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Calcul des variations/*Calculus of Variations*

Necessary conditions for extremals of Blake & Zisserman functional

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Abstract We show some necessary conditions for minimizers of a functional depending on free discontinuities, free gradient discontinuities and second derivatives, which is related to image segmentation. A candidate for minimality of main part of the functional is explicitly exhibited ***To cite this article: M. Carriero et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 343–348.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Conditions nécessaires d'extrémalité pour la fonctionnelle de Blake & Zisserman

Résumé On donne des conditions nécessaires de minimisation d'une fonctionnelle dépendant de discontinuités libres et de dérivées secondes, reliée à la segmentation d'images. On exhibe un candidat explicite vérifiant toutes les conditions d'extrémalité. ***Pour citer cet article : M. Carriero et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 343–348.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Nous envisageons la fonctionnelle de Blake & Zisserman (1) pour la segmentation d'images [2,3]. Dans [4–6] nous avons établi des conditions suffisantes pour l'existence des minima et de leur régularité. Ici, nous présentons des résultats nouveaux : conditions nécessaires d'extrémalité obtenues par plusieurs techniques variationnelles, en explicitant les conditions d'Euler et des conditions intégrales et géometriques pour la segmentation optimale.

Si le triplet (K_0, K_1, u) est minimisanté et $n = 2, 3$ alors $K_0 \cup K_1$ peut être interprétée comme segmentation optimale d'une image mono-chromatique d'intensité donnée g.

L'existence des minima a été prouvée par régularisation des solutions faibles en dimension 2, à condition que $g \in \mathrm{L}^{2q}_{\mathrm{loc}}(\Omega)$ (théorèmes 4, 5).

L'équation d'Euler au sens des distributions hors de la segmentation optimale $K_0 \cup K_1$ est :

$$\Delta^2 u = -\frac{q}{2}\mu|u-g|^{q-2}(u-g) \quad \text{dans } \Omega \setminus K_0 \cup K_1,$$

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couplée avec des conditions homogènes sur les opérateurs aux bords pour la décomposition du bilaplacien.

Les variations premières de l'énergie d'un minimum local par rapport à des déformations (à support compact) de la segmentation optimale donnent l'équation d'Euler globale (9) et des relations entre la courbure de la segmentation et les différences des traces du hessien.

À partir de cette analyse nous déduisons de nombreuses conditions nécessaires d'extrémalité. Enfin nous montrons un triplet, avec une segmentation non triviale, qui vérifie toutes les conditions pour être un minimum local de la partie principale de l'énergie dans $\mathbf{R}^2$, et en plus satisfait un principe variationnel d'equipartition entre l'énergie de volume et l'énergie de surface.

Nous conjecturons que ce triplet est un minimum local, unique à des déplacements rigides près et/ou addition de fonctions affines.

We focus the Blake & Zisserman functional in image segmentation [2,3]. In previous papers we proved the existence of minimizers and we showed some regularity properties [4–6]. Here we show necessary conditions for minimality by performing various kinds of first variations: Euler equations and several integral and geometric conditions on optimal segmentation set. The strong formulation of Blake & Zisserman functional F and its main part E are [5]:

$$F(K_0,K_1,u) := \int_{\Omega\setminus(K_0\cup K_1)} \big(|D^2u|^2+\mu|u-g|^q\big)\,\mathrm{d}\mathbf{y} + \alpha\mathcal{H}^{n-1}(K_0\cap\Omega) + \beta\mathcal{H}^{n-1}\big((K_1\setminus K_0)\cap\Omega\big), \tag{1}$$

$$E(K_0,K_1,u) := \int_{\Omega\setminus(K_0\cup K_1)} \big|D^2u\big|^2\,\mathrm{d}\mathbf{y} + \alpha\mathcal{H}^{n-1}(K_0\cap\Omega) + \beta\mathcal{H}^{n-1}\big((K_1\setminus K_0)\cap\Omega\big), \tag{2}$$

where $\Omega\subset\mathbf{R}^n$ is an open set, $n\geqslant 2$, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\alpha,\beta,\mu,q\in\mathbf{R}$, with, given,

$$q>1,\quad \mu>0,\quad 0<\beta\leqslant\alpha\leqslant 2\beta,\quad g\in \mathrm{L}^q(\Omega), \tag{3}$$

while $K_0, K_1 \subset\mathbf{R}^n$ are Borel sets (a priori unknown) with $K_0\cup K_1$ closed, $u\in \mathrm{C}^2(\Omega\setminus(K_0\cup K_1))$ and u is approximately continuous on $\Omega\setminus K_0$.

