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Elliptic equations in dimension three: a conjecture of H. Brezis

Olivier Druet

Département de mathématiques, Université de Cergy-Pontoise, site de Saint-Martin, 2, avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France

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Abstract We study the existence of minimizing solutions for an elliptic equation involving critical Sobolev exponent on domains of the three-dimensional Euclidean space. We solve in particular by the affirmative a conjecture of Haïm Brezis. The similar situation in higher dimensions was completely understood thanks to previous works by H. Brezis and L. Nirenberg. ***To cite this article: O. Druet, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 643–647.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Equations elliptiques en dimension 3 : une conjecture de H. Brezis

Résumé On étudie l'existence de solutions minimisantes à une EDP elliptique à croissance de Sobolev critique sur des domaines de l'espace euclidien de dimension trois. On résout en particulier une conjecture de H. Brezis sur le sujet. Les questions analogues en dimensions plus grandes étaient parfaitement comprises depuis les travaux de H. Brezis et L. Nirenberg. ***Pour citer cet article : O. Druet, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 643–647.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On fixe Ω un domaine lisse et borné de $\mathbf{R}^n$, $n \geqslant 3$, et on considère l'équation

$$\Delta u + au = u^{(n+2)/(n-2)} \quad \text{dans } \Omega, \quad u > 0 \quad \text{dans } \Omega, \quad u = 0 \quad \text{sur } \partial\Omega, \tag{E}$$

où $a \in C^\infty(\Omega) \cap L^\infty(\overline{\Omega})$ et Δ désigne le laplacien euclidien avec la convention de signe moins. On suppose également que l'opérateur $\Delta + a$, en tant qu'agissant sur $H_0^1(\Omega)$, l'espace des fonctions de $L^2(\Omega)$ dont le gradient est également dans $L^2(\Omega)$ et dont la trace sur le bord est nulle, est coercif. Cette hypothèse est nécessaire à l'existence de solutions au problème (E). Une technique naturelle pour trouver des solutions de (E) est de chercher des solutions minimisantes, c'est-à-dire des solutions du problème de minimisation suivant :

$$J_a = \inf_{u \in H_0^1(\Omega),\ u \not\equiv 0} \frac{\int_\Omega (|\nabla u|^2 + au^2)\,dx}{\left(\int_\Omega |u|^{2n/(n-2)}\,dx\right)^{(n-2)/n}}.$$

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En effet, si J_a est atteint par u_a, alors, quitte à changer u_a en $|u_a|$ et après une normalisation adéquate, u_a est une solution régulière de (E). Il est désormais bien connu qu'on a toujours $J_a \leqslant K_n^{-1}$ où K_n est la meilleure constante dans l'inégalité de Sobolev euclidienne dans tout l'espace $\mathbf{R}^n$. Des techniques de minimisation standard (basées sur le principe de concentration-compacité de Lions [12]) permettent de montrer que J_a est atteint si l'inégalité ci-dessus est stricte. Brezis et Nirenberg, dans [3], se sont intéressés précisément à la question de l'existence de solutions minimisantes de l'équation (E). Ils ont démontré qu'en dimensions $n \geqslant 4$, les trois propositions suivantes étaient équivalentes :

(1) il existe $x \in \Omega$ tel que $a(x) < 0$;
(2) $J_a < K_n^{-1}$;
(3) il existe une solution minimisante au problème (E).

