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Contrôle optimal/*Optimal Control*

A note on the stabilizability of stochastic heat equations with multiplicative noise

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Abstract We show that a stochastic heat equation with multiplicative noise on a bounded domain $\mathcal{D}$ can be stabilized by a control acting only on a subdomain $\mathcal{O}\subset\mathcal{D}$ if $\mathcal{D}\backslash\mathcal{O}$ is sufficiently 'thin'. We consider both linear and semilinear stochastic heat equations. ***To cite this article: V. Barbu et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 311–316.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur la stabilisabilité des équations de la chaleur stochastiques avec bruit multiplicatif

Résumé On démontre qu'une équation parabolique stochastique avec bruit multiplicatif sur un domaine $\mathcal{D}$ peut être stabilisée par un contrôle agissant seulement sur un sous-domaine $\mathcal{O}$ si $\mathcal{D}\backslash\mathcal{O}$ est « assez petit ». On considère le cas des équations linéaires et celui des équations semi-linéaires. ***Pour citer cet article : V. Barbu et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 311–316.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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L'étude du comportement asymptotique des équations stochastiques s'est revelé un problème difficile. Par exemple, même pour des équations controlées linéaires, on ne connaît pas d'analogue satisfaisant de la caractérisation « déterministe » de Hautus des systèmes stabilisables. Des résultats partiels ont été obtenus via l'équation de Riccati opérationelle, par exemple dans [4,5,8] et [9], mais tous ces résultats abstraits sont difficilement applicables à des équations concrètes, en particulier dans le cas de la dimension infinie.

On considère ici une équation parabolique du type suivant :

$$\begin{cases} \mathrm{d}_t y(t,x)=\Delta_x y(t,x)\,\mathrm{d}t+a(t,x)y(t,x)\,\mathrm{d}t+c(t,x)\cdot\nabla_x y(t,x)\,\mathrm{d}t \\ \qquad\qquad +I_{\mathcal{O}}(x)u(t,x)\,\mathrm{d}t+b(t,x)y(t,x)\,\mathrm{d}w_t, & t\in\mathbb{R}^+,\ x\in\mathcal{D}, \\ y(t,x)=0, & t\in\mathbb{R}^+,\ x\in\partial\mathcal{D}, \\ y(0,x)=y_0(x), & x\in\mathcal{D}, \end{cases} \tag{1}$$

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où y est l'état du système et u est le contrôle. On suppose que :

- $\mathcal{D} \subset \mathbb{R}^d$ est un domaine borné avec frontière régulière et $\mathcal{O} \subset \mathcal{D}$ est un sous-ensemble ouvert ;
- $(\Omega, \mathcal{E}, \mathbb{P})$ est un espace de probabilité muni d'une filtration $\{\mathcal{F}_t : t \geqslant 0\}$ et $\{w_t : t \geqslant 0\}$ est un $\mathcal{F}$-mouvement brownien ;
- a, b, c et div$c$ sont des fonctions continues et bornées sur $\mathbb{R}^+ \times \overline{\mathcal{D}}$;
- la donnée initiale y_0 est dans $\mathrm{L}^2(\mathcal{D})$;
- le contrôle u est un processus adapté à $\mathcal{F}$ appartenant à $\mathrm{L}^2(\Omega \times [0, +\infty[; \mathrm{L}^2(\mathcal{D}))$.

Il est bien connu que, sous ces hypothèses, l'équation (1) admet une unique solution « faible » $y \in \mathrm{L}^2(\Omega; \mathrm{C}([0, +\infty), \mathrm{L}^2(D))) \cap \mathrm{L}^2(\Omega; \mathrm{C}((\delta, +\infty), \mathrm{H}_0^1(D)))$, $\forall \delta > 0$.

On dit que l'équation (1) est *stabilisable à partir de* $\mathcal{O}$ s'il existe $M \geqslant 1$ et $\nu > 0$ tels que pour toute donnée initiale $y_0 \in \mathrm{L}^2(\mathcal{D})$ il existe un contrôle adapté $u \in \mathrm{L}^2(\Omega \times [0, +\infty); \mathrm{L}^2(\mathcal{D}))$ tel que

$$\mathbb{E} \int_{\mathcal{D}} y^2(t, x)\,\mathrm{d}x\,\mathrm{d}s \leqslant M\,\mathrm{e}^{-\nu t} \int_{\mathcal{D}} y_0^2(x)\,\mathrm{d}x, \quad \forall t \geqslant 0.$$

Le théorème suivant est le résultat principal de cette note.

