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Differential Geometry

Equivariant gerbes over compact simple Lie groups

Gerbes equivariantes sur les groupes de Lie simples compacts

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Abstract

Using groupoid S^1-central extensions, we present, for a compact simple Lie group G, an infinite dimensional model of S^1-gerbe over the differential stack G/G whose Dixmier–Douady class corresponds to the canonical generator of the equivariant cohomology $H^3_G(G)$. ***To cite this article: K. Behrend et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

En utilisant des extensions S^1-centrales de groupoïdes, nous présentons, dans le cas d'un groupe simple compact G, un modèle de dimension infinie d'une S^1-gerbe sur un champ différentiable G/G dont la classe de Dixmier–Douady correspond au générateur canonique de la cohomologie équivariante $H^3_G(G)$. ***Pour citer cet article : K. Behrend et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit G un groupe de Lie compact simple, le groupe de cohomologie équivariante $H^3_G(G)$ contient un générateur canonique $[\omega + \Omega]$ dont la classe est entière (voir Section 3.1 pour la définition de ω et Ω), où G agit sur lui-même par conjugaison. Nous réalisons cette classe en termes d'une extension S^1-centrale de groupoïdes, ou en tant que la classe de Dixmier–Douady d'une gerbe sur un champ différentiable. Le champ est celui correspondant à la transformation de groupoïde $G \times G \rightrightarrows G$. Cet exemple s'intègre dans la théorie générale développée dans [2,3]. Notre construction est divisée en deux étapes. La première étape s'inscrit dans le cadre de la géométrie de Poisson : pour une variété de Poisson affine $\mathfrak{g}^*$ induite par un 2-cocycle d'algèbre de Lie $\lambda \in \wedge^2\mathfrak{g}^*$, nous construisons son groupoïde symplectique ainsi qu'une extension S^1-centrale de groupoïdes. La construction se fait de la manière

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suivante. Soit $S^1 \to \widetilde{G} \xrightarrow{\pi} G$ une extension S^1-centrale au niveau des groupes de Lie. On forme les groupoïdes de transformations Γ $(= G \times \mathfrak{g}^*) \rightrightarrows \mathfrak{g}^*$ et R $(= \widetilde{G} \times \mathfrak{g}^*) \rightrightarrows \mathfrak{g}^*$ (G agit sur $\mathfrak{g}^*$ par l'action de jauge (2), tandis que $\widetilde{G}$ agit sur $\mathfrak{g}^*$ essentiellement par la même action obtenue par composition avec le morphisme de groupes $\pi : \widetilde{G} \to G$). En considérant R comme sous-groupoïde du groupoïde symplectique $T^*\widetilde{G} \rightrightarrows \tilde{\mathfrak{g}}^*$, on peut considérer les images réciproques sur R des formes symplectique et de Liouville sur $T^*\widetilde{G}$ et obtenir repectivement une 2-forme fermée $\omega_R \in \Omega^2(R)$ et une un-forme $\theta_R \in \Omega^1(R)$. Soit $\tilde{\pi} : R \to \Gamma$ la projection naturelle induite par $\pi : \widetilde{G} \to G$.

Théorème 0.1.

(1) *La* 2*-forme fermée* ω_R *est basique relativement au* S^1*-fibré* $R \to \Gamma$*, et donc se projette en une* 2*-forme fermée* ω_Γ *sur* Γ*, c.à.d.,* $\omega_R = \tilde{\pi}^*\omega_\Gamma$.
(2) ω_Γ *est symplectique et elle est compatible avec la structure de groupoïde de telle sorte que cela définisse une structure de groupoïde symplectique sur* Γ*. C'est le groupoïde symplectique de la variété de Poisson affine* $\mathfrak{g}^*$.
(3) $\tilde{\pi} : R \to \Gamma$ *est une extension* S^1*-centrale de groupoïdes de Lie.*
(4) θ_R *est une forme de connexion de préquantification sur le* S^1*-fibré* $\tilde{\pi} : R \to \Gamma$ *compatible avec la structure de groupoïde, c.à.d.,* $\partial\theta_R = 0$ *et* $d\theta_R = \tilde{\pi}^*\omega_\Gamma$.

