C. R. Acad. Sci. Paris, Ser. I 334 (2002) 629–634

Algèbre/*Algebra*
(Combinatoire/*Combinatorics*)

Invertible substitutions and local isomorphisms *

Zhi-Xiong Wen [a], Zhi-Ying Wen [b], Jun Wu [a]

[a] Nonlinear Science Center, Department of Mathematics, Wuhan University, Wuhan 430072, China
[b] Department of Mathematics, Tsinghua University, Beijing 10084, China

Received 24 October 2001; accepted 3 December 2001

Note presented by Jean-Pierre Kahane.

Abstract Let φ_1 and φ_2 be two primitive invertible substitutions over a two-letter alphabet. Let ξ_{φ_1} and ξ_{φ_2} be fixed points of φ_1 and φ_2, respectively. We show that ξ_{φ_1} and ξ_{φ_2} are locally isomorphic if and only if there exists a primitive invertible substitution φ_0 and two positive integers m and n such that $M_{\varphi_1} = M_{\varphi_0}^m$ and $M_{\varphi_2} = M_{\varphi_0}^n$, where M_φ is the substitutive matrix of the substitution φ. ***To cite this article: Z.-X. Wen et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 629–634.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Substitutions inversibles et isomorphismes locaux

Résumé Soient φ_1 et φ_2 deux substitutions primitives inversibles sur un alphabet de deux lettres. Soit ξ_{φ_1} (resp. ξ_{φ_2}) un point fixe de φ_1 (resp. φ_2). Nous montrons que ξ_{φ_1} et ξ_{φ_2} sont localement isomorphes si et seulement s'il existe une substitution primitive inversible φ_0 et deux entiers positifs m et n tels que $M_{\varphi_1} = M_{\varphi_0}^m$ et $M_{\varphi_2} = M_{\varphi_0}^n$, où M_φ est la matrice de la substitution φ. ***Pour citer cet article : Z.-X. Wen et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 629–634.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit $S = \{a, b\}$ un alphabet de deux lettres. Nous désignons par S^* (resp. par F_2) le monoïde libre (resp. le groupe libre) engendré par S. Posons $S^+ = S^* \setminus \{\varepsilon\}$ où ε est le mot vide et notons S^ω l'ensemble des mots infinis sur S.

Si $w \in S^*$, nous désignons par $|w|$ la *longueur* de w et par $|w|_a$ (resp. $|w|_b$) le nombre de fois que la lettre a (resp. b) figure dans le mot w.

Un mot $v \in S^*$ est dit *facteur* du mot w (et l'on note $v \prec w$) s'il existe $u, u' \in S^*$ tels que $w = uvu'$. Dans le cas où $u = \varepsilon$ (resp. $u' = \varepsilon$), nous disons que v est un *préfixe* (resp. *suffixe*) de w. Les notions de facteur et de préfixe gardent un sens même si w est un mot infini.

Si $\xi \in S^\omega$, nous désignons par $\Omega_n(\xi)$ l'ensemble des facteurs de longueur n de ξ et par $\Omega(\xi)$ la réunion des $\Omega_n(\xi)$.

E-mail addresses: zhxwen@whu.edu.cn (Z.-X. Wen); wenzy@tsinghua.edu.cn (Z.-Y. Wen); wujunyu@public.wh.hb.cn (J. Wu).

© 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés
S1631-073X(02)02234-3/FLA

Soit $w = x_1x_2 \cdots x_n \in S^*$ avec $x_i \in S$. Le mot $\overline{w} = x_n \cdots x_2x_1$ est appelé *mot miroir* de w.

Une *substitution* φ sur S est un endomorphisme de S^*. Dans la suite, nous identifierons la substitution φ au couple $(\varphi(a), \varphi(b))$ et nous supposerons toujours $\varphi(a)$ et $\varphi(b)$ différents de ε. Une substitution sur S agit également sur S^ω (par concaténation des images des lettres composant un mot infini).

