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On the special parabolic points and the topology of the parabolic curve of certain smooth surfaces in $\mathbb{R}^3$

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Abstract In this Note we give a class of real polynomials of degree $n \geqslant 3$ in two variables such that the parabolic curves of theirs graphs have at least $(n-1)(n-2)/2$ connected components diffeomorphic to the circle and exactly $n(n-2)$ special parabolic points. These polynomials are of the form $f = l_1 \cdots l_n$ where l_i, $i = 1, \ldots, n$, are generic real affine functions on the plane. ***To cite this article: A. Ortiz-Rodríguez, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 473–478.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur les points paraboliques spéciaux et sur la topologie de la courbe parabolique des certaines surfaces lisses en $\mathbb{R}^3$

Résumé Dans ce travail on donne une classe des polynômes réels de degré $n \geqslant 3$ à deux variables tels que les courbes paraboliques de leurs graphes contiennent au moins $(n-1)(n-2)/2$ composantes connexes difféomorphes au cercle et exactement $n(n-2)$ points paraboliques spéciaux. Ces polynômes sont de la forme $f = l_1 \cdots l_n$ où les l_i, $i = 1, \ldots, n$, sont des fonctions réelles affines génériques sur le plan. ***Pour citer cet article : A. Ortiz-Rodríguez, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 473–478.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

1. Introduction

Les points d'une surface lisse générique dans l'espace projectif réel de dimension trois peuvent être classifiés en 8 classes lesquelles sont invariantes par des transformations projectives de l'espace (7 peuvent être trouvées en [7,8] et la 8ème fut définie en 1998 par Panov [6]). La classification est basée sur le plus haut ordre de tangence des droites tangentes à la surface en chaque point. La description de ces classes dans une surface générique est la suivante : un *point elliptique* est un point où toute droite tangente à la surface a un ordre de tangence égal à un.

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On appelle *droite asymptotique* une droite tangente à la surface avec ordre de tangence plus grand que un. Un *point hyperbolique* est un point qui a exactement deux droites asymptotiques linéairement indépendentes. Un *point parabolique* est un point qui a exactement une droite asymptotique. Dans une surface générique l'ensemble de points paraboliques (il peut être vide) forme une courbe lisse appelée *courbe parabolique*. Les ensembles des points elliptiques et hyperboliques sont des domaines dans la surface et ils sont séparés par la courbe parabolique. Un *point parabolique spécial*, dénoté par PS, est un point où sa droite asymptotique est tangente à la courbe parabolique.

Un point hyperbolique est *d'inflexion* si au moins une de ses droites asymptotiques a ordre de tangence plus grand que deux. L'ensemble des points d'inflexion (il peut être vide) forme une courbe appelée *courbe flecnodale*. La courbe flecnodale est tangente à la courbe parabolique aux points paraboliques spéciaux. Un point est *hyperbolique spécial* si chacune de ses deux droites asymptotiques a ordre de tangence égal à trois (il est un point d'auto-intersection transversale de la courbe flecnodale). Un point est appelé *point de bi-inflexion* si seulement une de ses droites asymptotiques a ordre de tangence égal à quatre.

En 1981, Platonova [8] et Landis [5] ont donné, indepéndemment, les formes normales (jusqu'au jet d'ordre 5) de 7 classes et en 1998 Panov [6] a donné les formes normales (jusqu'au jet d'ordre 3) des 8 classes.

Dans ce travail on donne une classe des polynômes réels de degré $n \geqslant 3$ à deux variables tels que les courbes paraboliques de leurs graphes contiennent au moins $\frac{(n-1)(n-2)}{2}$ composantes connexes difféomorphes au cercle et exactement $n(n-2)$ points paraboliques spéciaux.

Considérons le graphe $\Gamma f := \{(x, y, z = f(x, y)\}$ d'un polynôme réel générique f de degré n à deux variables. La courbe parabolique de ce graphe projetée sur le plan xy le long de l'axe z est une courbe plane, denotée par $\mathcal{P}$, de degré au plus $2(n-2)$. Par ailleurs il existe une borne supérieure donnée par Harnack en 1876 [4,9] pour le nombre des composantes connexes d'une courbe réelle plane projective de degré m. Cette borne est $\frac{(m-1)(m-2)}{2} + 1$. Dénotons par $b_1(\mathcal{P})$ le nombre des composantes connexes compactes de la courbe parabolique $\mathcal{P} \subset \mathbb{R}^2$. Donc d'après Harnack, $b_1(\mathcal{P}) \leqslant (2n-5)(n-3)+1$. On ne connaît pas la constante minimale $C > 0$ telle que $b_1(\mathcal{P}) \leqslant Cn^2$ pour n suffisamment grand. Déjà pour $n = 4$, on ne sait pas s'il existe un polynôme réel de degré 4 à deux variables tel que $b_1(\mathcal{P}) = 4$.

