ *for all* $0 \leqslant \tau \leqslant t \leqslant T$; (d) $U_\mu(t,\tau) \in L(\mathrm{L}^p, BC)$ *for all* $0 \leqslant \tau < t \leqslant T$; (e) $U_\mu(t,\tau) \in L(\mathrm{L}^p,\mathrm{L}^q)$ *for all* $0 \leqslant \tau < t \leqslant T$ *and* $1 < p \leqslant q \leqslant \infty$. *Moreover, the following norm estimate holds*:

$$\left\|U_\mu(t,\tau)\right\|_{\mathrm{L}^p\to\mathrm{L}^q} \leqslant A\,\mathrm{e}^{\omega(t-\tau)}(t-\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}, \tag{10}$$

where $A > 0$ *is a constant independent of* t, τ, *and* μ. *The constant* ω *in estimate* (10) *does not depend on* t *and* τ.

We do not know whether Theorem 3 holds for $p = 1$.

Remark 1. – Let $\mu \in \mathcal{P}_{\mathrm{n,T}} \cap \mathcal{P}^*_{\mathrm{n,T}}$. Then Theorem 3 holds for $p = 1$. Moreover, the family Y_μ is a backward propagator for Eq. (3) for all $1 \leqslant p \leqslant \infty$, and the operators $U_\mu(t,\tau)$ and $Y_\mu(t,\tau)$ satisfy estimate (10) for all $1 \leqslant p \leqslant q \leqslant \infty$.

The following lemma is used in the proof of parts (b)–(d) of Theorem 3:

LEMMA 2. – *Let* $\mu \in \mathcal{P}_{\mathrm{n,T}}$. *Then*

$$\sup_{0\leqslant h\leqslant T}\sup_{x\in\mathbb{R}^n} E_x \sup_{0\leqslant t\leqslant h} C_\mu(t,h)^2 \leqslant c_T \sup_{0\leqslant\lambda\leqslant T}\sup_{x\in\mathbb{R}^n}\left\{\left|M_\mu(T-\lambda,T-\lambda,x)\right|\right\}\sup_{0\leqslant\lambda\leqslant T} \times \sup_{x\in\mathbb{R}^n}\left\{M_{|\mu|}(T-\lambda,T-\lambda,x)\right\}.$$

Our last result concerns the uniqueness of the distributional solutions.

THEOREM 4. – *Let* $\mu \in \mathcal{P}_{\mathrm{n,T}}$, $0 \leqslant \tau < T$, $f \in \mathrm{L}^p$ *with* $1 < p < \infty$, *and let* u *be a solution to initial value problem* (2) *in the* $D'((\tau,T)\times\mathbb{R}^n)$*-sense. Suppose that the following conditions hold*: (a) *the function* $t \to u(t)$ *is weakly continuous in* L^p *on the interval* $[\tau,T]$; (b) $u \in \mathrm{L}^\infty([\tau+\epsilon,T]\times\mathbb{R}^n)$ *for every* ϵ *with* $0 < \epsilon < T-\tau$. *Then* $u(t) = U_\mu(t,\tau)f$ *for all* $t \in [\tau,T]$.

For $p = \infty$, Theorem 4 holds if we assume the weak* continuity instead of the weak continuity in condition (a). Theorem 4 also holds for the spaces *BUC* or C_∞. If $\mu \in \mathcal{P}_{\mathrm{n,T}} \cap \mathcal{P}^*_{\mathrm{n,T}}$, then Theorem 4 holds for $p = 1$.

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