On manifolds supporting quasi-Anosov diffeomorphisms *

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Abstract   Let M be an n-dimensional manifold supporting a quasi-Anosov diffeomorphism. If $n = 3$ then either $M = \mathbb{T}^3$, in which case the diffeomorphisms is Anosov, or else its fundamental group contains a copy of $\mathbb{Z}^6$. If $n = 4$ then $\Pi_1(M)$ contains a copy of $\mathbb{Z}^4$, provided that the diffeomorphism is not Anosov. *To cite this article: J. Rodriguez Hertz et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 321–323.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur les variétés qui admettent des difféomorphismes de type quasi-Anosov

Résumé   Soit M une variété différentiable de dimension n qui admet un difféomorphisme de type quasi-Anosov. Si $n = 3$ alors on a l'alternative suivante, ou bien $M = \mathbb{T}^3$, et dans ce cas le difféomorphisme est en fait d'Anosov, ou bien le goupe fondamental de M contient une copie de $\mathbb{Z}^6$. Si $n = 4$, alors $\Pi_1(M)$ contient une copie de $\mathbb{Z}^4$, pourvu que le difféomorphisme ne soit pas d'Anosov. *Pour citer cet article : J. Rodriguez Hertz et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 321–323.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

In this work we obtain some restrictions on a manifold M in order to support *quasi-Anosov diffeomorphisms* (QAD), meaning diffeomorphisms $f : M \to M$ such that $\|(f^n)'(x)v\| \to +\infty$ for all non zero vectors $v \in T_x M$, either for $n \to \infty$ or for $n \to -\infty$.

These maps were introduced in [5] by Mañé, who showed they satisfy Axiom A and the no-cycle condition. Besides, they are the C^1 interior of the class of *expansive diffeomorphisms g* [6], those satisfying for some $\alpha > 0$, that $\sup_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) > \alpha$, if $x \neq y$.

We expect on one hand, that studying the relation between QAD's (seen as simplified examples of expansive diffeomorphisms) and the manifolds supporting them, will bring some ideas about the restrictions that expansive homeomorphisms impose on their ambient manifolds. Indeed, let us recall, for instance, that no expansive homeomorphism may be found on the 2-sphere, all expansive homeomorphisms are

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conjugated to Anosov diffeomorphisms on the 2-torus, and to pseudo-Anosov maps if they live in a surface with genus greater than 1 [2,4].

In the case M is a 3-manifold, an expansive $C^{1+\varepsilon}$ diffeomorphism is conjugated to an Anosov diffeomorphism if its non-wandering set is M; in which case M must be $\mathbb{T}^3$ [11].

On the other hand, Mañé [7] and Hiraide [3] have posed the question of whether a QAD on $\mathbb{T}^n$ is necessarily Anosov. This work provides a positive answer for $n = 3$, though we must point out that solving the problem for higher dimensions seems to involve more sophisticated tools.

However, there are also examples of QAD's on 3-dimensional manifolds having wandering points [1]. We present here necessary conditions on a 3-manifold to support this kind of example. Partial results on 4-manifolds are also obtained.

The main result in this work is the following:

THEOREM. – *Let $f : M^n \to M^n$ be a quasi-Anosov diffeomorphism which is not Anosov. If $n = 3$ then $\pi_1(M)$ contains a subgroup which is isomorphic to $\mathbb{Z}^6$. If $n = 4$, $\pi_1(M)$ contains a subgroup which is isomorphic to $\mathbb{Z}^4$.*

This result is strongly based on a theorem due to R. Plykin, which is stated below, and arguments concerning Axiom A diffeomorphisms with the no-cycle condition.

THEOREM 1.1 ([8,9]). – *Let $f : M \to M$ be a diffeomorphism on an n-manifold, where $n \geqslant 3$. If f has k expanding attractors or shrinking repellers of codimension 1 then $\pi_1(M)$ contains a subgroup isomorphic to $\bigoplus_k \mathbb{Z}^n$.*

We recall that a hyperbolic attractor is said to have *dimension u* or equivalently *codimension $n - u$* if the dimension of its unstable fibre bundle is u. Analogously we shall say that a hyperbolic repeller is of *dimension s* or *codimension $n - s$* if the dimension of its stable fiber bundle is s.