If (K_0,K_1,u) is a minimizing triplet for F and $n=2,3$, then $K_0\cup K_1$ can be interpreted as an optimal segmentation of the monochromatic image of brightness intensity g.

The existence of minimizers for functional (1) was proved by regularization of solution for a weak formulation, when $n=2$, provided the additional assumption $g\in \mathrm{L}^{2q}_{\mathrm{loc}}(\Omega)$ is satisfied (Theorems 4, 5). In general, when $n\geqslant 2$ and $g\notin \mathrm{L}^{nq}_{\mathrm{loc}}(\Omega)$, then the infimum may be not achieved [6].

1. Notation and definitions

For any Borel function $v:\Omega\to\mathbf{R}$ and $\mathbf{x}\in\Omega$, $z\in\overline{\mathbf{R}}:=\mathbf{R}\cup\{-\infty,+\infty\}$, we set $z=\operatorname{ap\,lim}_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ (that is to say z is the approximate limit of v at $\mathbf{x}$) if:

$$g(z)=\lim_{\rho\to 0} ⨍_{B_\rho(0)} g\big(v(\mathbf{x}+\mathbf{y})\big)\,\mathrm{d}\mathbf{y}\quad \forall g\in \mathrm{C}^0\big(\overline{\mathbf{R}}\big).$$

For $\nu\in S^{n-1}$, we denote by $v^+=\mathrm{tr}^+(\mathbf{x},v,\nu)$ (and $v^-=\mathrm{tr}^+(\mathbf{x},v,-\nu)$) if:

$$g\big(v^+\big)=\lim_{\rho\to 0} ⨍_{B_\rho(0)\cap\{\mathbf{y}\cdot\nu>0\}} g\big(v(\mathbf{x}+\mathbf{y})\big)\,\mathrm{d}\mathbf{y}\quad \forall g\in \mathrm{C}^0\big(\overline{\mathbf{R}}\big).$$

The set $S_v = \{\mathbf{x} \in \Omega : \nexists \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(y)\}$ is the singular set of v. By $Dv, \nabla v$ we denote, respectively, the distributional gradient and the approximate gradient of v (*see* [5]). $|\cdot|$ denotes the Euclidean norm and $\nabla_i v = (\mathbf{e}_i \cdot \nabla)v$, where $\{\mathbf{e}_i\}$ is the canonical basis of $\mathbf{R}^n$. When the right-hand side is meaningful, we set $\nabla^2_{ij} v = \nabla_i(\nabla_j v)$. We recall also the definitions of some classes of functions having derivatives which are special measures in the sense of De Giorgi, and we refer to [7,3–6,1] for their properties:

$$SBV(\Omega) := \left\{ v \in BV(\Omega) : \|Dv\|_{\mathcal{M}(\Omega)} = \int_\Omega |\nabla v| \,\mathrm{d}\mathbf{y} + \int_{S_v} \left|v^+ - v^-\right| \mathrm{d}\mathcal{H}^{n-1} \right\},$$

$$GSBV(\Omega) := \left\{ v : \Omega \to \mathbf{R} \text{ Borel function; } -k \vee v \wedge k \in SBV_{\mathrm{loc}}(\Omega)\ \forall k \in \mathbf{N} \right\},$$

$$GSBV^2(\Omega) := \left\{ v \in GSBV(\Omega),\ \nabla v \in \left(GSBV(\Omega)\right)^n \right\}.$$

The classes $GSBV(\Omega)$, $GSBV^2(\Omega)$ are neither vector spaces nor subsets of distributions in Ω; nevertheless smooth variations of a function in $GSBV^2(\Omega)$ belong to the same class. If $v \in GSBV(\Omega)$, then S_v is countably $\mathcal{H}^{n-1}$-rectifiable and ∇v exists a.e. in Ω. We set $S_{\nabla v} = \bigcup_{i=1}^n S_{\nabla_i v}$, and $K_v = \overline{S_v \cup S_{\nabla v}}$.