2. Résultats

DÉFINITION 1. – Un ensemble de n fonctions affines réelles $\{l_i\}_{i=1}^n$ sur le plan réel est *générique* si pour tous $i, j \in \{1, \ldots, n\}$, $i \neq j$, les droites d'équations $l_i = 0$, $l_j = 0$, ne sont pas parallèlles et si chaque droite $l_i = 0$ ne passe par aucun point critiques de la fonction $\prod_{j \neq i} l_j$.

Un polynôme réel f de degré n sur le plan est appelé *polynôme factorisable* s'il existe un ensemble générique de n fonctions affines réelles $\{l_i\}_{i=1}^n$ sur le plan tel que $f = l_1 \cdots l_n$.

Remarque 1. – L'ensemble des polynômes réels factorisables de degré n sur le plan est un ensemble de codimension $\frac{n(n-1)}{2}$ dans l'espace des polynômes réels de degré n sur le plan.

Remarque 2. – Soit $f = l_1 \cdots l_n$ un polynôme réel factorisable de degré n sur le plan. Un point $p \in \{l_i = 0\}$ est un point critique de la fonction $\prod_{j \neq i} l_j$ si et seulement si la deuxième forme fondamentale du graphe de f en ce point est identiquement nulle (p est un *point ombilic plat*).

THÉORÈME. – *Soit $f = l_1 \cdots l_n$ un polynôme réel factorisable de degré $n \geqslant 3$ sur le plan. Le graphe de f, Γf, contient au moins $n(n-2)$ points paraboliques spéciaux et sa courbe parabolique contient au moins $\frac{(n-1)(n-2)}{2}$ composantes connexes difféomorphes au cercle.*

La deuxième partie du théorème était une conjecture de Arnold [2] et elle a été aussi prouvée indépendamment par lui. Aicardi [1] a prouvé qu'il y a exactement $\frac{(n-1)(n-2)}{2}$ composantes connexes difféomorphes au cercle de la courbe parabolique.

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Remarque 3. – Chaque droite $l_i = 0$, $i = 1, \ldots, n$ (l_i est un facteur de f), dans le plan est divisée par les autres $n-1$ droites en n segments dont $n-2$ sont compacts et 2 non compacts.

PROPOSITION 1. – *Soit $f = l_1 \cdots l_n$ un polynôme réel factorisable de degré $n \geqslant 3$ sur le plan. Chaque droite d'équation $l_i = 0$, $i = 1, \ldots, n$, contient exactement $n-2$ points paraboliques spéciaux : un seul à l'intérieur de chaque segment compact considéré dans la remarque* 3 (voir *Fig.* 1).

DÉFINITION 2. – Soit $\{l_1, \ldots, l_n\}$ un ensemble générique des fonctions affines réelles sur le plan. Considérons l'ensemble $A = \mathbb{R}^2 \setminus \{l_1 \cdots l_n = 0\} \subset \mathbb{R}^2$. Si la fermeture d'une des composantes connexes de l'ensemble A est compacte, cette fermeture est appelée *polygone élémentaire*. L'union de tous les polygones élémentaires est appelée le *polygone principal.*

PROPOSITION 2. – *Le graphe d'un polynôme factorisable de degré* 3 *sur le plan est hyperbolique en dehors du polygone principal.*

CONJECTURE. – *Le graphe d'un polynôme factorisable de degré $n \geqslant 3$ sur le plan est hyperbolique en dehors du polygone principal.*

PROPOSITION 3. – *Le graphe d'un polynôme factorisable de degré $n \geqslant 3$ sur le plan contient exactement $n(n-2)$ points paraboliques spéciaux.*

Cette proposition découle immédiatement de la conjecture. Récemment cette conjecture a été prouvée par Aicardi [1]. Les démonstrations sont dans la partie anglaise.

