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Differential Geometry

# $S^1$-bundles and gerbes over differentiable stacks

# $S^1$-fibrés et gerbes sur des champs différentiables

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**Abstract**

We study $S^1$-bundles and $S^1$-gerbes over differentiable stacks in terms of Lie groupoids, and construct Chern classes and Dixmier–Douady classes in terms of analogues of connections and curvature. ***To cite this article: K. Behrend, P. Xu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


**Résumé**

Nous étudions les $S^1$-fibrés et les $S^1$-gerbes sur des champs différentiables en termes de groupoïdes de Lie et nous construisons les classes de Chern et Dixmier–Douady en termes d'analogues des connexions et de leur courbure. ***Pour citer cet article : K. Behrend, P. Xu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


## Version française abrégée

Soit $\mathfrak{X}$ un champ différentiable et $\mathfrak{P} \to \mathfrak{X}$ un $S^1$-fibré sur $\mathfrak{X}$. Soit $\Gamma \rightrightarrows M$ une présentation par un groupoïde de Lie pour $\mathfrak{X}$. Alors $\mathfrak{P}$ induit un $S^1$-fibré $P$ sur $M$ sur lequel agit $\Gamma \rightrightarrows M$. On réalise la classe de Chern de $\mathfrak{P}$ en termes de données de type connexion sur $P$ et prouve l'existence des préquantifications. Plus précisément, soit $\theta \in \Omega^1(P)$ une pseudo-connexion, et $\omega + \Omega \in Z^2_{DR}(\Gamma_\bullet)$ sa pseudo-courbure.

**Théorème 0.1.** *La classe $[\omega + \Omega] \in H^2_{DR}(\Gamma_\bullet)$ est indépendante du choix de la pseudo-connexion $\theta$ et correspond à la classe de Chern de $P$. Réciproquement, soit $\omega + \Omega \in C^2_{DR}(\Gamma_\bullet)$ un 2-cocycle entier. Alors il existe un $S^1$-fibré $P$ sur $\Gamma \rightrightarrows M$ et une pseudo-connexion $\theta \in \Omega^1(P)$ ayant $\omega + \Omega$ pour pseudo-courbure. De plus, l'ensemble des classes d'isomorphisme de tous ces $(P, \theta)$ est un $H^1(\Gamma_\bullet, \mathbb{R}/\mathbb{Z})$-ensemble.*

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Si $\mathfrak{G}$ est une $S^1$-gerbe sur $\mathfrak{X}$, et $R \rightrightarrows M$ une présentation du champ différentiable $\mathfrak{G}$ et soit $\Gamma \rightrightarrows M$ le groupoïde de Lie défini par la présentation induite $M \to \mathfrak{X}$ de $\mathfrak{X}$. Alors $R$ est une $S^1$-extension centrale du groupoïde de Lie $\Gamma \rightrightarrows M$. Ainsi les $S^1$-extensions centrales de $\Gamma \rightrightarrows M$ sont exactement les $S^1$-gerbes sur $\mathfrak{X}$ munies d'une trivialisation sur $M$. A nouveau, on peut réaliser les classes caractéristiques de la gerbe (que nous appelons classes de Dixmier–Douady) en termes de données de type connexion et prouver l'existence de préquantifications. Plus précisément, soit $\theta + B \in C^2_{DR}(R)$ une pseudo-connexion sur $R$, et $\theta + \omega + \Omega \in Z^3_{DR}(\Gamma_\bullet)$ sa pseudo-courbure.

**Théorème 0.2.** *La classe* $[\eta + \omega + \Omega] \in H^3_{DR}(\Gamma_\bullet)$ *est indépendante du choix de la pseudo-connexion* $\theta + B$ *sur* $R$ *et correspond à la classe de Dixmier–Douady de* $R$. *Réciproquement, pour tout* 3*-cocycle* $\eta + \omega + \Omega \in Z^3_{DR}(\Gamma_\bullet)$ *tel que* $[\eta + \omega + \Omega]$ *est une classe entière et* $\Omega$ *est exact, il existe une extension centrale* $R \rightrightarrows M$ *du groupoïde* $\Gamma \rightrightarrows M$, *et une pseudo-connexion* $\theta + B \in C^2_{DR}(R)$ *sur* $R$ *telle que* $\eta + \omega + \Omega$ *soit la pseudo-courbure. Les paires* $(R, \theta, B)$ *forment, à un isomorphisme près, un ensemble simplement transitif sous le groupe des extensions centrales plates.*

Dans le cas $s$-connexe, on obtient un construction explicite de l'extension centrale avec pseudo-connexion. Cela donne également un critère pour qu'une classe dans $H^3_{DR}(\Gamma_\bullet)$ soit entière. Ce théorème généralise le résultat de [3].