THÉORÈME. – *Soit $\lambda_{\mathcal{O}}$ la première valeur propre de l'opérateur $A_{\mathcal{O}} = -\Delta_x$ dans $\mathcal{D}\backslash\mathcal{O}$ avec conditions de Dirichlet homogènes au bord, plus precisément* :

$$D(A_{\mathcal{O}}) = \mathrm{H}_0^1\big(\mathcal{D}\backslash\overline{\mathcal{O}}\big) \cap \mathrm{H}^2\big(\mathcal{D}\backslash\overline{\mathcal{O}}\big) \subset \mathrm{L}^2\big(\mathcal{D}\backslash\overline{\mathcal{O}}\big) : \quad A_{\mathcal{O}}\phi = -\Delta_x\phi, \quad \forall \phi \in D(A_{\mathcal{O}})$$

et : $\mathbf{b} = \sup\{\frac{1}{2}b^2(t, x) + a(t, x) - \frac{1}{2}\operatorname{div} c(x) : x \in \mathcal{D},\ t \in \mathbb{R}^+\}$. *Si $\lambda_{\mathcal{O}} > \mathbf{b}$, alors* (1) *est stabilisable à partir de $\mathcal{O}$.* □

On remarque que si $\mathcal{O}$ se rapproche de $\mathcal{D}$, alors $\lambda_{\mathcal{O}}$ tend vers $+\infty$. Par conséquent, pour toutes fonctions a, b et c l'équation (1) est stabilisable à partir de $\mathcal{O}$ si $\mathcal{D}\backslash\mathcal{O}$ est « assez petit ».

Pour terminer, par un argument de comparaison, on obtient le même type de résultat pour l'équation semi-linéaire

$$\begin{cases} \mathrm{d}_t y(t, x) = \Delta_x y(t, x)\,\mathrm{d}t + c(t, x) \cdot \nabla_x y(t, x)\,\mathrm{d}t + f(x, y(t, x))y(t, x)\,\mathrm{d}t & \\ \qquad + g(y(t, x))\,\mathrm{d}t + I_{\mathcal{O}}(x)u(t, x)\,\mathrm{d}t + b(t, x)y(t, x)\,\mathrm{d}w_t, & x \in \mathcal{D}, \\ y(\sigma, x) = 0, & x \in \partial\mathcal{D}, \\ y(0, x) = y_0(x), & x \in \mathcal{D}. \end{cases} \tag{2}$$

1. Introduction

The study of asymptotic behavior of stochastic controlled equations has turned out to be a difficult problem (in the linear case as well). For instance, even for linear finite dimensional stochastic equations, no satisfactory analogue of the 'deterministic' Hautus spectral characterization of stabilizable systems is known so far. Partial results can be obtained via the Riccati operator equation (*see* [4,5,8] and [9]) but all such results are difficult to apply to specific controlled equations especially in the infinite dimensional case.

Here we are concerned with the stochastic parabolic equation (1) on a bounded domain $\mathcal{D}$ of $\mathbb{R}^d$ and we give a sufficient condition under which such equation can be stabilized acting on a subdomain $\mathcal{O} \subset \mathcal{D}$. Such a condition is given in terms of the norm of the noise coefficient (the supremum of function b in (1)) and of the first eigenvalue of the operator $(\phi \to \Delta\phi - a\phi)$ with Dirichlet boundary conditions on $\mathcal{D}\backslash\mathcal{O}$. Roughly speaking this spectral condition measures how 'thin' $\mathcal{D}\backslash\mathcal{O}$ is. Finally, for the related problem of the null controllability of Eq. (1), we refer the reader to [1,2] and [7].

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2. Setting of the problem and main result

We consider the following controlled stochastic parabolic equation

$$\begin{cases} \mathrm{d}_t y(t,x) = \Delta_x y(t,x)\,\mathrm{d}t + a(t,x)y(t,x)\,\mathrm{d}t + c(t,x)\cdot\nabla_x y(t,x)\,\mathrm{d}t & \\ \qquad\qquad + I_{\mathcal{O}}(x)u(t,x)\,\mathrm{d}t + b(t,x)y(t,x)\,\mathrm{d}w_t, & t\in\mathbb{R}^+,\ x\in\mathcal{D}, \\ y(t,x)=0, & t\in\mathbb{R}^+,\ x\in\partial\mathcal{D}, \\ y(0,x)=y_0(x), & x\in\mathcal{D}, \end{cases} \tag{1}$$

where y is the state of the system, u is the controller and $I_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}\subset\mathcal{D}$.