S'il existe $c \in S$ tel que c soit un préfixe de $\varphi(c)$ et tel que $|\varphi(c)| \geqslant 2$, alors la suite des mots finis $(\varphi^n(c))_{n\geqslant 1}$ converge vers un point fixe $\varphi^\omega(c) \in S^\omega$ de φ. Nous désignerons par $\xi_{\varphi,a}$ (resp. par $\xi_{\varphi,b}$) le point fixe qui débute par a (resp. par b), si un tel point existe. Dans le cas où il n'y pas de confusion possible, nous désignons tout simplement par ξ_φ n'importe quel point fixe de φ.

À chaque substitution φ sur S, on associe la matrice M_φ indexée par $S \times S$ dont le coefficient correspondant à (x, y) est le nombre de fois que la lettre x figure dans $\varphi(y)$. Si M_φ est primitive, la substitution φ est également dite *primitive*.

Une substitution φ est dite inversible si elle définit un automorphisme de F_2. L'ensemble des substitutions inversibles sur S est noté IS(S^*).

Soient u et v deux mots infinis sur S. Nous dirons qu'ils sont *localement isomorphes* (ce que l'on notera $u \simeq v$) si tout facteur de l'un, ou son mot miroir, est un facteur de l'autre.

THÉORÈME 1. – *Soient φ_1 et φ_2 deux substitutions primitives inversibles ayant ξ_{φ_1} et ξ_{φ_2} pour points fixes respectifs. Alors les deux assertions ci-dessous sont équivalentes* :

(1) *ξ_{φ_1} et ξ_{φ_2} sont localement isomor phes*;
(2) *il existe une substitution primitive inversible φ_0 et deux entiers positifs m, n tels que $M_{\varphi_1} = M_{\varphi_0}^m$ et $M_{\varphi_2} = M_{\varphi_0}^n$.*

1. Introduction

Given two infinite words which are fixed points of invertible primitive substitutions, it is interesting to ask whether they are locally isomorphic. This problem appears in several quite different contexts such as discrete dynamical system [9], one-dimensional quasicrystal chain [7] and computational complexity [4], etc.... The following theorem gives a complete answer to the above problem and generalizes the results of Wen and Wen [10,11]. We can also compare this result with those of Cobham [2], Fagnot [5] and Durand [3].

THEOREM 1. – *Let φ_1 and φ_2 be two invertible primitive substitutions having ξ_{φ_1} and ξ_{φ_2} as fixed points. Then the following assertions are equivalent*:

(1) *ξ_{φ_1} and ξ_{φ_2} are locally isomorphic*;
(2) *there exists a primitive invertible substitution φ_0 and positive integers m and n such that $M_{\varphi_1} = M_{\varphi_0}^m$, $M_{\varphi_2} = M_{\varphi_0}^n$.*

2. Definitions and notations

Let $S = \{a, b\}$ be a two-letter alphabet. We denote by S^* (resp. by F_2) the free monoid (resp. the free group) generated by S with the empty word ε as the identity element. Let S^ω be the set of infinite words over S and $S^+ = S^* \setminus \{\varepsilon\}$.

If $w \in S^*$ is a word, $|w|$ denotes its *length* and $|w|_a$ (resp. $|w|_b$) the number of times the letter a (resp. b) appears in the word w.

A word v is called a *factor* of w (and this is denoted $v \prec w$) if there exist u and u' in S^* such that $w = uvu'$. When $u = \varepsilon$ (resp. $u' = \varepsilon$), v is called a *prefix* (resp. *suffix*) of w.

Let $w = x_1x_2 \cdots x_n \in S^*$ with $x_i \in S$. The word $\overline{w} = x_n \cdots x_2x_1$ is called the *mirror word* $\overline{w}$ of w.

A *substitution* over S is an endomorphism of S^*. We identify a substitution φ with the couple $(\varphi(a), \varphi(b))$, and always suppose that $\varphi(a)$ and $\varphi(b)$ are both different from ε. Clearly φ can be extended uniquely as an endomorphism of F_2.