1. Introduction

The points of a generic smooth surface embedded in the three dimensional real projective space can be classified in 8 classes which are invariant under projective transformations of this space (7 of them can be founded in [7,8] and the 8th one was defined in 1998 by Panov [6]). The classification is given by looking to the highest order of tangency of the straight lines tangent to the surface at each point. The description of these classes is the following: an *elliptic point* is a point where every straight line tangent to the surface has order of tangency one. An *asymptotic line* is a straight line tangent to the surface having order of tangency at least two. An *hyperbolic point* is a point having exactly two different asymptotic lines. A *parabolic point* is a point having exactly one asymptotic line. In a generic surface the set of parabolic points (may be empty) is a smooth curve called *parabolic curve*. The sets of elliptic and hyperbolic points are domains in the surface and these sets are separated by the parabolic curve. A *special parabolic point*, denoted by SP, is a point where its asymptotic line is tangent to the parabolic curve. An hyperbolic point is an *inflection point* if at least one of its asymptotic lines has order of tangency at least three. The set of inflection points (may be empty) in a generic surface is a curve called *flecnodal curve*. The flecnodal curve is tangent to the parabolic curve at the special parabolic points. A point is *special hyperbolic* if its two asymptotic lines have order of tangency equal to three (it is a point of transversal self intersection of the flecnodal curve). A *bi-inflection point* is a point having only one asymptotic line with order of tangency four. In 1981, Platonova [8] and Landis [5] gave, independently, normal forms (up to the 5-jet) of 7 classes and in 1998 Panov gave [6] normal forms (up to the 3-jet) of 8 classes.

In this Note we give a class of real polynomials of degree n in two variables such that the parabolic curves of theirs graphs have at least $\frac{(n-1)(n-2)}{2}$ connected components diffeomorphic to the circle and have exactly $n(n-2)$ special parabolic points.

Consider the graph $\{(x, y, z = f(x, y))\}$ of a generic real polynomial f of degree n in the x, y variables. The parabolic curve of this graph projected to the xy plane along the z axis is a plane curve denoted by $\mathcal{P}$, of degree at most $2(n-2)$. There exists an upper bound given by Harnack in 1876 [4,9] for the number of connected components of a real projective plane curve of degree m. This bound is $\frac{(m-1)(m-2)}{2} + 1$. Denote by $b_1(\mathcal{P})$ the number of compact conneceted components of the curve $\mathcal{P} \subset \mathbb{R}^2$. So, the Harnack's inequality says that $b_1(\mathcal{P}) \leqslant (2n-5)(n-3)+1$. We do not know the minimal constant $C > 0$ such that $b_1(\mathcal{P}) \leqslant Cn^2$ for n sufficiently large. For $n = 4$ we do not if there exists a real polynomial of degree 4 in two variables such that $b_1(\mathcal{P}) = 4$.

2. Results

DEFINITION 1. – A set of n real affine functions $\{l_i\}_{i=1}^n$ on the real plane is *generic* if for any $i, j \in \{1, \ldots, n\}$, $i \neq j$, the straight lines with equations $l_i = 0$, $l_j = 0$ are not parallel and if each straight line $l_i = 0$ do not pass through any critical point of the function $\prod_{j \neq i} l_i$.

A real polynomial f of degree n on the plane is called *factorisable polynomial* if there exists a generic set of n real affine functions $\{l_i\}_{i=1}^n$ on the plane such that $f = l_1 \cdots l_n$.

Remark 1. – The set of factorisable real polynomials of degree n on the plane is a set of codimension $\frac{n(n-1)}{2}$ in the space of real polynomials of degree n on the plane.

Remark 2. – Let $f = l_1 \cdots l_n$ be a factorisable real polynomial of degree n on the plane. A point $p \in \{l_i = 0\}$ is a critical point of the function $\prod_{j \neq i} l_i$ if and only if the second fundamental form of the graph of f at this point is zero (p is a flat umbilic point).

THEOREM. – *Let $f = l_1 \cdots l_n$ be a factorisable real polynomial of degree $n \geqslant 3$ on the plane. The graph of $f, \Gamma f$, has at least $n(n-2)$ special parabolic points and its parabolic curve has at least $\frac{(n-1)(n-2)}{2}$ connected components diffeomorphic to the circle.*

The second part of the theorem was a conjecture of Arnold [2] and it was proved independently by him. Aicardi [1] proved that there are exactly $\frac{(n-1)(n-2)}{2}$ connected components diffeomorphic to the circle of the parabolic curve.

Remark 3. – Each straight line $l_i = 0$, $i = 1, \ldots, n$ (l_i is a factor of f), is divided by the other $n-1$ straight lines into n segments, $n-2$ of them are compact and the other 2 are not compact.

PROPOSITION 1. – *Let $f = l_1 \cdots l_n$ be a factorisable real polynomial of degree $n \geqslant 3$ on the plane. Each straight line with equation $l_i = 0$, $i = 1, \ldots, n$ contains exactly $n-2$ special parabolic points: one special parabolic point in the interior of each compact segment considered in Remark* 3 (see *Fig.* 1).