**Théorème 0.3.** *Soit* $\Gamma \rightrightarrows M$ *un groupoïde de Lie* $s$*-connexe, et* $\eta + \omega \in C^3_{DR}(\Gamma_\bullet)$ *un* 3*-cocycle, où* $\eta \in \Omega^1(\Gamma_2)$ *and* $\omega \in \Omega^2(\Gamma)$. *Supposons que* $\omega$ *représente une classe de cohomologie entière dans* $H^2_{DR}(\Gamma)$, *de telle sorte qu'il existe un* $S^1$*-fibré* $\pi : R \to \Gamma$ *avec une connexion* $\theta \in \Omega^1(R)$, *dont la courbure est* $\omega$. *Supposons que* $\varepsilon^* R$, *doté d'une connexion plate* $\varepsilon^*\theta + \pi^*\varepsilon_2^*\eta$ *soit sans holonomie.* (*Ici* $\varepsilon : M \to \Gamma$ *et* $\varepsilon_2 : M \to \Gamma_2$ *sont les morphisme d'identité respectifs.*) *Alors* $R \rightrightarrows M$ *admet de façon naturelle une structure de groupoïde, telle que* $R$ *soit une extension* $S^1$*-centrale de* $\Gamma \rightrightarrows M$ *et* $\eta + \omega$ *la pseudo-courbure de* $\theta$.

Puisque les extensions centrales de groupoïdes décrivent les gerbes sur $\mathfrak{X}$ avec des trivialisations données sur $M$, on peut seulement décrire les gerbes qui sont effectivement triviales sur $M$ en terme d'extensions centrales de groupoïdes de $\Gamma \rightrightarrows M$. Pour décrire toutes les gerbes sur $\mathfrak{X}$, on doit passer en général à un groupoïde de Lie Morita-équivalent $\Gamma' \rightrightarrows M'$.

## 1. Introduction

We study $S^1$-bundles and $S^1$-gerbes over differentiable stacks in terms of Lie groupoids.

Let $\mathfrak{X}$ be a differentiable stack and $\mathfrak{P} \to \mathfrak{X}$ an $S^1$-bundle over $\mathfrak{X}$. Let $\Gamma \rightrightarrows M$ be a Lie groupoid presentation for $\mathfrak{X}$, i.e., $\mathfrak{X}$ is (isomorphic to) the stack of $\Gamma \rightrightarrows M$-torsors. Then $\mathfrak{P}$ gives rise to an $S^1$-bundle $P$ over $M$ on which $\Gamma \rightrightarrows M$ acts. We realize the Chern class of $\mathfrak{P}$ in terms of connection-like data on $P$ and prove that prequantizations exist.

Note that $H^2(\Gamma_\bullet, \Omega^0)$ contains the obstructions to the existence of $\mathfrak{P}$ for an arbitrary integer cohomology class and $H^1(\Gamma_\bullet, \Omega^1)$ contains the obstructions to the existence of a connection on $\mathfrak{P}$ if $\mathfrak{P}$ exists. The possibility of non-vanishing of these cohomology groups distinguishes our case from the standard case of manifolds.

If $\mathfrak{G}$ is an $S^1$-gerbe over $\mathfrak{X}$, and $\Gamma \rightrightarrows M$ a presentation for $\mathfrak{X}$ as above, then $\mathfrak{G}$ gives rise to a gerbe over $M$. So we do not immediately get a description of $\mathfrak{G}$ in terms of groupoids. Instead, we can start with a presentation $R \rightrightarrows M$ of the differentiable stack $\mathfrak{G}$ and let $\Gamma \rightrightarrows M$ be the Lie groupoid defined by the induced presentation $M \to \mathfrak{X}$ of $\mathfrak{X}$, in other words, $\Gamma = M \times_{\mathfrak{X}} M$. In this situation, we get a morphism of groupoids from $R \rightrightarrows M$ to $\Gamma \rightrightarrows M$, and, moreover, $R \to \Gamma$ is an $S^1$-principal bundle. In fact, $R$ is an $S^1$-central extension of the Lie groupoid $\Gamma \rightrightarrows M$.