A substitution φ over S also acts on S^ω: if $w = x_1x_2x_3\cdots \in S^\omega$, then $\varphi(w)$ is the infinite concatenation of the words $\varphi(x_j)$.

If for some $c \in S$, the word $\varphi(c)$ begins with c and has length at least 2, then the sequence of $(\varphi^n(c))_{n\geqslant 1}$ converges to a fixed point $\varphi^\omega(c) \in S^\omega$ of φ. When no confusion can occur, any fixed point of φ will be denoted by ξ_φ.

One associates with a substitution φ over S a $S \times S$-matrix M_φ: its entry corresponding to x, y is the number of times that x appears in $\varphi(y)$. When M_φ is primitive (i.e., one of its power has only positive coefficients), the substitution φ is also said to be *primitive*. Clearly if $\varphi = \varphi_1\varphi_2$, then we have $M_\varphi = M_{\varphi_1}M_{\varphi_2}$.

A substitution on S which extends as an automorphism of F_2 is called an *invertible substitution*. The set of invertible substitutions over S (denoted by $\mathrm{IS}(S^*)$) is a submonoid of the group of automorphisms of F_2, and has a very simple structure (*see* [10]).

THEOREM 2. – *The monoid* $\mathrm{IS}(S^*)$ *is generated by the invertible substitutions* $\sigma = (ab,\ a)$, $\tau = (ba,\ a)$ *and* $\pi = (b,\ a)$.

Two infinite words $u, v \in S^\omega$ are said *locally isomorphic* (and then one writes $u \simeq v$) if, for any factor w of u, the word w or its mirror word $\overline{w}$ is a factor of v and vice versa. For any $\Omega \subset S^*$, define $\overline{\Omega} := \{\overline{w} : w \in \Omega\}$. By [12], for every $\varphi \in \mathrm{IS(S^*)}$, one has $\Omega(\xi_\varphi) = \overline{\Omega(\xi_\varphi)}$, where $\Omega(\xi)$ stands for the set of factors of the infinite word ξ. So for any $\varphi_1, \varphi_2 \in \mathrm{IS(S^*)}$, one has $\xi_{\varphi_1} \simeq \xi_{\varphi_2}$ if and only if $\Omega(\xi_{\varphi_1}) = \Omega(\xi_{\varphi_2})$.

An infinite word $\xi \in S^\omega$ is called *Sturmian* if $\#\Omega_n(\xi) = n + 1$ for any $n \geqslant 1$ where $\Omega_n(\xi)$ denotes the set of factors of length n of ξ. We can give a geometrical interpretation of Sturmian words. Consider the square grid consisting of all vertical and horizontal lines through integer points in the first quadrant. Let α be irrational in $(0, \infty)$ and let β be real. Label the intersections of the line $y = \alpha x + \beta$ with the above grid, using a if a vertical line is crossed, and b otherwise. The sequence of labels obtained in this way is called the *cutting word* associated with the line $y = \alpha x + \beta$ which we denote by $\mathbf{c}_{\alpha,\beta}$. In the case $\beta = 0$, the cutting sequence $\mathbf{c}_{\alpha,0}$ is called a *characteristic word*.

The following result tells us that Sturmian words and cutting words are identical and presents in the meantime some interesting properties of them (*see* [1,8] and [10]).

PROPOSITION 1. – *With above definitions and notations, we have*:

(1) *A word is Sturmian if and only if it is a cutting word*;

(2) *We have* $\mathbf{c}_{\alpha_1,\beta_1} \simeq \mathbf{c}_{\alpha_2,\beta_2}$ *if and only* $\alpha_1 = \alpha_2$. *So, as far as factors are concerned, it is sufficient to consider* $\beta = 0$;

(3) *A Sturmian word* ξ *is characteristic if and only if both* $a\xi$ *and* $b\xi$ *are Sturmian*;

(4) *Let* φ *be a primitive substitution over* S, *then* ξ_φ *is a Sturmian word if and only if* φ *is invertible.*

3. Proof of the main result

Let $\mathcal{G}$ be the monoid generated by the two invertible substitutions σ and π. Set $\mathcal{G}' = \mathcal{G} \setminus \{\pi, \rho\}$ with $\rho = (a, b)$. Then it is easily seen that each $\varphi \in \mathcal{G}'$ has at most one fixed point.