En d'autres termes, $\theta_R \in C^2_{DR}(R_\bullet)$ est une pseudo-connexion sur R au sens de [2] dont $\omega_\Gamma \in Z^3_{DR}(\Gamma_\bullet)$ est la pseudo-courbure.

Pour un groupe de Lie simple compact G d'algèbre de Lie $\mathfrak{g}$, la forme basique sur $\mathfrak{g}$ induit une 2-cocycle naturel d'algèbre de Lie sur l'algèbre de Lie de lacets $L\mathfrak{g}$. En appliquant la construction ci-dessus, on obtient un groupoïde symplectique $(LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}, \omega_{LG\times L\mathfrak{g}})$ ainsi qu'une extension S^1-centrale $\widetilde{LG} \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ de ce dernier. Ici nous identifions $L\mathfrak{g}$ avec $L\mathfrak{g}^*$ par la forme de Killing. L'application d'holonomie $\mathrm{Hol}: L\mathfrak{g} \to G$ induit un morphisme naturel de Morita du groupoïde $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ vers $G \times G \rightrightarrows G$, dont le morphisme induit en cohomologie de de Rham envoie $[\omega + \Omega]$ sur la classe $[\omega_{LG\times L\mathfrak{g}}]$. Nous prouvons donc que :

Théorème 0.2. *Soit* $\omega + \Omega$ *un* 3*-cocycle comme ci-dessus, définissant une classe entière dans* $H^3_G(G)$*. L'extension* S^1*-centrale* $\widetilde{LG} \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ *de* $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ *correspond à une* S^1*-gerbe sur le champ* G/G *de classe de Dixmier–Douady* $[\omega + \Omega] \in H^3_G(G)$.

1. Introduction

Let G be a compact simple Lie group, the equivariant cohomology group $H^3_G(G)$ contains a canonical generator of integer class, where G acts on itself by conjugation. We realize this class in terms of a Lie groupoid S^1-central extension, or as the Dixmier–Douady class of an S^1-gerbe over a differential stack. The stack is the one which corresponds to the transformation groupoid $G \times G \rightrightarrows G$. This example fits into the general theory developed in [2,3]. Recall that associated to every Lie groupoid $\Gamma \rightrightarrows M$, there are De Rham cohomology groups defined as follows. Define $\Gamma_p = \underbrace{\Gamma \times_M \cdots \times_M \Gamma}_{p \text{ times}}$, i.e., Γ_p is the manifold of composable sequences of p arrows in the groupoid $\Gamma \rightrightarrows M$ $(\Gamma_1 = \Gamma,\ \Gamma_0 = M)$. We have $p + 1$ canonical maps $\Gamma_p \to \Gamma_{p-1}$ (each leaving out one of the $p + 1$ objects involved a sequence of composable arrows), giving rise to a diagram $\ldots \Gamma_2 \,\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\, \Gamma_1 \,\substack{\longrightarrow\\ \longrightarrow}\, \Gamma_0$. In fact, $\Gamma_\bullet$ is a simplicial manifold. We introduce the double complex $\Omega^\bullet(\Gamma_\bullet)$. Its boundary maps are $d : \Omega^k(\Gamma_p) \to \Omega^{k+1}(\Gamma_p)$, the usual exterior derivative of differential forms and $\partial : \Omega^k(\Gamma_p) \to \Omega^k(\Gamma_{p+1})$, the alternating sum of the pull back maps of the above diagram. We denote the total complex by $C^*_{DR}(\Gamma_\bullet)$ and the total differential by $\delta = (-1)^p d + \partial$. The total cohomology groups of $\Omega^\bullet(\Gamma_\bullet)$, $H^k_{DR}(\Gamma_\bullet) = H^k(\Omega^\bullet(\Gamma_\bullet))$ are called the *De Rham cohomology* groups of $\Gamma \rightrightarrows M$. In the case that $\Gamma \rightrightarrows M$ is a transformation groupoid $G \times M \rightrightarrows M$, these are the G-equivariant cohomology groups. In [2], we discussed the general question of how to realize a De Rham integer 3-cocycle in terms of an analogue of curvature on a Lie groupoid S^1-central extension. When this 3-cocycle consists of only one term $\omega \in \Omega^2(\Gamma)$, the 3-cocycle condition is equivalent to that $(\Gamma \rightrightarrows M, \omega)$ is a symplectic groupoid if ω is further assumed to be non-degenerate. S^1-central extensions of a symplectic groupoid were studied extensively by Weinstein and one of us in [9], to which we refer the readers for details.

