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Équations aux dérivées partielles/*Partial Differential Equations*

On some infinite sums of integer valued Dirac's masses

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Abstract We give a simple proof of a result obtained by Bourgain, Brezis and Mironescu [2] concerning special distributions arising as singular Jacobian determinants. The strong relation of the problem with boundary rectifiability theorems is discussed, and an interesting question remains open. ***To cite this article: D. Smets, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 371–374.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur certaines sommes infinies de masses de Dirac entières

Résumé On donne une démonstration simple d'un résultat obtenu par Bourgain, Brezis et Mironescu [2] concernant certains déterminants jacobiens singuliers. La preuve utilise la relation forte du problème avec les théoremes de rectifiabilité du bord en théorie géometrique de la mesure. Un problème intéressant reste ouvert. ***Pour citer cet article : D. Smets, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 371–374.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Let Ω be the boundary of a domain in $\mathbb{R}^3$. To each function $u \in H^{1/2}(\Omega, \mathbb{R})$, one can associate a distribution $T(u) \in \mathcal{D}'(\Omega, \mathbb{R})$ which, if u is smooth, is given by $T(u) = \det Du$ (*see* [2] for the construction). If u is smooth and takes its values in S^1, then clearly $T(u) = 0$. Instead, if one only has $u \in \mathrm{H}^{1/2}(\Omega, S^1) \cap \mathrm{H}^1_{\mathrm{loc}}(\Omega \setminus \bigcup_{i=1}^k \{a_j\}, S^1)$, then

$$T(u) = 2\pi \sum_{j=1}^{k} d_j \, \delta_{a_j}, \tag{1}$$

where $d_j = \deg(u, a_j) \in \mathbb{Z}$ and δ_{a_j} is the unit Dirac's mass at point a_j. Roughly said, the distribution T describes the location and the topological degree of the singularities of u. These singularities play a crucial role in the 3-dimensional Ginzburg–Landau model with the function u as boundary data. Using an approximation theorem due to T. Rivière [7], it is shown in [2] that if $u \in \mathrm{H}^{1/2}(\Omega, S^1)$, then there are two sequences of points (P_i) and (Q_i) in Ω such that

$$\sum_i \operatorname{dist}(P_i, Q_i) < +\infty \quad \text{and} \quad \big\langle T(u), f \big\rangle = 2\pi \sum_i \big(f(P_i) - f(Q_i) \big). \tag{2}$$

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If moreover T is a measure, then

$$T(u) = 2\pi \sum_{\text{finite}} d_j\, \delta_{a_j}, \tag{3}$$

with $d_j \in \mathbb{Z}$, and $a_j \in \Omega$.

A related result was also obtained by Jerrard and Soner [5,6] for maps in $\mathrm{W}^{1,1}(\Omega, S^1) \cap \mathrm{L}^\infty(\Omega, S^1)$.

The proof of the last assertion, when T is a measure, takes advantage of some additional properties that T inherits from its construction. The question was raised by Haïm Brezis whether if such a result could be obtained without these further informations, and in more general spaces. The precise question is stated in the following section.

1. Statement of the problem

Assume that X is a complete metric space, we will denote by $\mathrm{BC}(X)$ the Banach space of bounded continuous functions on X with the supremum norm, and by $\mathrm{BLip}(X)$ the Banach space of bounded Lipschitz continuous functions on X with norm $\|f\|_{\mathcal{C}^{0,1}} := |f|_\infty + \mathrm{Lip}(f)$, where $\mathrm{Lip}(f)$ is the best Lipschitz constant for f. The norms in the duals of $\mathrm{BC}(X)$ and $\mathrm{BLip}(X)$ will be denoted respectively by $\|\cdot\|_0$ and $\|\cdot\|_1$. When X is a Banach space, $\mathcal{C}^1_b(X)$ is the closed subspace of $\mathrm{BLip}(X)$ consisting of continuously differentiable functions. Also, when Y and Z are two Banach spaces, the norm on $Y \times Z$ is chosen to be the maximum of the norms of the two canonical projections.

Let (P_i), (Q_i) be two sequences in X such that $\sum_{i\in\mathbb{N}} \mathrm{dist}(P_i, Q_i) < +\infty$. The following functional is well defined on $\mathrm{BLip}(X)$:

$$T(f) := \sum_{i\in\mathbb{N}} \bigl(f(P_i) - f(Q_i)\bigr), \tag{4}$$

and is continuous, with $|T(f)| \leqslant \sum_{i\in\mathbb{N}} \mathrm{dist}(P_i, Q_i)\,\mathrm{Lip}(f)$.

PROBLEM 1.1. – *Assume that T is a measure, in the sense that there exists $C > 0$ such that*

$$\bigl|T(f)\bigr| \leqslant C|f|_\infty \quad \forall f \in \mathrm{BLip}(X).$$

Does there exists a finite number of points $x_i \in X$ and integers a_i such that

$$T(f) = \sum_i a_i f(x_i).$$

Remark 1. – (a) Notice that the topology induced by the norm $\|\cdot\|_1$ is equivalent to the one induced by the minimal connection length for the pair (P_i), (Q_i). By definition, this minimal connection length is the infimum of $\sum_{i\in\mathbb{N}} \mathrm{dist}(P_i, Q_{\sigma(i)})$ when σ is a permutation of $\mathbb{N}$ (see the work of Brezis, Coron and Lieb [3] for the proof (Lemma 4.2) and more).