PROPOSITION 2. – *If* $\varphi \in \mathrm{IS}(S^*)$ *is primitive and has* ξ_φ *as its fixed point, then* ξ_φ *is characteristic if and only if* $\varphi \in \mathcal{G}$.

Proof. – If $\varphi \in \mathcal{G}$ is primitive, then we have $\varphi \in \mathcal{G}'$ and it must take one of the four forms with $n_j \geqslant 1$ $(1 \leqslant j \leqslant k)$:

$$\text{(i) } \sigma^{n_1}\pi\sigma^{n_2}\pi\cdots\sigma^{n_{k-1}}\pi\sigma^{n_k}, \qquad \text{(ii) } \sigma^{n_1}\pi\sigma^{n_2}\pi\cdots\sigma^{n_k}\pi,$$

$$\text{(iii) } \pi\sigma^{n_1}\pi\sigma^{n_2}\pi\cdots\sigma^{n_{k-1}}\pi\sigma^{n_k}, \qquad \text{(iv) } \pi\sigma^{n_1}\pi\sigma^{n_2}\pi\cdots\sigma^{n_{k-1}}\pi\sigma^{n_k}\pi.$$

For the cases (i) and (ii), we have a^2 and $ba \prec \xi_\varphi$. Moreover the last letter of $\varphi^n(a)$ and the last one of $\varphi^n(b)$ are distinct. Hence $\varphi^n(a^2) = \varphi^n(a)\varphi^n(a) \prec \xi_\varphi$ and $\varphi^n(ba) = \varphi^n(b)\varphi^n(a) \prec \xi_\varphi$ for any $n \geqslant 1$. Thus $a\varphi^n(a) \prec \xi_\varphi$ and $b\varphi^n(a) \prec \xi_\varphi$, which yields $a\xi_\varphi \simeq \xi_\varphi$ and $b\xi_\varphi \simeq \xi_\varphi$. Therefore $a\xi_\varphi$ and $b\xi_\varphi$ are both Sturmian and, by Proposition 1, ξ_φ is characteristic. Cases (iii) and (iv) can be treated similarly by replacing a^2 and ba respectively by b^2 and ab.

Conversely suppose $\varphi \in \mathrm{IS}(S^*) \setminus \mathcal{G}$. Decompose $\varphi = \varphi_1\varphi_2\cdots\varphi_n$ with $\varphi_j \in \{\sigma, \tau, \pi\}$. Then there exists at least one index j such that $\varphi_j = \tau$. As a result, the last letter of $\varphi(a)$ is the same as the last one of $\varphi(b)$. Define $\varphi' = \varphi'_1\varphi'_2\cdots\varphi'_n$ where $\varphi'_j = \sigma$ if $\varphi_j = \tau$ and $\varphi'_j = \varphi_j$ otherwise. Since $M_\sigma = M_\tau$, so $M_\varphi = M_{\varphi'}$ and $\xi_\varphi \simeq \xi_{\varphi'}$ by Theorem 2 in [10].

If ξ_φ is characteristic, then $\xi_{\varphi'} = \xi_\varphi$ by Proposition 1, for $\xi_{\varphi'}$ is also characteristic. However the last letter of $\varphi'(a)$ differs from the last one of $\varphi'(b)$ but the last letter of $\varphi(a)$ agree with the last one of $\varphi(b)$. Remark also that $|\varphi'(a)| = |\varphi(a)|$ and $|\varphi'(b)| = |\varphi(b)|$. Therefore $\xi_\varphi \neq \xi_{\varphi'}$. Then ξ_φ is not characteristic.

The following corollary is a direct consequence of Propositions 1 and 2.