When $\Gamma \rightrightarrows M$ is the transformation groupoid $G \times M \rightrightarrows M$, then $\omega + \Omega \in C^3_{DR}(\Gamma_\bullet)$, where $\omega \in \Omega^2(\Gamma)$ and $\Omega \in \Omega^3(M)$, is a 3-cocycle if and only if $d\Omega = 0$, $d\omega = \alpha^*\Omega - \beta^*\Omega$ and $(\partial_0^* - \partial_1^* + \partial_2^*)\omega = 0$. In this case, the class of $\omega + \Omega$ defines an element in the G-equivariant cohomology group $H^3_G(M)$. When $M = G$ is a compact simple Lie group and G acts on itself by conjugation, an explicit formula for both an $\omega \in \Omega^2(\Gamma)$ and an $\Omega \in \Omega^3(M)$ appeared in [1] in the study of group valued momentum maps and the moduli spaces of flat connections over two surfaces. See also [8] and [5] for the related topics and motivation. However, the fact that $\omega + \Omega$ is a 3-cocycle in $C^3_{DR}(\Gamma_\bullet)$ was overlooked in the literature. In this Note, we reinstate this fact, and show that $[\omega + \Omega]$ is an integer class by constructing an S^1-gerbe over the stack G/G (also called a G-equivariant gerbe over G) which has $[\omega + \Omega]$ as its Dixmier–Douady class. Our method is to pass from the groupoid $G \times G \rightrightarrows G$ to a Morita equivalent infinite dimensional symplectic groupoid, where an S^1-central extension can be readily constructed by the methods of Poisson geometry.

Applications of this construction to momentum map theory and twisted K-theory will be discussed elsewhere.

2. Symplectic groupoids of affine Poisson manifolds

2.1. General construction

Let $\mathfrak{g}$ be a (finite or infinite dimensional) Lie algebra over $\mathbb{R}$, and $\lambda \in \wedge^2\mathfrak{g}^*$ a Lie algebra 2-cocycle. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ be the corresponding central extension. Assume that $S^1 \to \widetilde{G} \xrightarrow{\pi} G$ is a central extension on the level of Lie groups, which exists if $\omega_G \in \Omega^2(G)$, the left invariant closed two-form corresponding to λ, is of integer class. It is well-known that the transformation groupoid $\widetilde{\Gamma}$ $(= \widetilde{G} \times \tilde{\mathfrak{g}}^*) \rightrightarrows \tilde{\mathfrak{g}}^*$, where $\widetilde{G}$ acts on $\tilde{\mathfrak{g}}^*$ by coadjoint action: $\tilde{g} \cdot \tilde{\xi} = Ad^*_{\tilde{g}^{-1}}\tilde{\xi}$, $\forall \tilde{g} \in \widetilde{G}, \xi \in \tilde{\mathfrak{g}}^*$ ($Ad^*_{\tilde{g}^{-1}}$ stands for the dual of $Ad_{\tilde{g}^{-1}}$), is a symplectic groupoid. The symplectic structure on $\widetilde{G} \times \tilde{\mathfrak{g}}^*$ is the canonical cotangent symplectic structure when $\widetilde{G} \times \tilde{\mathfrak{g}}^*$ is being identified with $T^*\widetilde{G}$ via the right translation.