(b) The positive answer in the case of $X = \mathbb{R}^n$ can be seen as consequence of a closure theorem in Federer's book ([4], Theorem 4.2.16, part 1). The proof in Section 2 below follows rather the same lines. See also [1] for a more general setting.

(c) The completeness of the space X is essential for a positive answer, as easy examples show.

We should mention that there is no easy transfer in general metric spaces between measures in the set theoretic sense and measures in the functional sense. However, this transfer is available in locally compact spaces due to Riesz's representation theorem. It is also available in general metric spaces if T satisfies a Daniell's condition (see Section 4), which will be added to our assumptions. We reduce the cases of metric spaces to that of the real line by using a pull-back process.

2. The real line case

Let T_n denote the partial sum $T_n := \sum_{i=0}^n (\delta_{P_i} - \delta_{Q_i})$ and $\mu_n := T - T_n$.

Clearly, the condition $\sum_{i\in\mathbb{N}} \operatorname{dist}(P_i, Q_i) < +\infty$ implies that $\|\mu_n\|_1 \to 0$ as $n \to +\infty$. Going if necessary to a subsequence, we can assume that

$$\sum_{n\in\mathbb{N}} \|\mu_n\|_1 < +\infty. \tag{5}$$

Let $a, b \in \mathbb{R}$, we will prove that $T([a,b]) \in \mathbb{Z}$.

On the closed subspace $V := \{(\varphi, \varphi'), \varphi \in \mathcal{C}^1_b(\mathbb{R})\} \subset \mathrm{BC}(\mathbb{R}) \times \mathrm{BC}(\mathbb{R})$, we define $A_n(\varphi, \varphi') := \mu_n(\varphi)$. Clearly, $\|A_n\| = \|\mu_n\|_1$. Using the Hahn–Banach theorem, A_n can be extended to a linear map on $\mathrm{BC}(\mathbb{R}) \times \mathrm{BC}(\mathbb{R})$ without increasing its norm, we still denote it by A_n. An easy calculation shows that the measures $\nu_n(\varphi) := A_n(0, \varphi)$ and $(\nu_n \circ d)(\varphi) := A_n(0, \varphi')$ satisfy

$$\big\|\mu_n - (\nu_n \circ d)\big\|_0 + \|\nu_n\|_0 = \|\mu_n\|_1. \tag{6}$$

Assume that $\chi_{r,h}$ is a smooth function which is equal to 1 on $[a-r, b+r]$, to 0 outside $[a-r-h, b+r+h]$ and which satisfies $|\chi_{r,h}|_\infty \leqslant 1$, $\operatorname{Lip}(\chi_{r,h}) \leqslant 2/h$.

The function $g(r) := |\nu_n|([a-r, b+r])$ being increasing in r, it is almost everywhere differentiable. In those points r where it is, we compute:

$$\big|\mu_n(\chi_{r,h})\big| \leqslant \big|\mu_n(\chi_{r,h}) - (\nu_n \circ d)(\chi_{r,h})\big| + \big|\nu_n(\chi'_{r,h})\big| \leqslant \big\|\mu_n - (\nu_n \circ d)\big\|_0 + \frac{2}{h} \cdot \big(g(r+h) - g(r)\big).$$

Taking the limit $h \to 0$ first and then integrating for $r \in [0,1]$, we obtain:

$$\int_0^1 \big|\mu_n\big([a-r, b+r]\big)\big| \,\mathrm{d}r \leqslant \big\|\mu_n - (\nu_n \circ d)\big\|_0 + \|\nu_n\|_0,$$

and using Eqs. (5), (6),

$$\int_0^1 \sum_{n\in\mathbb{N}} \big|\mu_n\big([a-r, b+r]\big)\big| \,\mathrm{d}r < +\infty. \tag{7}$$

Hence, for almost every $r \in [0,1]$, $\mu_n([a-r, b+r]) \to 0$, so that $T([a-r, b+r]) = \lim_{n\to+\infty} T_n([a-r, b+r]) \in \mathbb{Z}$ and $T([a,b]) = \lim_{r\to 0} T([a-r, b+r]) \in \mathbb{Z}$.

PROPOSITION 2.1. – *The answer to Problem* 1.1 *is yes when* $X = \mathbb{R}$.

Proof. – The result immediately follows from the previous analysis. □

3. The locally compact case

When X is a locally compact metric space, we can associate to the functional T a Borel regular measure (also denoted T) by mean of the Riesz's representation theorem.

LEMMA 3.1. – *Assume* X *is locally compact and* T *is a measure as in Problem* 1.1, *then for each closed ball* $B[x,r] \subseteq X$, $T(B[x,r]) \in \mathbb{Z}$.