COROLLARY 1. – *Let $\varphi_1, \varphi_2 \in \mathcal{G}$ be primitive. Then $\xi_{\varphi_1} \simeq \xi_{\varphi_2}$ if and only if $\xi_{\varphi_1} = \xi_{\varphi_2}$.*

The following result is well known (*see* Proposition 1.3.1 in [6], p. 7).

LEMMA 1. – *Let $u, v \in S^+$. If there exist integers $k, l \geqslant 1$ such that $u^k = v^l$, then we can find $w \in S^+$ and integers $m, n \geqslant 1$ such that $u = w^m$ and $v = w^n$.*

Let $u, v \in S^*$ such that $2 < |u| \leqslant |v|$. We call u a *weak prefix* of v if $u = u_1\alpha\beta$ where $\alpha, \beta \in S$ and $u_1 \in S^*$ is a prefix of v. Clearly u is a weak prefix of v if and only if $\pi(u)$ is a weak prefix of $\pi(v)$. Moreover if u is a prefix of v, then it is also a weak prefix of v.

LEMMA 2. – *Let $u, v \in S^*$ such that $2 < |u| \leqslant |v|$ and b^3 is not a suffix of u. If u is not a weak prefix of v, then $\sigma(u)$ is not a weak prefix of $\sigma(v)$.*

Proof. – Write $u = u_1u_2\cdots u_{|u|}$ and $v = v_1v_2\cdots v_{|v|}$ where, $u_i, v_j \in S$. Let j be the least integer i such that $1 \leqslant i \leqslant |u|$ and $u_i \neq v_i$. Because u is not a weak prefix of v, one has $j \leqslant |u| - 2$ and $v = u_1u_2\cdots u_{j-1}v_j\cdots v_{|v|}$.

To prove this result, we have to distinguish two cases.

(1) $u_j = a$, $v_j = b$. Then from $\sigma(a) = ab$ and $\sigma(b) = a$, we obtain:

$$\sigma(u) = \sigma(u_1u_2\cdots u_{j-1})ab\sigma(u_{j+1})\sigma(u_{j+2})\cdots\sigma(u_{|u|}),$$
$$\sigma(v) = \sigma(u_1u_2\cdots u_{j-1})a\sigma(v_{j+1})\sigma(v_{j+2})\cdots\sigma(v_{|v|}).$$

Comparing these two equalities and noting that b is a prefix neither of $\sigma(a)$ nor of $\sigma(b)$, we see that $\sigma(u)$ is not a weak prefix of $\sigma(v)$.

(2) $u_j = b$, $v_j = a$. Then we obtain:

$$\sigma(u) = \sigma(u_1u_2\cdots u_{j-1})a\sigma(u_{j+1})\sigma(u_{j+2})\cdots\sigma(u_{|u|}),$$
$$\sigma(v) = \sigma(u_1u_2\cdots u_{j-1})ab\sigma(v_{j+1})\sigma(v_{j+2})\cdots\sigma(v_{|v|}).$$

However b^3 is not a suffix of u, so we must have $|\sigma(u_{j+1})\sigma(u_{j+2})\cdots\sigma(u_{|u|})| \geqslant 3$ and $\sigma(u)$ is not a weak prefix of $\sigma(v)$.

Proof of Theorem 1. – We first show that (2) implies (1).

Assume (2) holds. Then $M_{\varphi_1^n} = M_{\varphi_1}^n = M_{\varphi_0}^{mn} = M_{\varphi_2}^m = M_{\varphi_2^m}$ which implies $\xi_{\varphi_1^n} \simeq \xi_{\varphi_2^m}$ by virtue of Theorem 2 in [10]. However $\xi_{\varphi_1^n} = \xi_{\varphi_1}$ and $\xi_{\varphi_2^m} = \xi_{\varphi_2}$, hence $\xi_{\varphi_1} \simeq \xi_{\varphi_2}$.