DEFINITION 1 (Weak formulation of Blake & Zisserman functional [4]). – For $\Omega \subset \mathbf{R}^n$ open set, under the assumption (3), we define $\mathcal{F} : X(\Omega) \to [0, +\infty]$ by:

$$\mathcal{F}(v) := \int_\Omega \left(\left|\nabla^2 v\right|^2 + \mu|v-g|^q\right) \mathrm{d}\mathbf{y} + \alpha\mathcal{H}^{n-1}(S_v) + \beta\mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v), \tag{4}$$

where $X(\Omega) := GSBV^2(\Omega) \cap \mathrm{L}^q(\Omega)$. We consider also localization $\mathcal{F}_A$ on any Borel set $A \subseteq \Omega$.

We observe that the subset of $GSBV^2(\Omega)$ where $\mathcal{F}$ is finite is a vector space.

DEFINITION 2 (Local minimizer). – We say that u is a local minimizer of the functional $\mathcal{F}$ in Ω if:

$$u \in GSBV^2(A), \quad \mathcal{F}_A(u) < +\infty, \quad \mathcal{F}_A(u) \leqslant \mathcal{F}_A(u+\varphi),$$

for every open subset $A \Subset \Omega$ and for every $\varphi \in GSBV^2(\Omega)$ with compact support in A.

We introduce also the weak form of functional (2):

$$\mathcal{E}(v) := \int_\Omega \left|\nabla^2 v\right|^2 \mathrm{d}\mathbf{y} + \alpha\mathcal{H}^{n-1}(S_v) + \beta\mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v). \tag{5}$$

We say that u is a local minimizer of the functional $\mathcal{E}$ in Ω if, by denoting $\mathcal{E}_A$ the localization,

$$u \in GSBV^2(A), \quad \mathcal{E}_A(u) < +\infty, \quad \mathcal{E}_A(u) \leqslant \mathcal{E}_A(u+\varphi)$$

for every open subset $A \Subset \Omega$ and for every $\varphi \in GSBV^2(\Omega)$ with compact support in A.

Remark 3. – If u is a local minimizer of $\mathcal{E}$ in Ω then also the function $u(x) + a\cdot x + b$ is a local minimizer in Ω for every $a \in \mathbf{R}^n$, $b \in \mathbf{R}$, moreover, if $B_\rho(x_0) \subset \Omega$, then the re-scaling,

$$u_\rho(x) = \rho^{-3/2} u(x_0 + \rho x),$$

defines a local minimizer in $B_1(0)$ such that $\mathcal{E}_{B_\rho(x_0)}(u) = \rho^{n-1}\, \mathcal{E}_{B_1(0)}(u_\rho)$.

About the minimization of (1) and (4), the two following statements are known:

THEOREM 4 (Existence of weak solutions (*see* [4])). – *Let* $\Omega \subset \mathbf{R}^n$ *be an open set and assume* (3). *Then there is* $v_0 \in X(\Omega)$ *such that* $\mathcal{F}(v_0) \leqslant \mathcal{F}(v)\ \forall v \in X(\Omega)$.

We recall that assumption $\beta \leqslant \alpha \leqslant 2\beta$ is necessary for lower semi-continuity of $\mathcal{F}$.

THEOREM 5 (Existence of strong solutions (*see* [5])). – *Let* $n = 2$, $\Omega \subset \mathbf{R}^2$ *be an open set. Assume* (3) *and* $g \in \mathrm{L}^{2q}_{\mathrm{loc}}(\Omega)$. *Then there is at least one triplet among* $K_0, K_1 \subset \mathbf{R}^2$ *Borel sets with* $K_0 \cup K_1$ *closed and* $u \in \mathrm{C}^2(\Omega \setminus (K_0 \cup K_1))$ *approximately continuous on* $\Omega \setminus K_0$ *minimizing the functional* (1) *with finite energy. Moreover the sets* $K_0 \cap \Omega$ *and* $K_1 \cap \Omega$ *are* $(\mathcal{H}^1, 1)$ *rectifiable.*

2. New results

THEOREM 6 (Euler equation and regularity outside the optimal segmentation K_u). – *Assume* (3) *and* $u \in GSBV^2(\Omega)$ *a local minimizer of* $\mathcal{F}$ *in* $\Omega \subset \mathbf{R}^n$, $n \geqslant 2$, $g \in \mathrm{L}^s(\Omega)$, $1 < q \leqslant s$, *then*

(i) $\Delta^2 u = -\frac{q}{2}\mu|u-g|^{q-2}(u-g)$ *in* $\Omega \setminus K_u$;

(ii) $u \in \mathrm{W}^{4,s/(s-1)}_{\mathrm{loc}}(\Omega \setminus K_u)$;

(iii) $u \in \mathrm{C}^{1,1/2}_{\mathrm{loc}}(\Omega \setminus K_u)$.