Ils mirent également en lumière le phénomène inattendu des petites dimensions : la situation change complètement lorsqu'on passe des grandes dimensions ($n \geqslant 4$) aux petites dimensions (ici, $n = 3$). En particulier, la situation est beaucoup plus compliquée en dimension 3 et la question de la caractérisation de l'existence de solutions minimisantes était ouverte jusqu'à présent. Quand $n = 3$, Brezis et Nirenberg ont étudié dans [3] le cas où la fonction a est une constante λ. Ils ont montré qu'il existe $\lambda^\star(\Omega) \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ la première valeur propre de Δ dans Ω avec condition de Dirichlet au bord, tel que $J_\lambda = K_3^{-1}$ pour $\lambda \geqslant -\lambda^\star(\Omega)$ et $J_\lambda < K_3^{-1}$ pour $\lambda < -\lambda^\star(\Omega)$ et tel que J_λ n'est pas atteint pour $\lambda > -\lambda^\star(\Omega)$. Si Ω est une boule B, ils ont montré que $\lambda^\star(B) = \frac{1}{4}\lambda_1(B)$ et que $J_{-\frac{1}{4}\lambda_1(B)}$ n'est pas atteint. Au vu du résultat en dimensions $n \geqslant 4$ et du résultat sur les boules de $\mathbf{R}^3$, Brezis demandait ultérieurement, dans [2], si l'existence de solutions minimisantes était équivalente à l'inégalité stricte $J_a < K_3^{-1}$ en dimension 3.

On note maintenant $G_a : \Omega \times \Omega \backslash \{(x, x),\ x \in \Omega\} \to \mathbf{R}$ la fonction de Green de $\Delta + a$, opérateur que l'on suppose coercif. C'est une fonction symétrique en x et y qui vérifie

$$\Delta_y G_a(x, y) + a G_a(x, y) = \delta_x \quad \text{dans } \Omega, \quad G_a(x, y) = 0 \quad \text{si } y \in \partial\Omega$$

pour tout $x \in \Omega$. On peut décomposer G_a en une partie singulière et une partie régulière, i.e. écrire

$$G_a(x, y) = \frac{1}{\omega_2 |x - y|} + g_a(x, y),$$

où $g_a \in C^0(\Omega \times \Omega)$. Des calculs de fonctions tests, menés dans [14] pour la première fois, donnent que $J_a < K_3^{-1}$ si la partie régulière de G_a prend une valeur strictement positive sur la diagonale. Brezis demandait dans [2] si la réciproque était vraie ou non. On répond par l'affirmative aux deux questions ci-dessus posées :

THÉORÈME. – *Soit Ω un domaine lisse de $\mathbf{R}^3$; soit $a \in C^\infty(\Omega) \cap L^\infty(\overline{\Omega})$ tel que $\Delta + a$ soit un opérateur coercif. Les trois propositions suivantes sont équivalentes* :

(1) *il existe $x \in \Omega$ tel que $g_a(x, x) > 0$* ;
(2) $J_a < K_3^{-1}$;
(3) *il existe une solution minimisante au problème* (E).

La preuve de ce théorème est basée sur une fine analyse asymptotique de suites de solutions d'équations elliptiques à croissance de Sobolev critique qui explosent en un point. De telles études ont été initiées par Atkinson–Peletier [1] et Brezis–Peletier [4] puis reprises entre autres dans [7–9,11,13].

We let Ω be a smooth bounded domain of $\mathbf{R}^n$, $n \geqslant 3$, and we consider the following equation

$$\Delta u + au = u^{(n+2)/(n-2)} \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{E}$$

where $a \in C^\infty(\Omega) \cap L^\infty(\overline{\Omega})$ and Δ is the positive Euclidean Laplacian. We assume also that $\Delta + a$ is a coercive operator when acting on $H_0^1(\Omega)$, the set of functions which are in $L^2(\Omega)$, whose gradient is also in $L^2(\Omega)$ and which vanish on the boundary of Ω. The coercivity of $\Delta + a$ is a necessary condition for problem (E) to have a solution. A natural way to find solutions to problem (E) is to look for solutions of the following minimization problem:

$$J_a = \inf_{u \in H_0^1(\Omega),\ u \not\equiv 0} \frac{\int_\Omega (|\nabla u|^2 + a u^2)\,dx}{\left(\int_\Omega |u|^{2n/(n-2)}\,dx\right)^{(n-2)/n}}.$$