Now show that (1) implies (2). Since $\mathrm{IS}(S^*)$ is generated by σ, τ and π and $M_\sigma = M_\tau$, we can find $\psi_1, \psi_2 \in \mathcal{G}$ such that $M_{\psi_1} = M_{\varphi_1}$ and $M_{\psi_2} = M_{\varphi_2}$. By Theorem 2 of [10], we have $\xi_{\psi_1} \simeq \xi_{\varphi_1}$ and $\xi_{\psi_2} \simeq \xi_{\varphi_2}$, which yields $\xi_{\psi_1} \simeq \xi_{\psi_2}$. So $\xi_{\psi_1} = \xi_{\psi_2}$ by Corollary 1.

Since each primitive $\varphi \in \mathcal{G}$ takes one of the four forms discussed in Proposition 2, so by symmetry and by similarity, we can concentrate our attention only on the case where $\psi_1 = \sigma^{n_1}\pi\sigma^{n_2}\pi\cdots\sigma^{n_{k-1}}\pi\sigma^{n_k}$ with $n_i \geqslant 1$ $(1 \leqslant i \leqslant k)$, i.e., ψ_1 takes the form (i) and where ψ_2 is arbitrary since the other cases can be discussed likewise.

Let θ be the homomorphism from S^* onto $\mathcal{G}$ such that $\theta(a) = \sigma$ and $\theta(b) = \pi$. Put $u = a^{n_1}ba^{n_2}b\cdots a^{n_{k-1}}ba^{n_k}$. Then we have $\theta(u) = \psi_1$.

According to the form of ψ_2, we distinguish three cases:

Case (1). If ψ_2 takes the form (iii) or (iv), then bb is a factor of ξ_{ψ_2} but not of ξ_{ψ_1}. This is absurd for $\xi_{\psi_1} = \xi_{\psi_2}$. So ψ_2 can only take the form (i) or (ii).

Case (2). Let $v = a^{m_1}ba^{m_2}b\cdots a^{m_{l-1}}ba^{m_l}$ with $m_j \geqslant 1$ $(1 \leqslant j \leqslant l)$ and assume $\psi_2 = \theta(v)$, i.e., ψ_2 takes the form (i). Put $p = |v|$, $q = |u|$ and write

$$u^p = a^{s_1}ba^{s_2}b\cdots ba^{s_{\tilde{n}}} \quad \text{and} \quad v^q = a^{t_1}ba^{t_2}b\cdots ba^{t_{\tilde{m}}}.$$

Then we have $\psi_1^p = \sigma^{s_1}\pi\sigma^{s_2}\pi\cdots\pi\sigma^{s_{\tilde{n}}}$ and $\psi_2^q = \sigma^{t_1}\pi\sigma^{t_2}\pi\cdots\pi\sigma^{t_{\tilde{m}}}$.

Below we will prove that $s_j = t_j$ $(1 \leqslant j \leqslant \min\{\widetilde{n}, \widetilde{m}\})$.

Set $\Psi_1 := \psi_1^p \circ \psi_1^r$ and $\Psi_2 := \psi_2^q \circ \psi_2^s$ with $r \geqslant 1$ and $s \geqslant 1$ large enough so that $|\psi_1^r(a)| > 3$, $|\psi_2^s(a)| > 3$ and that $\Psi_1(a)$ is a prefix of $\Psi_2(a)$, a weak prefix of the latter. The existence of r, s is clear for ψ_1, ψ_2 have the same fixed point which begins with a.

First we prove $s_1 = t_1$. Suppose this is false; we can suppose $s_1 < t_1$ without loss of generality. Then b is a prefix of $\sigma^{-s_1}\Psi_1(a)$ (for π is the first component of $\sigma^{-s_1}\Psi_1$) and a is a prefix of $\sigma^{-s_1}\Psi_2(a)$ (since σ is the first component of $\sigma^{-s_1}\Psi_2$). Thus $\sigma^{-s_1}\Psi_1(a)$ is not a weak prefix of $\sigma^{-s_1}\Psi_2(a)$ by noting $|\sigma^{-s_1}\Psi_1(a)| > 3$. But a is a suffix of $\sigma^{-s_1}\Psi_1(a)$, then by Lemma 2, $\Psi_1(a)$ is not a weak prefix of $\Psi_2(a)$. This is absurd. So we have $s_1 = t_1$.