Thus the $S^1$-central extensions of $\Gamma \rightrightarrows M$ are exactly the $S^1$-gerbes over $\mathfrak{X}$, endowed with a trivialization over $M$. Therefore, the central extension case is not entirely analogous to the bundle case.

Again, we can realize the characteristic class of the gerbe (which we call the Dixmier–Douady class) in terms of connection-like data and prove that prequantizations exist. Note that there are again obstructions to the existence of honest connective structures and curvings. More precisely, $H^3(\Gamma_\bullet, \Omega^0)$ contains the obstructions to the existence of $\mathfrak{G}$, given an integer degree-3 cohomology class. Assuming $\mathfrak{G}$ exists, $H^2(\Gamma_\bullet, \Omega^1)$ contains the obstructions to the existence of a connective structure on $\mathfrak{G}$. If we assume the existence of a connective structure, $H^1(\Gamma_\bullet, \Omega^2)$ contains the obstructions to the existence of a curving.

Because groupoid central extensions describe gerbes over $\mathfrak{X}$ together with given trivializations over $M$, we can only describe those gerbes that are indeed trivial over $M$ in terms of groupoid central extensions of $\Gamma \rightrightarrows M$. To describe all gerbes over $\mathfrak{X}$, we need to pass in general to a Morita equivalent Lie groupoid $\Gamma' \rightrightarrows M'$.

## 2. Homology and cohomology

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Define $\Gamma_p = \underbrace{\Gamma \times_M \cdots \times_M \Gamma}_{p \text{ times}}$, i.e., $\Gamma_p$ is the manifold of composable sequences of $p$ arrows in the groupoid $\Gamma \rightrightarrows M$. We have $p+1$ canonical maps $\Gamma_p \to \Gamma_{p-1}$ (each leaving out one of the $p+1$ objects involved a sequence of composable arrows), giving rise to a diagram

$$\ldots \Gamma_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \Gamma_1 \;\substack{\longrightarrow\\\longrightarrow}\; \Gamma_0 . \tag{1}$$

In fact, $\Gamma_\bullet$ is a simplicial manifold.

The *piecewise differentiable chain complex* of $\Gamma_\bullet$ is the total complex associated to the double complex $C_\bullet(\Gamma_\bullet)$. Here $C_k(\Gamma_p)$ is the free Abelian group generated by the piecewise differentiable maps $\Delta_k \to \Gamma_p$. Its homology groups $H_k(\Gamma_\bullet, \mathbb{Z}) = H_k(C_\bullet(\Gamma_\bullet))$ are called the *homology groups* of $\Gamma \rightrightarrows M$.

We denote the dual of the double complex $C_\bullet(\Gamma_\bullet)$ by $C^\bullet(\Gamma_\bullet)$. It's total cohomology groups $H^k(\Gamma_\bullet, \mathbb{Z}) = H^k(C^\bullet(\Gamma_\bullet))$ are called the *integer cohomology groups* of $\Gamma \rightrightarrows M$. In the case that $\Gamma \rightrightarrows M$ is a transformation groupoid $G \times M \rightrightarrows M$, these are the $G$-equivariant cohomology groups.

Finally, we introduce the double complex $\Omega^\bullet(\Gamma_\bullet)$. It's boundary maps are $d : \Omega^k(\Gamma_p) \to \Omega^{k+1}(\Gamma_p)$, the usual exterior derivative of differentiable forms and $\partial : \Omega^k(\Gamma_p) \to \Omega^k(\Gamma_{p+1})$, the alternating sum of the pull back maps of (1). We denote the total differential by $\delta = (-1)^p d + \partial$. The total cohomology groups of $\Omega^\bullet(\Gamma_\bullet)$, $H^k_{DR}(\Gamma_\bullet) = H^k(\Omega^\bullet(\Gamma_\bullet))$ are called the *De Rham cohomology* groups of $\Gamma \rightrightarrows M$.

Recall that a *Morita morphism* from the Lie groupoid $\Gamma' \rightrightarrows M'$ to $\Gamma \rightrightarrows M$ is a morphism of Lie groupoids satisfying the two conditions

(i) the diagram

$$\begin{array}{ccc} \Gamma' & \longrightarrow & M' \times M' \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & M \times M \end{array}$$

is Cartesian, i.e., a pullback diagram,

(ii) $M' \to M$ is a surjective submersion.