The following hypotheses will be assumed.

HYPOTHESIS 1. –

(1) $(\Omega,\mathcal{E},\mathbb{P})$ *is a probability space with filtration* $\{\mathcal{F}_t : t\geqslant 0\}$. *Moreover* $\{w_t : t\geqslant 0\}$ *is a standard* (*real valued*) $\mathcal{F}$*-Brownian motion.*

(2) $\mathcal{D}$ *is a bounded open domain in* $\mathbb{R}^d$ *with smooth boundary and* $\mathcal{O}$ *is an open subset of* $\mathcal{D}$.

(3) $a, b\in \mathrm{C}(\mathbb{R}^+\times\overline{\mathcal{D}};\mathbb{R})$ *are bounded. Moreover,* $c\in\mathrm{C}(\mathbb{R}^+\times\overline{\mathcal{D}};\mathbb{R}^d)$ *and is differentiable with respect to* x *with continuous and bounded derivative* c_x.

(4) *The initial datum* y_0 *belongs to* $\mathrm{L}^2(\mathcal{D})$.

(5) *The control* u *is an* $\mathcal{F}$*-adapted process belonging to* $\mathrm{L}^2(\Omega\times[0,+\infty[;\mathrm{L}^2(\mathcal{D}))$.

Under the above assumptions it is well known that equation (1) has a unique adapted 'mild' solution $y\in\mathrm{L}^2(\Omega;\mathrm{C}([0,+\infty),\mathrm{L}^2(D)))\cap\mathrm{L}^2(\Omega;\mathrm{C}((\delta,+\infty),\mathrm{H}^1_0(D)))$, $\forall\delta>0$.

DEFINITION 2. – Let $\mathcal{O}\subset\mathcal{D}$ be an open subset. We say that equation (1) is stabilizable from $\mathcal{O}$ if there exist $M\geqslant 1$ and $\nu>0$ such that for all initial datum $y_0\in\mathrm{L}^2(\mathcal{D})$ there exists an adapted control $u\in\mathrm{L}^2(\Omega\times[0,+\infty);\mathrm{L}^2(\mathcal{D}))$ such that

$$\mathbb{E}\int_{\mathcal{D}} y^2(t,x)\,\mathrm{d}x\,\mathrm{d}s \leqslant M\,\mathrm{e}^{-\nu t}\int_{\mathcal{D}} y_0^2(x)\,\mathrm{d}x,\quad \forall t\geqslant 0. \tag{2}$$

THEOREM 3 (Main result). – *Let* $\lambda_{\mathcal{O}}$ *be the first eigenvalue of* $A_{\mathcal{O}}=-\Delta_x$ *on* $\mathcal{D}\backslash\mathcal{O}$ *with Dirichlet boundary conditions. Namely*:

$$D(A_{\mathcal{O}})=\mathrm{H}^1_0\big(\mathcal{D}\backslash\overline{\mathcal{O}}\big)\cap\mathrm{H}^2\big(\mathcal{D}\backslash\overline{\mathcal{O}}\big);\quad A_{\mathcal{O}}\phi=-\Delta_x\phi,\quad\forall\phi\in D(A_{\mathcal{O}})$$

and let: $\mathbf{b}=\sup\{\frac12 b^2(t,x)+a(t,x)-\frac12\operatorname{div}c(x) : x\in\mathcal{D},\ t\in\mathbb{R}^+\}$. *If* $\lambda_{\mathcal{O}}>\mathbf{b}$ *then* (1) *is stabilizable from* $\mathcal{O}$.

The proof will be sketched in the next section.

Remark 4. – It is easily seen that if $\partial\mathcal{O}$ is regular then

$$\lambda_{\mathcal{O}}\geqslant \mathrm{const.d}_{\mathrm{H}}^{-2},\quad\text{where}\quad \mathrm{d}_{\mathrm{H}}=\sup\big\{\operatorname{dist}\big(x,\partial(\mathcal{D}\backslash\mathcal{O})\big) : x\in\mathcal{D}\backslash\mathcal{O}\big\}.$$

Thus the above result implies that (1) is stabilizable from $\mathcal{O}$ for every a, b and c if $\mathcal{D}\backslash\mathcal{O}$ is 'thin' enough.