Second we can deduce the equality $s_2 = t_2$ from above by only replacing Ψ_1 (resp. Ψ_2) by $\pi^{-1}\sigma^{-s_1}\Psi_1$ (resp. $\pi^{-1}\sigma^{-t_1}\Psi_2$) in the preceding discussion.

Finally by using the same argument as above, we obtain $s_j = t_j$ $(1 \leqslant j \leqslant \min\{\widetilde{n}, \widetilde{m}\})$. However $|u^p| = |v^q|$, then we have $\widetilde{n} = \widetilde{m}$ and $u^p = v^q$. Now by Lemma 1, we can find $w \in S^*$ and positive integers m, n such that $u = w^m$ and $v = w^n$. Put $\theta(w) = \varphi_0$. Then $\varphi_0 \in \mathcal{G}$ and $\psi_1 = \varphi_0^m$, $\psi_2 = \varphi_0^n$. Hence $M_{\varphi_1} = M_{\psi_1} = M_{\varphi_0}^m$ and $M_{\varphi_2} = M_{\psi_2} = M_{\varphi_0}^n$.

Case (3). Assume that ψ_2 takes the form (ii). As above we can prove $\psi_1^p = \psi_2^q\pi^{-1}$. However, since a is a prefix of $\psi_1^r(a)$ and b is a prefix of $\pi\psi_2^s(a)$, then $\psi_1^r(a)$ is not a weak prefix of $\pi\psi_2^s(a)$. So by Lemma 2, $\Psi_1(a) = \psi_1^p \circ \psi_1^r(a)$ is not a weak prefix of $\Psi_2(a) = \psi_1^p \circ \pi\psi_2^s(a)$. This is impossible because, by construction, $\Psi_1(a)$ is a prefix of $\Psi_2(a)$.

* Supported by the Special Funds for Major State Basic Research Projects of China and Morningside Center of Mathematics (CAS).

References

[1] J. Berstel, Recent results in Sturmian words, in: Developments in Language Theory, II (Magdeburg, 1995), World Sci. Publishing, River Edge, NJ, 1996, pp. 13–24.
[2] A. Cobham, On the base-dependence of sets of numbers recognizable by finite automata, Math. Systems Theory 3 (1969) 186–192.
[3] F. Durand, A generalization of Cobham's theory, Theory Comput. Systems 31 (1998) 169–185.
[4] I. Fagnot, On the subword equivalence problem for morphic words, Discrete Appl. Math. 75 (1997) 231–253.
[5] I. Fagnot, Sur les facteurs des mots automatiques, Theoret. Comput. Sci. 172 (1997) 67–89.
[6] M. Lothaire, Combinatorics on words, Encyclopedia of Mathematics and its Applications, Vol. 17, Addison-Wesley, 1983.

[7] J.M. Luck, C. Godrèche, A. Janner, T. Janssen, The nature of the atomic surfaces of quasiperiodic self-similar structures, J. Phys. A 26 (8) (1993) 1951–1999.
[8] F. Mignosi, P. Séébold, Morphismes Sturmians et règles de Rauzy, J. de Théorie des Nombres de Bordeaux 5 (1993) 221–233.
[9] M. Queffélec, Substitution Dynamical Systems – Spectral Analysis, Lecture Notes in Math., Vol. 1294, Springer-Verlag, 1987.
[10] Z.X. Wen, Z.Y. Wen, Local isomorphisms of the invertible substitutions, C. R. Acad. Sci. Paris, Série I 318 (1994) 299–304.
[11] Z.X. Wen, Z.Y. Wen, Some properties of the singular words of the Fibonacci word, European J. Combin. 15 (1994) 587–598.
[12] Z.X. Wen, Z.Y. Wen, Factor properties of the infinite words generated by a class of invertible substitutions, in: 5th Conference Formal Power Series and Algebraic Combinatorics, Florence, 1993, pp. 455–466.