Our strategy is to see that a QAD on a 3-manifold must have at least two codimension one attractors or repellers, while on a 4-manifold, it must have at least one. This fact will allow us to prove our main theorem, getting the following as an immediate corollary:

COROLLARY 1.1. – *Every quasi-Anosov diffeomorphism on $\mathbb{T}^3$ is Anosov.*

Corollary 1.1 provides an easier way of recognizing Anosov diffeomorphisms on $\mathbb{T}^3$: it suffices to check that each non zero vector in the tangent bundle goes to infinity under the action of Df^n.

2. Relation among dimension 1 attractors and codimension 1 repellers

From now on, we shall assume M is a compact smooth n-manifold without boundary, and $f : M \to M$ is a QAD. The non-wandering set of f will be denoted by $\Omega(f)$, and Λ_i will stand for the basic sets arising from the Spectral Decomposition Theorem [10]. We shall also maintain the standard notation $W^s(x)$ for the stable manifold of x, i.e., the set of points $y \in M$ such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$.

DEFINITION 2.1. – Letting $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_k$, we shall say that Λ_i is of type (s, u) if the stable dimension of Λ_i is s, and the unstable dimension of Λ_i is u. The expression $W^\sigma(\Lambda_i)$ will be used to denote the set $\bigcup_{x \in \Lambda_i} W^\sigma(x)$, for $\sigma = s, u$, in which case the relation $\Lambda_i \prec \Lambda_j$ will mean that $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$.

We recall from [5] that a QAD is Anosov, if it satisfies the strong transversality condition. Notice that this condition is satisfied if all basic sets are of the same type. We state this conclusion as a lemma

LEMMA 2.1. – *If all basic sets of a QAD f are of the same type then f is Anosov.*

Therefore we get the following:

PROPOSITION 2.1. – *Let f be a QAD such that $\Omega(f) \neq M$. If some basic set of f is of type $(1, n-1)$ (or $(n-1, 1)$), then f must have a codimension one attractor (repeller).*

Proof. – Let Λ_1 be a basic set of f, of type $(1, n-1)$ then $W^u(\Lambda_1)$ must meet $W^s(\Lambda_2)$ where Λ_2 is another basic set of f (the no cycle condition yields $\Lambda_1 \neq \Lambda_2$). Hence $\Lambda_1 \prec \Lambda_2$, implying that the stable dimension of Λ_2 is one. Indeed, let $x \in \Lambda_1$ and $y \in \Lambda_2$ such that $z \in W^s(x) \cap W^u(y)$. If the stable dimension of Λ_2 were greater than one, then we would find a non zero vector $v \in T_z M$ which would be tangent both to $W^u(z)$ and $W^s(z)$, what would yield $\|f^n(z)v\| \to 0$ as $|n| \to \infty$, contradicting the fact that f is quasi-Anosov.

This leaves us only two possibilities: either $W^u(\Lambda_2)$ meets $W^s(\Lambda_3)$, for some other basic set Λ_3, or Λ_2 contains the whole set $W^u(\Lambda_2)$, whence it would be a codimension one attractor. In this way we inductively obtain a chain

$$\Lambda_1 \prec \Lambda_2 \prec \cdots \Lambda_r$$

which must end after a finite number of steps, due to the no cycle condition. Besides, the previous argument shows that Λ_i has stable dimension one, for each $i = 1, \ldots, r$.

If we suppose the previous chain is maximal, then Λ_r must be a codimension one attractor. $\quad\square$

3. Proof of the theorem

We shall see that each QAD on a 3-manifold must have an attractor *and* a repeller of codimension one, provided that it is not Anosov. This, together with Theorem 1.1 finishes the case $\dim M = 3$.

Let $f : M^3 \to M^3$, since quasi-Anosov maps do not have attracting or repelling periodic points, their basic sets must be all of type $(1, 2)$ or $(2, 1)$. If f has a basic set of type $(1, 2)$ then Proposition 2.1 implies the existence of a codimension one attractor. Now, if f is not Anosov, then it must also have a basic set of type $(2, 1)$, otherwise, all the basic sets would be of the same type. But in this case, Proposition 2.1 guarantees the existence of a codimension one repeller, as well.

Observe that the previous argument also shows that if $f : M^4 \to M^4$ is not an Anosov diffeomorphism, then it must have (at least) a basic set of type $(1, 3)$ or $(3, 1)$, otherwise, all basic sets would be of type $(2, 2)$ what would imply f is Anosov. Proposition 2.1 again, implies the existence of an attractor or a repeller of codimension one.

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