COROLLARY 5. – *Assume that* $c=0$ *and let* $\mathbf{b}_1=\sup\{\frac12 b^2(t,x)+a(t,x) : x\in\mathcal{D}\backslash\mathcal{O},\ t\in\mathbb{R}^+\}$. *If* $\lambda_{\mathcal{O}}>\mathbf{b}_1$ *then* (1) *is stabilizable from* $\mathcal{O}'$ *for all open sets* $\mathcal{O}'$ *with* $\overline{\mathcal{O}}\subset\mathcal{O}'\subset\mathcal{D}$.

Proof. – We choose a second open set $\mathcal{O}''$ with $\overline{\mathcal{O}}\subset\mathcal{O}''$ and $\overline{\mathcal{O}''}\subset\mathcal{O}'$ and fix two functions ℓ, m in $\mathrm{C}^2(\mathcal{D};[0,1])$ such that $\ell=1$ on $\mathcal{O}''$, $\ell=0$ on $\mathcal{D}\backslash\mathcal{O}'$, $m=1$ on $\mathcal{D}\backslash\mathcal{O}''$ and $m=0$ on $\mathcal{O}$. Since

$$\sup_{x\in\mathcal{D},\ t\geqslant 0}\left\{\frac12 m^2(x)b^2(t,x)+m(x)a(t,x)\right\}\leqslant\sup_{x\in\mathcal{D}\backslash\mathcal{O},\ t\geqslant 0}\left\{\frac12 b^2(t,x)+a(t,x)\right\}\vee 0$$

and $\lambda_{\mathcal{O}} > 0$. Theorem 3 implies that, if we replace functions a and b by ma and mb respectively, then equation (1) becomes stabilizable from $\mathcal{O}$. Moreover since $\ell = 1$ on $\mathcal{O}$ the controlled equation

$$\begin{cases} \mathrm{d}_t y(t,x) = \Delta_x y(t,x)\,\mathrm{d}t + m(x)a(t,x)y(t,x)\,\mathrm{d}t \\ \qquad\qquad + \ell(x)u(t,x)\,\mathrm{d}t + m(x)b(t,x)y(t,x)\,\mathrm{d}w_t, & t \in \mathbb{R}^+,\ x \in \mathcal{D}, \\ y(t,x) = 0, & t \in \mathbb{R}^+,\ x \in \partial\mathcal{D}, \\ y(0,x) = y_0(x), & x \in \mathcal{D}, \end{cases} \tag{3}$$

is stabilizable (the definition of stabilizability of (3) is totally analogous to Definition 2). Since $1 - m \neq 0$ only on $\mathcal{O}''$ where $\ell = 1$ Example 2.2 in [8] yields that the stabilizability of (3) is not affected if we replace function mb by b. For the same reason it can be easily seen that the stabilizability of (3) is not affected if we replace function ma by a. Finally, being $\ell = 0$ on $\mathcal{D}\backslash\mathcal{O}'$, the stabilizability of (3) with a and b instead of ma and mb implies the stabilizability of equation (1) from $\mathcal{O}'$. □

3. Proof of the main result

The proof is based on the following lemma

LEMMA 6. – *For all* $\varepsilon > 0$ *there exists* $k_\varepsilon > 0$ *such that* $\forall \phi \in \mathrm{H}_0^1(\mathcal{D})$, $\forall k \geqslant k_\varepsilon$

$$\int_{\mathcal{D}} \left|\nabla_x \phi(x)\right|^2 \mathrm{d}x + k \int_{\mathcal{O}} \phi^2(x)\,\mathrm{d}x \geqslant (\lambda_{\mathcal{O}} - \varepsilon)|\phi|^2_{\mathrm{L}^2(\mathcal{D})}.$$

Proof. – Let λ_k be the first eigenvalue of the operator A_k defined by

$$D(A_k) = \mathrm{H}_0^1(\mathcal{D}) \cap \mathrm{H}^2(\mathcal{D}); \quad A_k\phi = -\Delta_x\phi + kI_{\mathcal{O}}\phi.$$

As it is well known

$$\lambda_k = \inf\left\{ \int_{\mathcal{D}} \left[|\nabla_x\phi|^2 + kI_{\mathcal{O}}\phi^2\right]\mathrm{d}x : \phi \in \mathrm{H}_0^1(\mathcal{D}),\ |\phi|_{\mathrm{L}^2(\mathcal{D})} = 1 \right\}.$$

Clearly $\lambda_k \leqslant \lambda_{\mathcal{O}}$. Therefore, if ϕ_k is the corresponding eigenfunction we have

$$\int_{\mathcal{D}} |\nabla_x\phi_k|^2\,\mathrm{d}x + k\int_{\mathcal{O}} |\phi_k|^2\,\mathrm{d}x \leqslant \lambda_{\mathcal{O}}.$$