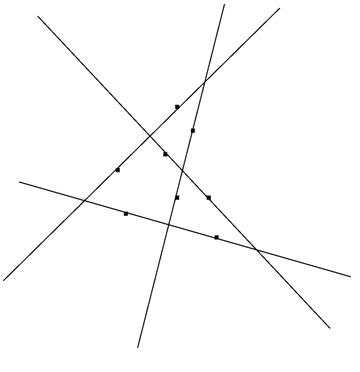

Figure 1. – The distribution of SP points for $n = 4$.

Figure 1. – La distribution des points PS pour $n = 4$.

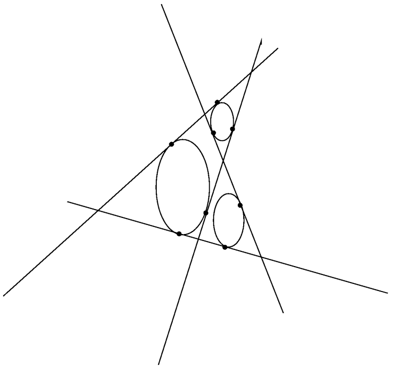

Figure 2. – An exemple for $n = 4$ of the parabolic curve on the plane.

Figure 2. – Un exemple pour $n = 4$ de la courbe parabolique dans le plan.

DEFINITION 2. – Let $\{l_1, \ldots, l_n\}$ be a generic set of n real affine functions on the plane. Consider the set $A = \mathbb{R}^2 \setminus \{l_1 \cdots l_n = 0\}$. When the closure of a connected component of the set A is compact we call this closure *elementary polygon*. The union of all elementary polygons is called the *principal polygon*.

PROPOSITION 2. – *The graph of a factorisable real polynomial of degree* 3 *on the plane is hyperbolic outside the principal polygon.*

CONJECTURE. – *The graph of a factorisable real polynomial of degree* $n \geqslant 3$ *on the plane is hyperbolic outside the principal polygon.*

PROPOSITION 3. – *The graph of a factorisable real polynomial of degree* $n \geqslant 3$ *on the plane has exactly* $n(n-2)$ *special parabolic points.*

This proposition is a direct consequence of the conjecture. Recently, this conjecture was proved by Aicardi [1].

3. Proofs

Proof of Remark 1. – The dimension of the space of factorisable polynomials of degree n is $2n+1$ and the dimension of the space of polynomials of degree n is $\frac{(n+1)(n+2)}{2}$.

Remark 4. – The points of the straight line $l_i = 0$ are not elliptic points. By an appropriate choice of coordinates in the affine plane we may suppose that the line $l_i = 0$ coincides with the line $y = 0$. So, $f_{xx}|_{y=0} = 0$. The Hessian of f, $\mathrm{hess}(f) := f_{xx} f_{yy} - f_{xy}^2$ is a real polynomial on the plane of degree at most $2(n-2)$. The points (x, y) where the Hessian is positive, zero or negative correspond to elliptic, parabolic and hyperbolic points, respectively (it can be found in [3]) of the graph of f. The Hessian of f, restraint to the line $y = 0$ is not positive: $\mathrm{hess}(f)|_{y=0} = -(f_{xy})^2|_{y=0} \leqslant 0$. So, generically, the points of the line $y = 0$ are hyperbolic points and only at isolated points are they parabolic points.

LEMMA 1. – *Let* $f = l_1 \cdots l_n$ *be a factorisable real polynomial of degree* n *on the plane. The straight line with equation* $l_i = 0$, $i \in \{1, \ldots, n\}$, *is a component of the flecnodale curve.*

Proof of Lemma 1. – By Remark 4, generically the points of the lines $\{l_i = 0\}_{i=1}^n$ are hyperbolic. Moreover, these straight lines are tangent lines to the surface with order of tangency at least three, so theirs points are inflection points.