Proof. – Let

$$\sigma_x : \mathrm{BC}(\mathbb{R}) \to \mathrm{BC}(X) \quad \text{and} \quad f \mapsto \sigma_x f(\cdot) := f\big(\operatorname{dist}(x, \cdot)\big). \tag{8}$$

The pull-back measures,

$$\widetilde{T}(f) := T(\sigma_x f) \quad \text{and} \quad \widetilde{T}_n(f) := T_n(\sigma_x f)$$

satisfy the assumptions of Proposition 2.1. Indeed,

$$\big|\widetilde{T}(f) - \widetilde{T}_n(f)\big| = \big|T(\sigma_x f) - T_n(\sigma_x f)\big| \leqslant \|T - T_n\|_1 \operatorname{Lip}(\sigma_x f) \leqslant \|T - T_n\|_1 \operatorname{Lip}(f). \tag{9}$$

Using Proposition 2.1, $\widetilde{T}$ is a finite sum of integer valued Dirac's masses. Hence, $T(B[x,r]) \in \mathbb{Z}$, whatever r is. As x was arbitrary, the proof is complete. □

PROPOSITION 3.2. – *The answer to Problem* 1.1 *is yes if* X *is locally compact.*

Proof. – Let T_{cont} be the non atomic part of T. That is, $T - \sum_i T(x_i)\delta_{x_i}$ where the x_i are the points in X with a nonzero T measure (of course there are a finite number of them). Again, $T_{\mathrm{cont}}(B[x,r]) \in \mathbb{Z}$ for each $x \in X$, $r \geqslant 0$. Assume that the theorem is false, so that $T_{\mathrm{cont}} \neq 0$. Then, for each $n \in \mathbb{N}$, there exists a ball $B[y_n, 1/n]$ such that $0 \neq T_{\mathrm{cont}}(B[y_n, 1/n]) \in \mathbb{Z}$. Indeed, X has a nonzero T_{cont} measure and can be covered by a countable family of such balls (X is even separable). Now, the sequence (y_n) must have an accumulation point $y^* \in X$, otherwise T would have an infinite total variation. Then, $T_{\mathrm{cont}}(y^*) \neq 0$, a contradiction. This proves the theorem. □

4. The general case

As explained in the introduction, there is no easy correspondence in non locally compact metric spaces between measures in the sense of Problem 1.1 and measures in a set theoretic sense. However, it is not clear that the right definition of a functional measure in the case of a non locally compact space is that of Problem 1.1. Instead, the following definition of a functional measure (often called a Daniell integral) seems more appropriate because of Theorem 4.2.

DEFINITION 4.1. – A finite Daniell integral on a complete metric space X is a linear functional $T : \mathrm{BC}(X) \to \mathbb{R}$ such that

$$\bigl|T(f)\bigr| \leqslant C|f|_\infty \quad \forall f \in \mathrm{BC}(X) \tag{10}$$

and

$$f_n \uparrow f \text{ everywhere } \Rightarrow T(f_n) \to T(f), \tag{11}$$

whenever $f_n, f \in \mathrm{BC}(X)$.

Remark 2. – When X is locally compact and the f_n have all their supports in a same compact set, then condition (11) is implied by condition (10).

The following theorem can be found in Federer's book ([4], Theorem 2.5.5).

THEOREM 4.2. – *Assume that T is a finite Daniell integral, then there exists a finite Borel regular (signed) measure μ_T such that*

$$T(f) = \int_X f \, \mathrm{d}\mu_T \quad \textit{whenever } f \in \mathrm{BC}(X).$$

In this framework of Daniell integral, we have:

THEOREM 4.3. – *If X is a complete metric space and T is a finite Daniell integral, the answer to Problem* 1.1 *is yes.*

Proof. – First notice that the support of T is contained in the closure of the set $(\bigcup_i P_i) \cup (\bigcup_i Q_i)$, which, with the induced topology, is a complete separable metric space. Without loss of generality, we can thus assume that X is separable. The proof then goes on along the same lines as in Proposition 3.2. □

References

[1] L. Ambrosio, B. Kirchheim, Currents in metric spaces, Acta Math. 185 (1) (2000) 1–80.
[2] J. Bourgain, H. Brezis, P. Mironescu, On the structure of the Sobolev space $\mathrm{H}^{1/2}$ with values into the circle, C. R. Acad. Sci. Paris, Sér. I 331 (2) (2000) 119–124.
[3] H. Brezis, J.-M. Coron, E. Lieb, Harmonic maps with defects, Comm. Math. Phys. 7 (1986) 649–705.
[4] H. Federer, Geometric Measure Theory, Springer-Verlag, Berlin, 1969.
[5] R.L. Jerrard, H.M. Soner, Rectifiability of the distributional Jacobian for a class of functions, C. R. Acad. Sci. Paris, Sér. I 329 (8) (1999) 683–688.
[6] R.L. Jerrard, H.M. Soner, Functions of bounded higher variation, Preprint, 1999.
[7] T. Rivière, Dense subsets of $\mathrm{H}^{1/2}(S^2, S^1)$, Ann. Global Anal. Geom. 18 (5) (2000) 517–528.