Thus (passing to a subsequence if necessary) there exist $\phi_\infty \in \mathrm{H}_0^1(\mathcal{D})$ and $\lambda_\infty \in \mathbb{R}$ such that, as $k \nearrow +\infty$:

$$\phi_k \to \phi_\infty \quad \text{strongly in } \mathrm{L}^2(\mathcal{D}); \quad \phi_k \rightharpoonup \phi_\infty \quad \text{weakly in } \mathrm{H}_0^1(\mathcal{D}); \quad \lambda_k \nearrow \lambda_\infty.$$

Moreover $\phi_\infty \equiv 0$ on $\mathcal{O}$ and ϕ_∞ solves (in $\mathrm{H}_0^1(\mathcal{D})$) the equation:

$$-\Delta_x\phi_\infty = \lambda_\infty\phi_\infty.$$

Therefore $\lim_{k\to\infty}\lambda_k = \lambda_\infty = \lambda_{\mathcal{O}}$ and the claim follows. □

Now we can conclude the proof of Theorem 3. We fix $\varepsilon > 0$ such that $\lambda_{\mathcal{O}} - \mathbf{b} \geqslant 4\varepsilon$ and choose $k > k_\varepsilon$. Next we consider the feedback $u = -ky$ and the corresponding closed loop equation

$$\begin{cases} \mathrm{d}_t y(t,x) = \Delta_x y(t,x)\,\mathrm{d}t + a(t,x)y(t,x)\,\mathrm{d}t + c(t,x)\cdot\nabla_x y(t,x)\,\mathrm{d}t \\ \qquad\qquad - kI_{\mathcal{O}}(x)y(t,x)\,\mathrm{d}t + b(t,x)y(t,x)\,\mathrm{d}w_t, & t \in \mathbb{R}^+,\ x \in \mathcal{D}, \\ y(t,x) = 0, & t \in \mathbb{R}^+,\ x \in \partial\mathcal{D}, \\ y(0,x) = y_0(x), & x \in \mathcal{D}. \end{cases} \tag{4}$$

Finally we define $\hat{y}(t) = \exp(\varepsilon t) y(t)$. Fix an arbitrary $\tau > 0$, if we compute by Itô's rule $\mathrm{d}_t |\hat{y}(t)|^2_{\mathrm{L}^2(\mathcal{D})}$ and integrate in $[0, \tau]$ we get (noticing that $\hat{y}$ solves (4) with a replaced by $a + \varepsilon$):

$$\mathbb{E}\int_{\mathcal{D}} \hat{y}^2(\tau, x)\,\mathrm{d}x = \int_{\mathcal{D}} \hat{y}^2(x)\,\mathrm{d}x + \mathbb{E}\int_0^\tau\int_{\mathcal{D}} \left[b^2(\sigma, x) + 2a(\sigma, x) + 2\varepsilon - \operatorname{div} c(\sigma, x)\right]\hat{y}^2(\sigma, x)\,\mathrm{d}x\,\mathrm{d}\sigma$$
$$- 2\mathbb{E}\int_0^\tau\int_{\mathcal{D}} \left[\left|\nabla_x \hat{y}(\sigma, x)\right|^2 + kI_{\mathcal{O}}(x)\hat{y}^2(\sigma, x)\right]\mathrm{d}x\,\mathrm{d}\sigma$$

and by Lemma 6

$$\mathbb{E}\int_{\mathcal{D}} \hat{y}^2(\tau, x)\,\mathrm{d}x \leqslant \int_{\mathcal{D}} y_0^2(x)\,\mathrm{d}x.$$

Thus $\mathbb{E}|\hat{y}(\tau, \cdot)|^2_{\mathrm{L}^2(\mathcal{D})}$ is uniformly bounded. The claim follows since $y(t) = \exp(-\varepsilon t)\hat{y}(t)$. □

4. Semilinear case

By monotonicity arguments we can easily extend the main result to semilinear state equations. Namely, consider the controlled equation

$$\begin{cases} \mathrm{d}_t y(t, x) = \Delta_x y(t, x)\,\mathrm{d}t + c(t, x)\cdot\nabla_x y(t, x)\,\mathrm{d}t + f(t, x, y(t, x))y(t, x)\,\mathrm{d}t \\ \qquad + g(y(t, x))\,\mathrm{d}t + I_{\mathcal{O}}(x)u(t, x)\,\mathrm{d}t + b(t, x)y(t, x)\,\mathrm{d}w_t, & t \in \mathbb{R}^+,\ x \in \mathcal{D}, \\ y(t, x) = 0, & t \in \mathbb{R}^+,\ x \in \partial\mathcal{D}, \\ y(0, x) = y_0(x), & x \in \mathcal{D}, \end{cases} \tag{5}$$

where $\mathcal{D}$, $\mathcal{O}$, b, $\{w_t : t \geqslant 0\}$, y_0 and u verify Hypothesis 1. Moreover, f and g are assumed to verify the following conditions