Two Lie groupoids are Morita equivalent, if and only if there exist a third Lie groupoid together with a Morita morphism to each of them.

**Proposition 2.1.** *Let $f : [\Gamma' \rightrightarrows M'] \to [\Gamma \rightrightarrows M]$ be a Morita morphism of Lie groupoids. Then we get induced isomorphisms $f^* : H^k(\Gamma_\bullet, \mathbb{Z}) \xrightarrow{\sim} H^k(\Gamma'_\bullet, \mathbb{Z})$ and $f^* : H^k_{DR}(\Gamma_\bullet) \xrightarrow{\sim} H^k_{DR}(\Gamma'_\bullet)$.*

In particular, if $\Gamma \rightrightarrows M$ is a *banal* groupoid, i.e., there exists a surjective submersion $\pi : M \to X$, for some manifold $X$, and $\Gamma \rightrightarrows M$ is isomorphic to $M \times_X M \rightrightarrows M$, then we have canonical isomorphisms $f^* : H^k(X, \mathbb{Z}) \xrightarrow{\sim} H^k(\Gamma_\bullet, \mathbb{Z})$ and

$$f^* : H^k_{DR}(X) \xrightarrow{\sim} H^k_{DR}(\Gamma_\bullet) . \tag{2}$$

The canonical homomorphism $\Omega^\bullet(\Gamma_\bullet) \to C^\bullet(\Gamma_\bullet) \otimes \mathbb{R}$ induces isomorphisms

$$H^k_{DR}(\Gamma_\bullet) \xrightarrow{\sim} H^k(\Gamma_\bullet, \mathbb{R}) \tag{3}$$

and pairings $Z_k(\Gamma_\bullet, \mathbb{Z}) \otimes Z^k_{DR}(\Gamma_\bullet) \to \mathbb{R}$; $\gamma \otimes \omega \mapsto \int_\gamma \omega$.

We call a De Rham cocycle an *integer cocycle*, if it maps under (3) into the image of the canonical map $H^k(\Gamma_\bullet, \mathbb{Z}) \to H^k(\Gamma_\bullet, \mathbb{R})$.

**Proposition 2.2.** *Let $\omega \in Z^k_{DR}(\Gamma_\bullet)$ be a De Rham cocycle. The following are equivalent*: (1) *$\omega$ is an integer cocycle*; (2) *$\int_\gamma \omega \in \mathbb{Z}$, for all $\gamma \in Z_k(\Gamma_\bullet, \mathbb{Z})$*; (3) *for every closed surface $S$ and every $\Gamma \rightrightarrows M$-torsor $T$ over $S$, giving rise to a morphism of groupoids $g$ from $T \times_S T \rightrightarrows T$ to $\Gamma \rightrightarrows M$, we have $\int_S g^*\omega \in \mathbb{Z}$. Here we use the isomorphism* (2), *to make sense of the integral.*

(Recall that a $\Gamma \rightrightarrows M$*-torsor* over $S$ is a surjective submersion $T \to S$, together with an action of $\Gamma \rightrightarrows M$ on $T$, such that $S$ is the quotient of $T$ by this action.)

For any Abelian sheaf $F$ on the category of differentiable manifolds, we have the cohomology groups $H^k(\Gamma_\bullet, F)$. One way to define them is by choosing for every $p$ an injective resolution $F_p \to I^\bullet_p$ of sheaves on $\Gamma_p$, where $F_p$ is the sheaf induced by $F$ on $\Gamma_p$; then choosing homomorphisms $f^{-1}I^\bullet_{p-1} \to I^\bullet_p$ for every map $f : \Gamma_p \to \Gamma_{p-1}$ in (1). This gives rise to a double complex $I^\bullet(\Gamma_\bullet)$, whose total cohomology groups are the $H^k(\Gamma_\bullet, F)$.

Examples of Abelian sheaves on the category of manifolds are: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\Omega^k$ and $S^1$. The first three are sheaves of locally constant functions, $S^1$ is the sheaf of differentiable $S^1$-valued functions. With respect to the first three, the notation $H^k(\Gamma, F)$ does not conflict with the notation introduced before.