Denote by $\chi : G \to \mathfrak{g}^*$ the group 1-cocycle integrating the Lie algebra 1-cocycle $\lambda^b : \mathfrak{g} \to \mathfrak{g}^*$, $\langle \lambda^b(v), u \rangle = \lambda(v, u)$, $\forall v, u \in \mathfrak{g}$, where G acts on $\mathfrak{g}^*$ by the coadjoint action. We assume that χ exists, which is true, for instance, when G is simply connected. Since $\widetilde{G}$ is a central extension of G, its adjoint action on $\tilde{\mathfrak{g}}$ descends to an action of G given by $g \cdot (X, t) = (Ad_g X,\ t + \langle \chi(g^{-1}), X \rangle)$, $\forall g \in G$, $(X, t) \in \tilde{\mathfrak{g}}$ $(\cong \mathfrak{g} \oplus \mathbb{R})$, and therefore the induced coadjoint action is

$$g \cdot (\xi, t) = \left(Ad^*_{g^{-1}}\xi + t\chi(g),\ t\right), \quad \forall g \in G,\ (\xi, t) \in \tilde{\mathfrak{g}}^* \ (\cong \mathfrak{g}^* \oplus \mathbb{R}). \tag{1}$$

Embed $\mathfrak{g}^*$ as a hyperplane of $\tilde{\mathfrak{g}}^*$ via the map $\phi : \xi \to (\xi, 1)$, $\forall \xi \in \mathfrak{g}^*$. Clearly $\mathfrak{g}^*$ is a Poisson submanifold of $\tilde{\mathfrak{g}}^*$ with the affine Poisson relation: $\{l_X, l_Y\} = l_{[X,Y]} + \lambda(X, Y)$, $\forall X, Y \in \mathfrak{g}$. By Eq. (1), this hyperplane is invariant under the coadjoint action of G, on which it takes the form:

$$g \cdot \xi = Ad^*_{g^{-1}}\xi + \chi(g), \quad \forall g \in G,\ \xi \in \mathfrak{g}^*. \tag{2}$$

Let Γ be the corresponding transformation groupoid $G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$. One may also form the transformation groupoid $R : \widetilde{G} \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ (here the $\widetilde{G}$-action on $\mathfrak{g}^*$ is essentially the same action (2) composing with the group morphism $\pi : \widetilde{G} \to G$). Then $R \rightrightarrows \mathfrak{g}^*$ is a subgroupoid of the symplectic groupoid $\widetilde{\Gamma} \rightrightarrows \tilde{\mathfrak{g}}^*$ under the natural embedding $i : (\tilde{g}, \xi) \to (\tilde{g}, \phi(\xi))$. By $\theta_{\tilde{\Gamma}}$ and $\omega_{\tilde{\Gamma}}$, we denote the Liouville one-form and symplectic two-form on $\widetilde{\Gamma}$ $(\cong T^*\widetilde{G})$ respectively, and set $\omega_R = i^*\omega_{\tilde{\Gamma}} \in \Omega^2(R)$ and $\theta_R = i^*\theta_{\tilde{\Gamma}} \in \Omega^1(R)$. By $\tilde{\pi} : R \to \Gamma$, we denote the natural projection: $\tilde{\pi}(\tilde{g}, \xi) = (\pi(\tilde{g}), \xi)$, $\forall (\tilde{g}, \xi) \in \widetilde{G} \times \mathfrak{g}^*$.

Theorem 2.1.