(1) $f : \mathbb{R}^+ \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is continuous. For all $x \in \mathcal{D}$ and $t \in \mathbb{R}^+$ the map $y \to f(t, x, y)y$ is monotone decreasing and there exists a bounded function $a \in \mathrm{C}(\mathbb{R}^+ \times \overline{\mathcal{D}}; \mathbb{R})$ such that $f(t, x, y) \leqslant a(t, x)$ for all $t \in \mathbb{R}^+$, $x \in \mathcal{D}$, and $y \in \mathbb{R}$.

(2) $g : \mathbb{R} \to \mathbb{R}$ is continuous, monotone decreasing and $g(0) = 0$.

It is well known, see [6], that under the above assumptions (5) admits a unique mild solution $y \in \mathrm{L}^2(\Omega; \mathrm{C}([0, +\infty), \mathrm{L}^2(D))) \cap \mathrm{L}^2(\Omega; \mathrm{C}((\delta, +\infty), \mathrm{H}^1_0(D)))$, $\forall\delta > 0$. The following is the semilinear version of Theorem 3

THEOREM 7. – *Let* $\mathbf{b}$ *and* $\lambda_{\mathcal{O}}$ *be as in Theorem* 3. *If* $\lambda_{\mathcal{O}} > \mathbf{b}$ *then equation* (5) *is stabilizable from* $\mathcal{O}$.

Proof. – Choose k as in the proof of Theorem 3 and let y be the solution to the equation:

$$\begin{cases} \mathrm{d}_t y(t, x) = \Delta_x y(t, x)\,\mathrm{d}t + c(t, x)\cdot\nabla_x y(t, x)\,\mathrm{d}t + f(t, x, y(t, x))y(t, x)\,\mathrm{d}t \\ \qquad + g(x, y(t, x))\,\mathrm{d}t - kI_{\mathcal{O}}(x)y(t, x)\,\mathrm{d}t + b(t, x)y(t, x)\,\mathrm{d}w_t, & t \in \mathbb{R}^+,\ x \in \mathcal{D}, \\ y(t, x) = 0, & t \in \mathbb{R}^+,\ x \in \partial\mathcal{D}, \\ y(0, x) = y_0(x), & x \in \mathcal{D}. \end{cases}$$

Moreover, let y^+ be the solution of

$$\begin{cases} \mathrm{d}_t y(t, x) = \Delta_x y(t, x)\,\mathrm{d}t + a(t, x)y(t, x)\,\mathrm{d}t + c(t, x)\cdot\nabla_x y(t, x)\,\mathrm{d}t \\ \qquad - kI_{\mathcal{O}}(x)y(t, x)\,\mathrm{d}t + b(t, x)y(t, x)\,\mathrm{d}w_t, & t \in \mathbb{R}^+,\ x \in \mathcal{D}, \\ y(t, x) = 0, & t \in \mathbb{R}^+,\ x \in \partial\mathcal{D}, \\ y(0, x) = y_0(x) \vee 0, & x \in \mathcal{D} \end{cases}$$

(respectively, let y^- be the solution of the above equation with initial datum $y^-(0, x) = y_0(x) \wedge 0$). By well known comparison results, see for instance [6] or [3], $y^+(t, x) \geqslant 0$ $\mathbb{P}$-a.s. for a.e. $x \in \mathcal{D}$, $y^-(t, x) \leqslant 0$ $\mathbb{P}$-a.s.

for a.e. $x \in \mathcal{D}$ and

$$y^-(t,x) \leqslant y(t,x) \leqslant y^+(t,x) \quad \mathbb{P}\text{-a.s. for a.e. } x \in \mathcal{D}.$$

By Theorem 3, $\mathbb{E}|y^+(t,\cdot)|^2_{L^2(\mathcal{D})} \leqslant M\,\mathrm{e}^{-\nu t}\mathbb{E}|y_0|^2_{L^2(\mathcal{D})}$ for a suitable $M \geqslant 1$ and $\nu > 0$ (similarly for y^-). This completes the proof. □

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