It is well-known that $H^1(\Gamma_\bullet, S^1)$ classifies principal $S^1$-bundles over $\Gamma_\bullet$, whereas $H^2(\Gamma_\bullet, S^1)$ classifies $S^1$-gerbes over $\Gamma_\bullet$.

## 3. $S^1$-bundles

**Definition 3.1.** Let $\Gamma \rightrightarrows M$ be a Lie groupoid. A (right) $S^1$*-bundle* over $\Gamma \rightrightarrows M$ is a (right) $S^1$-bundle $P$ over $M$, together with a (left) action of $\Gamma$ on $P$, which respects the $S^1$-action, i.e., we have $(\gamma \cdot x) \cdot t = \gamma \cdot (x \cdot t)$, for all $t \in S^1$ and all compatible pairs $(\gamma, x) \in \Gamma \times_{t,M} P$.

Let $Q = \Gamma \times_{t,M} P$ be the manifold of compatible pairs. Action and projection form a diagram $Q \rightrightarrows P$ and it is easy to check that $Q \rightrightarrows P$ is in a natural way a groupoid (called the transformation groupoid of the $\Gamma$-action). Moreover, there is a natural morphism of groupoids $\pi$ from $Q \rightrightarrows P$ to $\Gamma \rightrightarrows M$. Of course, $Q$ is an $S^1$-bundle over $\Gamma$.

More is true: the $S^1$-bundle $P$ over $\Gamma \rightrightarrows M$ gives rise to an $S^1$-bundle on the simplicial manifold $\Gamma_\bullet$. As such it has an associated class in $H^1(\Gamma_\bullet, S^1)$ and, in fact, $S^1$-bundles over $\Gamma \rightrightarrows M$ are classified by $H^1(\Gamma_\bullet, S^1)$. The exponential sequence $\mathbb{Z} \to \Omega^0 \to S^1$ induces a boundary map $H^1(\Gamma_\bullet, S^1) \to H^2(\Gamma_\bullet, \mathbb{Z})$; the image of the class of $P$ under this boundary map is called the *Chern class* of $P$.

Let $\theta \in \Omega^1(P)$ be a connection form for the $S^1$-bundle $P \to M$. One checks that $\delta\theta \in C^2_{DR}(Q_\bullet)$ descends to $C^2_{DR}(\Gamma_\bullet)$. In other words, there exist unique $\omega \in \Omega^1(\Gamma)$ and $\Omega \in \Omega^2(M)$ such that $\pi^*(\omega + \Omega) = \delta\theta$.

**Proposition 3.2.** *The class $[\omega + \Omega] \in H^2_{DR}(\Gamma_\bullet)$ is independent of the choice of the connection $\theta$ on $P \to M$. Under the canonical homomorphism $H^2(\Gamma_\bullet, \mathbb{Z}) \to H^2_{DR}(\Gamma_\bullet)$, the Chern class of $P$ maps to $[\omega + \Omega]$.*

Here is a converse.

**Proposition 3.3.** *Let $\omega + \Omega \in C^2_{DR}(\Gamma_\bullet)$ as above be an integer 2-cocycle. Then there exists an $S^1$-bundle $P$ over $\Gamma \rightrightarrows M$ and a connection form $\theta \in \Omega^1(P)$ for the bundle $P \to M$, such that $\pi^*(\omega + \Omega) = \delta\theta$.*

*Moreover, the set of isomorphism classes of all such $(P, \theta)$ is a simply transitive $H^1(\Gamma_\bullet, \mathbb{R}/\mathbb{Z})$-set. Here $(P, \theta)$ and $(P', \theta')$ are isomorphic if $P$ and $P'$ are isomorphic as $S^1$-bundles over $\Gamma \rightrightarrows M$ and under such an isomorphism $\theta$ is identified with $\theta'$.*

These two propositions indicate that $\theta$ can be thought of as an analogue of a connection on $P$ and $\omega + \Omega$ as an analogue of the curvature of this connection.

On the other hand, we do not call $\theta$ a connection on the $S^1$-bundle over $\Gamma \rightrightarrows M$, because this term should be reserved for $\theta$ satisfying $\partial\theta = 0$.

Thus we suggest the name *pseudo-connection* for a connection on the underlying bundle over $M$. If $\theta$ is such a pseudo-connection, we call $\omega + \Omega \in Z^2_{DR}(\Gamma_\bullet)$ such that $\pi^*(\omega + \Omega) = \delta\theta$ the *pseudo-curvature* of $\theta$.