(1) *The closed two-form ω_R is basic for the S^1-bundle $R \to \Gamma$, and therefore descends to a closed two-form ω_Γ on Γ, i.e., $\omega_R = \tilde{\pi}^*\omega_\Gamma$.*
(2) *ω_Γ is symplectic and compatible with the groupoid structure so that it defines a symplectic groupoid on Γ. This is the symplectic groupoid of the affine Poisson manifold $\mathfrak{g}^*$.*
(3) *$\tilde{\pi}: R \to \Gamma$ is a S^1-central extension of Lie groupoids.*
(4) *θ_R is a prequantization connection form on the S^1-bundle $\tilde{\pi}: R \to \Gamma$ compatible with the groupoid structure, i.e., $\partial\theta_R = 0$ and $d\theta_R = \tilde{\pi}^*\omega_\Gamma$.*

In other words, $\theta_R \in C^2_{DR}(R_\bullet)$ is a pseudo-connection on R in the sense of [2] with $\omega_\Gamma \in Z^3_{DR}(\Gamma_\bullet)$ being its pseudo-curvature. Since $\Gamma \rightrightarrows \mathfrak{g}^*$ is a transformation groupoid, its De Rham cohomology is the equivariant cohomology $H^*_G(\mathfrak{g}^*)$, where G acts on $\mathfrak{g}^*$ by the gauge action (2). The Lie groupoid S^1-central extension $R \to \Gamma$ can be considered as a geometrical model realizing the class $[\omega_\Gamma] \in H^3_G(\mathfrak{g}^*)$. Indeed, if $\mathfrak{R}$ and $\mathfrak{X}$ are the differential stacks corresponding to the Lie groupoids $R \rightrightarrows \mathfrak{g}^*$ and $\Gamma \rightrightarrows \mathfrak{g}^*$, respectively, then $\mathfrak{R}$ is an S^1-gerbe over $\mathfrak{X}$ whose Dixmier–Douady class is equal to $[\omega_\Gamma] \in H^3_{DR}(\mathfrak{X})$ [2,3].

Remark 1. Note that $R \to \Gamma$ is indeed the pull back S^1-central extension of $\widetilde{G} \xrightarrow{\pi} G$ via the groupoid morphism ψ from Γ $(= G \times \mathfrak{g}^*) \rightrightarrows \mathfrak{g}^*$ to $G \rightrightarrows \cdot$ defined by the natural projection. As a consequence, ω_Γ and $\psi^*\omega_G$ define the same class in $H^3_G(\mathfrak{g}^*)$: the Dixmier–Douady class of the S^1-gerbe $\mathfrak{R} \to \mathfrak{X}$. It would be interesting to investigate whether this class is non-trivial when $\lambda \in \wedge^2\mathfrak{g}^*$ is assumed to be a non-trivial 2-cocycle (otherwise it is obvious that $\psi^*\omega$ is a trivial class). In the case of loop groups below, one indeed obtains a non-trivial class.

2.2. Loop group case

We will apply the above construction to the case of loop groups.

Let $(\cdot,\cdot)$ be an ad-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. It is well-known that $(\cdot,\cdot)$ induces a Lie algebra 2-cocycle on the loop Lie algebra $\lambda \in \wedge^2(L\mathfrak{g}^*)$ defined by [7]:

$$\lambda(X, Y) = \frac{1}{2\pi}\int_0^{2\pi} \big(X(s), Y'(s)\big)\,\mathrm{d}s, \quad \forall X(s), Y(s) \in L\mathfrak{g}. \tag{3}$$

By $\widetilde{L\mathfrak{g}}$ we denote its corresponding Lie algebra central extension. Assume that λ satisfies the integrability condition (i.e., the corresponding closed two-form $\omega_{LG} \in \Omega^2(LG)^{LG}$ is of integer class). It defines a loop group central extension $S^1 \to \widetilde{LG} \xrightarrow{\pi} LG$. By identifying $L\mathfrak{g}^*$ with $L\mathfrak{g}$ via the bilinear form $(\cdot,\cdot)$, the 1-cocycle χ admits the form: $\chi(g(s)) = g'(s)g(s)^{-1},\ \forall g(s) \in LG$, and the gauge action (2) becomes

$$g \cdot \xi = Ad^*_{g^{-1}}\xi + g'g^{-1}, \quad \forall g \in LG,\ \xi \in L\mathfrak{g}. \tag{4}$$

This is the standard gauge transformation when $L\mathfrak{g}$ is identified with the space of connections on the trivial bundle over the unit circle S^1.