Proof of Proposition 1. – We can suppose that the straight line $l_n = 0$ coincides with the line $y = 0$. By Lemma 1 this line is a component of the flecnodal curve. Denote by x_i, $i = 1, \ldots, n-1$ the intersection point of the straight lines $y = 0$ and $l_i = 0$. We recall that $f = y \cdot l_1 \cdots l_{n-1}$, so $f_y|_{y=0} = l_1 \cdots l_{n-1}|_{y=0}$. Denote $f_y|_{y=0} = F(x)$. This real polynomial F of degree $n-1$ in one variable is zero at the points x_i, $i = 1, \ldots, n-1$. So, F has $n-1$ simple real roots and $n-2$ nondegenerated critical points, i.e., the function $F'(x) = f_{xy}|_{y=0}$ has $n-2$ simple real roots. Each critical point of F is a double real root of the hessian of f restraint to $y = 0$ because $\mathrm{hess}(f)|_{y=0} = -(f_{xy})^2|_{y=0} = -(F'(x))^2$. Moreover, $f_{xxy} \neq 0$ at the critical points of F because they are nondegenerated. Let $\tilde{x}$ be a critical point of F. By the hypothesis Γf has no flat umbilic points then $f_{yy}(\tilde{x}, 0) \neq 0$ (by the Remark 4, $f_{xx} = f_{xy} = 0$). This implies that $\frac{\partial}{\partial y}\mathrm{hess}(f)(\tilde{x}, 0) = f_{yy}(\tilde{x}, 0) f_{xxy}(\tilde{x}, 0) \neq 0$, i.e., the parabolic curve is smooth at $(\tilde{x}, 0)$ and its asymptotic line, $y = 0$, is tangent to the line $y = 0$. Finally the point $(\tilde{x}, 0)$ is a SP point.

LEMMA 2. – *Let* f *be a factorisable real polynomial of degree* n *on the plane. There are* $\frac{(n-1)(n-2)}{2}$ *elementary polygons.*

LEMMA 3. – *Let* f *be a factorisable real polynomial of degree* $n \geqslant 3$ *on the plane. Then* f *has exactly one local extremum in the interior of each elementary polygon. This extremum is nondegenerated.*

Proof of theorem. – The first part is a direct consequence of the Proposition 1.

Denote by Δ an elementary polygon and $\tilde{\Delta} = \{(x, y, f(x, y)) \mid (x, y) \in \Delta\} \subset \Gamma f$. By Lemma 3 the quadratic form of the surface is positive defined or negative defined at the critical point p of f which is in the interior of Δ. So there is an elliptic domain containing the point p. By Lemma 1 the surface is not elliptic on the boundary of $\tilde{\Delta}$. So, the elliptic domain containing p is in the interior of $\tilde{\Delta}$ and the graph of f has at least one parabolic component in $\tilde{\Delta}$. By Proposition 1 the intersection points of the parabolic curve and the boundary of $\tilde{\Delta}$ are SP points, i.e., the parabolic curve is smooth and tangent to the boundary of $\tilde{\Delta}$ at these points. So, the parabolic curve do not crosses the boundary of $\tilde{\Delta}$. By Lemma 2 there are at least $\frac{(n-1)(n-2)}{2}$ compact connected parabolic components.

Proof of Lemma 2. – The set $A = \mathbb{R}^2 \setminus \{l_1 \cdots l_n = 0\}$ is formed by $1 + \sum_{i=1}^{n} i = \frac{n^2+n+2}{2}$ components. In this set there are $2n$ components whose closures are non compact. So, there are $\frac{n^2+n+2}{2} - 2n = \frac{(n-1)(n-2)}{2}$ elementary polygons.

Proof of Lemma 3. – The function f has at least one local extrema in the interior of each elementary polygon: the function is continue and $f = 0$ on the boundary of each elementary polygon. We shall prove that these local extrema are nondegenerated. We remark that these points are critical points of f, so by Lemma 2 f has at least $\frac{(n-1)(n-2)}{2}$ critical points. Otherwise, the intersection points of the straight lines $l_i = 0$ and $l_j = 0$, $i < j$; $i, j \in \{1, \ldots, n\}$ are critical points of the function f: the tangent plane to the graph of f at these points is parallel to xy plane. So, f has at least $\frac{n(n-1)}{2}$ critical points. The number of critical points of f is $\geqslant \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} = (n-1)^2$. By Bézout's theorem the function f has at most $(n-1)^2$ critical points: the equations $f_x = 0$ and $f_y = 0$ represent two real plane curves of degree $n - 1$. So, f has exactly $(n-1)^2$ critical points. This implies that the local extremum being in the interior of each elementary polygon is nondegenerated.

Proof of Proposition 2. – By an appropriate choice of coordinates on affine plane we may suppose that the straight lines $l_i = 0$, $i = 1, 2, 3$, coincide with the straight lines $x = 0$, $y = 0$, $y + x - 1 = 0$. So, $f = xy(y + x - 1)$. The hessian of function f is negative outside the principal polygon. This fact implies that the surface is hyperbolic outside the principal polygon.

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