## 4. $S^1$-central extensions

**Definition 4.1.** Let $\Gamma \rightrightarrows M$ be a Lie groupoid. An $S^1$-*central extension* of $\Gamma \rightrightarrows M$ consists of (1) a Lie groupoid $R \rightrightarrows M$, together with a morphism of Lie groupoids $(\pi, \mathrm{id}) : [R \rightrightarrows M] \to [\Gamma \rightrightarrows M]$; (2) a left $S^1$-action on $R$, making $\pi : R \to \Gamma$ a (left) principal $S^1$-bundle. These two structures are compatible in the sense that $(s \cdot x)(t \cdot y) = st \cdot (xy)$, for all $s, t \in S^1$ and $(x, y) \in R \times_M R$.

Since $S^1$ is Abelian, any left principal $S^1$-bundle is a right principal $S^1$-bundle in a natural way. Thus, if $R$ and $R'$ are central extensions of $\Gamma \rightrightarrows M$ as in the definition, we may form the associated bundle $R \times_{S^1} R'$, which is again an $S^1$-bundle over $\Gamma$. It has a natural groupoid structure making it into another $S^1$-central extension of $\Gamma \rightrightarrows M$. We denote this central extension by $R \otimes R'$. This operation turns the set of isomorphism classes of $S^1$-central extensions into an Abelian group.

Central extensions of groupoids pull back via morphisms of groupoids.

Groupoid central extensions of $\Gamma \rightrightarrows M$ give rise to $S^1$-gerbes over $\Gamma_\bullet$, which are trivialized over $M$. Thus we have the

**Proposition 4.2.** *There is a natural exact sequence*

$$H^1(\Gamma_\bullet, S^1) \longrightarrow H^1(M, S^1) \longrightarrow \{S^1\text{-central extensions of } \Gamma \rightrightarrows M\} \longrightarrow H^2(\Gamma_\bullet, S^1) \longrightarrow H^2(M, S^1).$$

Given a central extension $R$ of $\Gamma \rightrightarrows M$, then a connection form $\theta \in \Omega^1(R)$ for the bundle $R \to \Gamma$, such that $\partial\theta = 0$ is a *connective structure* on $R$. Given $(R, \theta)$, a 2-form $B \in \Omega^2(M)$, such that $d\theta = \partial B$ is a *curving* on $R$, and given $(R, \theta, B)$, the 3-form $\Omega = dB \in H^0(\Gamma_\bullet, \Omega^3) \subset \Omega^3(M)$ is called the *curvature* of $(R, \theta, B)$. If $\Omega = 0$, then $(R, \theta, B)$ is called a *flat* $S^1$-central extension of $\Gamma \rightrightarrows M$. Note that the flat central extensions form an Abelian group.

**Proposition 4.3.** *There is a natural exact sequence*

$$\begin{aligned} H^1(\Gamma_\bullet, \mathbb{R}/\mathbb{Z}) &\longrightarrow H^1(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \{\text{flat } S^1\text{-central extensions of } \Gamma \rightrightarrows M\} \\ &\longrightarrow H^2(\Gamma_\bullet, \mathbb{R}/\mathbb{Z}) \longrightarrow H^2(M, \mathbb{R}/\mathbb{Z}). \end{aligned}$$

The exponential sequence gives rise to a homomorphism $H^2(\Gamma_\bullet, S^1) \to H^3(\Gamma_\bullet, \mathbb{Z})$. The image of a central extension $R$ in $H^3(\Gamma_\bullet, \mathbb{Z})$ is called the *Dixmier–Douady class* of $R$. The Dixmier–Douady class behaves well with respect to pullbacks and the tensor operation.

Let $R$ be a central extension of $\Gamma \rightrightarrows M$. Choose a connection form $\theta \in \Omega^1(R)$ for the $S^1$-bundle $\pi : R \to \Gamma$. One checks that $\delta\theta \in Z^3_{DR}(R_\bullet)$ descends to $Z^3_{DR}(\Gamma_\bullet)$, i.e., there exist unique $\eta \in \Omega^1(\Gamma_2)$ and $\omega \in \Omega^2(\Gamma)$ such that $\pi^*(\eta + \omega) = \delta\theta$.