Moreover, if $s \geqslant nq$, *then, by setting* $\gamma = 1 - n(q-1)/s$,

(iv) $u \in \mathrm{W}^{4,s/(q-1)}_{\mathrm{loc}}(\Omega \setminus K_u) \subset \mathrm{C}^{3,\gamma}_{\mathrm{loc}}(\Omega \setminus K_u)$.

If A is a C^2 uniformly regular open subset of Ω, N is the outward unit normal to ∂A and $\{t_k = t_k(x);\ k = 1, \ldots, n-1,\ x \in \partial A\}$ denotes a system of local tangential coordinates, then for every $\varphi \in \mathrm{W}^{2,2}(A)$ and $u \in \mathrm{W}^{2,2}(A)$ with $\Delta^2 u \in \mathrm{L}^2(A)$ the following *Green formula* holds true:

$$\int_A \left(D^2 u\right) : \left(D^2 \varphi\right) \mathrm{d}\mathbf{y} = \int_A \left(\Delta^2 u\right)\varphi \, \mathrm{d}\mathbf{y} + \int_{\partial A}\left(S(u) - \frac{\partial}{\partial N}\Delta u\right)\varphi \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial A} T(u)\frac{\partial \varphi}{\partial N}\, \mathrm{d}\mathcal{H}^{n-1},$$

where the natural boundary operators $T(u)$ and $S(u)$ are defined by:

$$T(u) := \sum_{i,j=1}^{n} \nabla^2_{ij} u N_i N_j, \qquad S(u) := -\sum_{i,j=1}^{n}\sum_{k=1}^{n-1} \frac{\partial}{\partial t_k}\left(\nabla^2_{ij} u N_j \frac{\partial t_k}{\partial x_i}\right). \tag{6}$$

By evaluating the first variation of the energy functional (4) around a local minimizer u (Definition 2) under compactly supported deformations of u, which are smooth outside K_u, we get Theorems 7 and 8.

THEOREM 7 (Necessary conditions on S_u for natural boundary operators). – *Assume* (3), $n \geqslant 2$, $q > 1$ *and* u *is a local minimizer of* $\mathcal{F}$, $B \Subset \Omega$ *an open ball such that* $S_u \cap B$ *is the graph of a* C^3 *function and* $(S_{\nabla u} \setminus S_u) \cap B = \emptyset$. *Denote by* B^+, B^- *the two connected components of* $B \setminus S_u$ *and by* N *the unit normal vector to* S_u *pointing toward* B^+. *Assume that* $u \in \mathrm{C}^3(\overline{B^+}) \cap \mathrm{C}^3(\overline{B^-})$. *Then,*

$$\left(T(u)\right)^{\pm} = 0, \quad \left(S(u) - \frac{\partial}{\partial N}\Delta u\right)^{\pm} = 0 \quad \text{on } S_u \cap B. \tag{7}$$

THEOREM 8 (Necessary conditions on $S_{\nabla u}$ for jumps of natural boundary operators). – *Assume* (3), $n \geqslant 2$, $q > 1$ *and* u *is a local minimizer of* $\mathcal{F}$, $B \Subset \Omega$ *an open ball such that* $S_{\nabla u} \cap B$ *is the graph of a* C^3 *function and* $S_u \cap B = \emptyset$. *Denote by* B^+, B^- *the two connected components of* $B \setminus S_{\nabla u}$ *and by* N *the unit normal vector to* $S_{\nabla u}$ *pointing toward* B^+. *Assume that* $u \in \mathrm{C}^3(\overline{B^+}) \cap \mathrm{C}^3(\overline{B^-})$. *Then,*

$$\left(T(u)\right)^{\pm} = 0, \quad \left[\!\left[S(u) - \frac{\partial}{\partial N}\Delta u \right]\!\right] = 0 \quad \text{on } S_{\nabla u} \cap B, \tag{8}$$

where for a function w *we set* $[\![w]\!] = w^+ - w^-$.

By evaluating the first variation of the energy functional (4) around a local minimizer u, under compactly supported smooth deformation of S_u and $S_{\nabla u}$, we find the global Euler equation.