Indeed, if J_a is achieved by some $u_a \in H_1^0(\Omega)$, up to changing u_a into $|u_a|$ and up to normalization, one gets by standard elliptic theory a smooth solution of (E). Such a solution is refered to as a minimizing solution of (E). It is well known that, in any case,

$$J_a \leqslant K_n^{-1},$$

where K_n is the best constant in the Euclidean Sobolev inequality

$$\left(\int_{\mathbf{R}^n} |u|^{2n/(n-2)}\,dx\right)^{(n-2)/n} \leqslant K_n \int_{\mathbf{R}^n} |\nabla u|^2\,dx.$$

The exact value of K_n is also well known. One has that

$$K_n = \frac{4}{n(n-2)} \omega_n^{-2/n},$$

where ω_n denotes the volume of the standard unit n-sphere. The question of finding minimizing solutions to (E) was handled by Brezis and Nirenberg in [3]. They show in particular that the situation is rather different in dimensions $n \geqslant 4$ and in dimension 3. In dimensions $n \geqslant 4$, Brezis and Nirenberg prove that the following properties are equivalent:

(1) there exists $x \in \Omega$ such that $a(x) < 0$;
(2) $J_a < K_n^{-1}$;
(3) there exists a minimizing solution to problem (E).

The equivalence of propositions (1) and (2) is a consequence of the optimal Euclidean inequality on the whole space and of test functions computations. The fact that the second proposition implies the third may now be seen as a consequence of the concentration-compactness principle of Lions [12]. The converse is due to the fact that all the extremal functions for the optimal Euclidean Sobolev inequality in the whole space are known and are non compactly supported. The situation is more tricky in dimension 3 and remained open for many years. In [3], Brezis and Nirenberg studied the case where a is a constant function and they prove that for any smooth domain Ω of $\mathbf{R}^3$, there exists $\lambda^\star(\Omega) \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ being the first eigenvalue of Δ in Ω with Dirichlet boundary conditions, such that $J_\lambda = K_3^{-1}$ when $\lambda \geqslant -\lambda^\star(\Omega)$ and $J_\lambda < K_3^{-1}$ when $\lambda < -\lambda^\star(\Omega)$ and such that J_λ is not achieved for $\lambda > -\lambda^\star(\Omega)$. If Ω is a ball B, they also proved that $\lambda^\star(B) = \frac{1}{4}\lambda_1(B)$ and that $J_{-\frac{1}{4}\lambda_1(B)}$ is not achieved. This result gives in particular a quantitative improvement of the Sobolev inequality on bounded domains of $\mathbf{R}^3$. Looking at what happens on the ball in dimension 3 and on any domain in dimensions $n \geqslant 4$, Brezis asked in [2] the following question:

If $n = 3$, is J_a achieved if and only if $J_a < K_3^{-1}$? (Question 5 of [2].)

From now on, we fix some smooth bounded domain Ω of $\mathbf{R}^3$ and some function a in $C^\infty(\Omega) \cap L^\infty(\overline{\Omega})$ such that $\Delta + a$ is coercive. We let $G_a : \Omega \times \Omega \setminus \{(x,x), x \in \Omega\} \to \mathbf{R}$ be the Green function of $\Delta + a$ on Ω

with Dirichlet boundary condition. We have in the sense of distributions

$$\Delta_y G_a(x,y) + aG_a(x,y) = \delta_x$$

in Ω and $G_a(x,y) = 0$ as soon as one of the two variables is on the boundary of Ω. Moreover, G_a is symmetric with respect to the two variables. We may write

$$G_a(x,y) = \frac{1}{\omega_2|x-y|} + g_a(x,y),$$

where $g_a \in \mathrm{C}^0(\Omega \times \Omega)$ is the regular part of the Green function G_a. By test functions computations (*see* [14]), if there exists $x \in \Omega$ such that $g_a(x,x) > 0$, the strict inequality $J_a < K_3^{-1}$ holds. Brezis asked in [2] whether the converse holds or not (question 7 of [2]). We answer by the affirmative to these two questions of Brezis:

THEOREM. – *Let Ω be a smooth domain of $\mathbf{R}^3$ and let $a \in \mathrm{C}^\infty(\Omega) \cap \mathrm{L}^\infty(\overline{\Omega})$ be such that $\Delta + a$ is coercive. The three following properties are equivalent*:

(1) *there exists $x \in \Omega$ such that $g_a(x,x) > 0$*;
(2) $J_a < K_3^{-1}$;
(3) *there exists a minimizing solution to problem* (E).

The proof of such a result is mainly based on a fine asymptotic analysis for sequences of solutions of elliptic PDE's in $\mathbf{R}^3$. There are many works about this kind of blow-up analysis : among them, [1,4,7–11, 13] and [15] were a great source of inspiration.

We briefly sketch the proof of the theorem. We let $a \in \mathrm{C}^\infty(\Omega) \cap \mathrm{L}^\infty(\overline{\Omega})$. Up to add a constant to a, we may assume that $J_{a-\varepsilon} < K_3^{-1}$ for any $\varepsilon > 0$ and that $J_a = K_3^{-1}$. Proving the theorem reduces to prove that:

(1) J_a is not achieved;
(2) there exists $x_0 \in \Omega$ such that $g_a(x_0,x_0) = 0$.

The first point is a consequence of the second variation formula and of a lemma concerning the action of the conformal group on the center of mass of probability measures on the sphere. This was first noticed by the author in [6] when dealing with the existence of extremal functions for optimal Sobolev inequalities on 3-dimensional compact manifolds. Let us now sketch the proof of the second point. For any $\varepsilon > 0$, $J_{a-\varepsilon} < K_3^{-1}$ so that it is attained by some smooth positive function u_ε which verifies

$$\Delta u_\varepsilon + (a-\varepsilon)u_\varepsilon = J_{a-\varepsilon} u_\varepsilon^5$$

in Ω with $u_\varepsilon = 0$ on $\partial\Omega$ and $\int_\Omega u_\varepsilon^6\,\mathrm{d}x = 1$. Since J_a is not attained, our sequence of functions (u_ε) must converge weakly, but not strongly, to 0 in $\mathrm{H}_1^0(\Omega)$. Thus the sequence (u_ε) develops a concentration point $(x_\varepsilon, \mu_\varepsilon)$ defined by $u_\varepsilon(x_\varepsilon) = \mu_\varepsilon^{-1/2} = \|u_\varepsilon\|_\infty$. The aim is then to study the asymptotic pointwise behaviour of u_ε around its concentration point and on the entire set Ω. The main difficulty is to prove that the concentration point x_ε converges to some point which is not on the boundary of Ω. Then one must prove that u_ε is controled from above by the standard bubble associated to the concentration point $(x_\varepsilon, \mu_\varepsilon)$. In other words, we prove that there exists some $C > 0$ such that for any $\varepsilon > 0$ and any $x \in \Omega$,

$$u_\varepsilon(x) \leqslant C\mu_\varepsilon^{1/2}\left(\mu_\varepsilon^2 + \frac{|x-x_\varepsilon|^2}{4\omega_3^{2/3}}\right)^{-1/2}.$$

The behaviour of (u_ε) far from the concentration points is ruled by the following convergence result:

$$\mu_\varepsilon^{-1/2} u_\varepsilon \to \lambda G_a(x_0,\cdot) \quad \text{in } \mathrm{C}^2_{\mathrm{loc}}\big(\Omega \setminus \{x_0\}\big),$$

where $x_0 = \lim_{\varepsilon\to 0} x_\varepsilon$ and λ is some positive real number. The conclusion that $g_a(x_0, x_0) = 0$ follows from the Pohozaev identity applied to u_ε on smaller and smaller balls $B(x_0, \delta)$.

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