**Proposition 4.4.** *The class $[\eta + \omega] \in H^3_{DR}(\Gamma_\bullet)$ is independent of the choice of the connection $\theta$ on $R \to \Gamma$. Under the canonical homomorphism $H^3(\Gamma, \mathbb{Z}) \to H^3_{DR}(\Gamma_\bullet)$, the Dixmier–Douady class of $R$ maps to $[\eta + \omega]$.*

Since the class $[\eta + \omega]$ does not change by adding a coboundary, we may choose, in addition to $\theta$, any $B \in \Omega^2(M)$, and then the Dixmier–Douady class of $R$ is represented by $\eta + \omega + \Omega$, such that $\pi^*(\eta + \omega + \Omega) = \delta(\theta + B)$.

**Proposition 4.5.** *Given any* 3*-cocycle $\eta + \omega + \Omega \in Z^3_{DR}(\Gamma_\bullet)$, as above, satisfying* (1) *$[\eta + \omega + \Omega]$ is integer;* (2) *$\Omega$ is exact, there exists a groupoid central extension $R \rightrightarrows M$ of the groupoid $\Gamma \rightrightarrows M$, a connection $\theta$ on the bundle $R \to \Gamma$ and a* 2*-form $B \in \Omega^2(M)$, such that $\delta(\theta + B) = \pi^*(\eta + \omega + \Omega)$. The pairs $(R, \theta, B)$ up to isomorphism form a simply transitive set under the group of flat central extensions.*

Because of these propositions, $\theta + B$ plays a role similar to a connection (connective structure plus curving) on a gerbe over a manifold. We therefore call $\theta + B$ a *pseudo-connection* on $R$, and $\theta + \omega + \Omega$ its *pseudo-curvature*.

**Remark 1.** Given a 3-cocycle $\eta + \omega + \Omega$ of integer class, we may have to pass to a Morita equivalent groupoid via a Morita morphism $[\Gamma' \rightrightarrows M'] \to [\Gamma \rightrightarrows M]$, in order to realize the condition that $\Omega$ be exact. For example, if $\Gamma = M$ we may have to pass to an open cover $\{U_\alpha\}$ of $M$ to construct a groupoid central extension. In this case we use the Morita morphism $[\coprod_{\alpha,\beta} U_{\alpha\beta} \rightrightarrows \coprod_\alpha U_\alpha] \to [M \rightrightarrows M]$. See [1]. If $M$ is connected, another possibility is to pass to the (infinite dimensional) path space $PM \to M$ and use the Morita homotopy morphism $[LM \rightrightarrows PM] \to [M \rightrightarrows M]$, where $LM$ is the space of based loops. See [2].

We close with a theorem that gives an explicit construction of the central extension with pseudo-connection in the $s$-connected case. It also gives a criterion for a class in $H^3_{DR}(\Gamma_\bullet)$ to be integer. This theorem generalizes the result of [3].

**Theorem 4.6.** *Let $\Gamma \rightrightarrows M$ be an $s$-connected Lie groupoid, and $\eta + \omega \in C^3_{DR}(\Gamma_\bullet)$ a 3-cocycle, where $\eta \in \Omega^1(\Gamma_2)$ and $\omega \in \Omega^2(\Gamma)$. Assume that $\omega$ represents an integer cohomology class in $H^2_{DR}(\Gamma)$, so that there exists an $S^1$-bundle $\pi : R \to \Gamma$ with a connection $\theta \in \Omega^1(R)$, whose curvature is $\omega$. Assume that $\varepsilon^* R$ endowed with the flat connection $\varepsilon^*\theta + \pi^*\varepsilon_2^*\eta$ is holonomy free.* (*Here $\varepsilon : M \to \Gamma$ and $\varepsilon_2 : M \to \Gamma_2$ are the respective identity morphisms.*) *Then $R \rightrightarrows M$ admits in a natural way the structure of a groupoid, such that $R$ becomes an $S^1$-central extension of $\Gamma \rightrightarrows M$ and $\eta + \omega$ the pseudo-curvature of $\theta$.*

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## References

[1] N. Hitchin, Lectures on special Lagrangian submanifolds, in: AMS/IP Stud. Adv. Math., Vol. 23, American Mathematical Society, Providence, RI, 2001, pp. 151–182.
[2] M.K. Murray, Bundle gerbes, J. London Math. Soc. (2) 54 (1996) 403–416.
[3] A. Weinstein, P. Xu, Extensions of symplectic groupoids and quantization, J. Reine Angew. Math. 417 (1991) 159–189.