THEOREM 9 (Global Euler equation). – *Let $u \in GSBV^2(\Omega)$ be a local minimizer of $\mathcal{F}$ in Ω with $\mathcal{H}^{n-1}(k_\mu)$ finite, $g \in \mathrm{C}^1(\Omega)$, then for every $\eta \in \mathrm{C}^2_0(\Omega, \mathbf{R}^n)$ the following equation holds*:

$$\begin{aligned}&\int_\Omega \left(\left|\nabla^2 u\right|^2 \operatorname{div}\eta - 2\left(\nabla^2 u D\eta + (D\eta)^t \nabla^2 u + \nabla u D^2\eta\right) : \nabla^2 u\right) \mathrm{d}\mathbf{y}\\ &\quad + \mu \int_\Omega \left(|u-g|^q \operatorname{div}\eta - q|u-g|^{q-2}(u-g)\, Dg \cdot \eta\right) \mathrm{d}\mathbf{y}\\ &\quad + \alpha \int_{S_u} \operatorname{div}_{S_u} \eta \,\mathrm{d}\mathcal{H}^{n-1} + \beta \int_{S_{\nabla u}\setminus S_u} \operatorname{div}_{S_{\nabla u}\setminus S_u} \eta \,\mathrm{d}\mathcal{H}^{n-1} = 0,\end{aligned} \tag{9}$$

where $(\nabla^2 u D\eta + (D\eta)^t \nabla^2 u + \nabla u D^2\eta)_{ik} = \sum_j \nabla^2_{ij} u D_k \eta_j + D_i \eta_j \nabla^2_{jk} u + \nabla_j u D^2_{ik} \eta_j$, and div_M denotes the tangential divergence on M.

We perform a qualitative analysis of the singular set by assuming enough regularity to deal with normal derivatives of u and of the traces of $|\nabla^2 u|$ on both sides of K_u, and to perform integration by parts in Theorem 9: by using compactly supported vector fields that are normal to S_u or $S_{\nabla u}$ as test functions we can prove the two following statements:

THEOREM 10 (Curvature of S_u and squared Hessian jump). – *Let u be a local minimizer of $\mathcal{F}$ in Ω, $g \in \mathrm{C}^1(\Omega)$ and $B \Subset U \subset \Omega$ two open balls, such that $S_u \cap U$ is the graph of a C^3 function and B_+ (resp. B_-) the open connected epigraph (resp. hypograph) of such function in B. Assume $\overline{S_{\nabla u}} \cap U = \emptyset$, $(\overline{S}_u \setminus S_u) \cap U = \emptyset$, and $u \in \mathrm{C}^3(\overline{B}_+) \cap \mathrm{C}^3(\overline{B}_-)$. Then,*

$$\left[\!\left[\left|\nabla^2 u\right|^2 + \mu|u-g|^q \right]\!\right] = (n-1)\alpha H(S_u) \quad \text{on } S_u \cap B,$$

where H is the scalar mean curvature evaluated by orienting the surface through the normal pointing toward B^+.

THEOREM 11 (Curvature of $S_{\nabla u}$ and squared Hessian jump). – *Let u be a local minimizer of $\mathcal{F}$ in Ω, $g \in \mathrm{C}^1(\Omega)$ and $B \Subset U \subset \Omega$ two open balls such that $S_{\nabla u} \cap U$ is the graph of a C^3 function and B_+ (resp. B_-) the open connected epigraph (resp. hypograph) of such function in B. Assume $\overline{S_u} \cap U = \emptyset$ and $u \in \mathrm{C}^3(\overline{B}_+) \cap \mathrm{C}^3(\overline{B}_-)$. Then,*

$$\left[\!\left[\left|\nabla^2 u\right|^2 + \mu|u-g|^q \right]\!\right] = (n-1)\beta H(S_{\nabla u}) \quad \text{on } S_{\nabla u} \cap B.$$

We perform a qualitative analysis of the "boundary" of the singular set, by assuming that it is a manifold as smooth as required by the computation of boundary operators. The strategy is a new choice of the test functions in the global Euler equation (9): a vector field η tangential to S_u. Here, for simplicity, we state the theorem only in the case $n = 2$.

THEOREM 12 (Crack-tip). – *Let $n = 2$, u be a local minimizer of $\mathcal{F}$ in Ω, and $B \Subset U \subset \Omega$ open balls, such that $S_u \cap U$ is an oriented C^3 arc, oriented by a normal vector field $\nu \in \mathrm{C}^2(U)$, and $S_{\nabla u} \cap U = \emptyset$. Assume $(\overline{S_u} \setminus S_u) \cap U = \{x_0\} \subset B$, and $u \in \mathrm{W}^{2,2}(B \setminus \overline{S_u})$. Let $\mathbf{n}$ be the unit vector tangent to S_u at x_0 and pointing toward S_u.*

Then, for every $\eta \in \mathrm{C}^\infty_0(B, \mathbf{R}^2)$ s.t. $\eta = \zeta\tau$, $\zeta \in \mathrm{C}^\infty_0(B)$, $\tau \in \mathrm{C}^\infty(B, S^1)$, s.t. $\eta \cdot \nu \equiv 0$ on S_u and $\tau \cdot \mathbf{n} = 1$ at x_0,

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x_0)} \left\{ \left(\left|\nabla^2 u\right|^2 + \mu|u-g|^q\right) \eta \cdot \mathbf{n}_\varepsilon - 2T^\varepsilon(u) \frac{\partial \eta}{\partial \mathbf{n}_\varepsilon} \cdot \nabla u - 2\left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon} \Delta u\right) \eta \cdot \nabla u \right\} \mathrm{d}\mathcal{H}^1 = \alpha \zeta(x_0),$$

where the natural boundary operators T^{ε} *and* S^{ε} *are defined as in* (6)*, but using* $\mathbf{n}_{\varepsilon}$ *instead of* N ($\mathbf{n}_{\varepsilon}$ *points inside* $B_{\varepsilon}(x_0)$)*.*

So far we have found many necessary conditions for minimality for the functional $\mathcal{F}$. Now we examine the main part $\mathcal{E}$ of the functional $\mathcal{F}$ in $\mathbf{R}^n$, which is a natural procedure in the study of regularity properties of $\mathcal{F}$. We emphasize that Theorems 6–12 hold true for local minimizers of $\mathcal{E}$ provided all the terms including $u-g$ are dropped.

Eventually we show a candidate local minimizer W of $\mathcal{E}$ in $\mathbf{R}^2$, which has a non trivial singular set. It is constructed by suitable combination of real parts of analytic branch of multi-valued functions with branching point at the origin and cut along the negative real axis, and by exploiting Almansi representation of bi-harmonic functions. Some of the long and boring computations were performed by using the symbolic calculation routines of Mathematica 4.1 ©. This function W exhibits the only homogeneity in ϱ compatible with the following: being bi-harmonic, scaling invariance of the energy, all the necessary conditions, local finiteness of energy and the proper decay rate of energy around the origin (tip of the crack). Notice that W is left unchanged by natural dilatations of homogeneity $-3/2$ (see Remark 3). We stress the fact that minimizers are not defined up to a free constant multiplier, due to the analysis around the crack-tip.

Actually a variational principle of equi-partition of bulk and surface energy is fulfilled by W, say, $\forall \varrho > 0$,

$$\int_{B_\varrho(0)} \left|\nabla^2 W\right|^2 \mathrm{d}x\, \mathrm{d}y = \alpha \mathcal{H}^1\left(S_W \cap B_\varrho(0)\right).$$

Such a candidate, using polar coordinates in $\mathbf{R}^2$, is given by:

$$W = \sqrt{\frac{\alpha}{193\pi}}\varrho^{3/2}\left(\sqrt{21}\left(\sin\frac{\theta}{2} - \frac{5}{3}\sin\left(\frac{3}{2}\theta\right)\right) + \left(\cos\frac{\theta}{2} - \frac{7}{3}\cos\left(\frac{3}{2}\theta\right)\right)\right), \quad \theta \in (-\pi, \pi).$$

The following properties show that W fulfills the necessary conditions of Theorems 6–12:

$$S_W = \text{negative real axis}, \quad S_{\nabla W} = \emptyset, \quad \Delta^2 W = 0 \quad \text{on } \mathbf{R}^2 \setminus \overline{S_W}, \quad W_{yy} = 0 = W_{yyy} + 2W_{xxy} \quad \text{on } S_W.$$

CONJECTURE 13. – *We conjecture that* W *is a local minimizer of* $\mathcal{E}$ *in* $\mathbf{R}^2$*, and there are no other nontrivial local minimizers besides* W*, up to sign charge, rigid motions in* $\mathbf{R}^2$ *and addition of affine functions.*

Complete proofs will be published elsewhere.

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