As above, we can form the transformation groupoids $\Gamma: LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ and $R: \widetilde{LG} \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$, and define $\omega_R \in \Omega^2(R)$ and $\theta_R \in \Omega^1(R)$.

According to Theorem 2.1, we see that the closed two-form ω_R is basic and descends to a closed two-form $\omega_{LG\times L\mathfrak{g}}$ on Γ.

Corollary 2.2. *$(LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g},\ \omega_{LG\times L\mathfrak{g}})$ is a symplectic groupoid integrating the affine Poisson structure on $L\mathfrak{g}$. Moreover, $\tilde{\pi}: \widetilde{LG} \times L\mathfrak{g} \to LG \times L\mathfrak{g}$ is a S^1-central extension of Lie groupoids, on which $\theta_R \in C^2_{DR}(R_\bullet)$ defines a pseudo-connection with $\omega_{LG\times L\mathfrak{g}}$ being its pseudo-curvature.*

3. An S^1-gerbe over G/G

3.1. AMM-groupoids

Let G be a Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Consider the transformation groupoid $G \times G \rightrightarrows G$, where G acts on itself by conjugation. As in [1], we denote by θ and $\bar\theta$ the left and right Maurer–Cartan forms on G respectively, i.e., $\theta = g^{-1}\,\mathrm{d}g$ and $\bar\theta = \mathrm{d}g g^{-1}$. Let $\Omega \in \Omega^3(G)$ denote the bi-invariant 3-form on G corresponding to the Lie algebra 3-cocycle $\frac{1}{12}(\cdot,[\cdot,\cdot]) \in \wedge^3\mathfrak{g}^*$, and $\omega \in \Omega^2(G\times G)$ the two-form:

$$\omega = -\tfrac{1}{2}\big[(Ad_x g^*\theta, g^*\theta) + \big(g^*\theta, x^*(\theta+\bar\theta)\big)\big], \tag{5}$$

where (g,x) denotes the coordinate in $G\times G$, and $g^*\theta$ and $x^*\theta$ are, respectively, the $\mathfrak{g}$-valued one-forms on $G\times G$ obtained by pulling back θ via the first and second projections, and similarly for $x^*\bar\theta$.

A simple computation leads to

Proposition 3.1. *$\omega+\Omega$ is a 3-cocycle of the De Rham total complex of the transformation groupoid $G\times G \rightrightarrows G$, and therefore it defines a class in the equivariant cohomology $H^3_G(G)$.*

Remark 2.

(1) When G is a compact simple Lie group with the basic form $(\cdot,\cdot)$, $[\omega+\Omega]$ is a generator of $H^3_G(G)$. In Cartan model, it corresponds to the class defined by the d_G-closed equivariant 3-form $\chi_G(\xi) = \Omega - \frac{1}{2}\langle\theta+\bar\theta,\xi\rangle : \mathfrak{g} \to \Omega^*(G)$, $\forall \xi\in\mathfrak{g}$.

(2) In general, given a transformation groupoid $G\times M \rightrightarrows M$ (G is assumed to be compact), and a d_G-closed equivariant 3-form $\chi_G = \Omega + E(\xi)$, where $\Omega\in\Omega^3(M)$ is an invariant closed 3-form on M and $E:\mathfrak{g}\to\Omega^1(M)$ a G-equivariant linear map, an explicit formula for a two-form $\omega\in\Omega^2(G\times M)$ can be found [6], using the Bott–Shulman construction, such that $\omega+\Omega \in Z^3_{DR}(G\times M \rightrightarrows M)$ defines the same class of χ_G.

3.2. S^1-central extensions

Next we want to construct an S^1-central extension of Lie groupoids which realizes the class of the 3-cocycle $\omega+\Omega$ in Proposition 3.1 as its Dixmier–Douady class. Since $\Omega\in\Omega^3(G)$ is not exact, first of all we need to pass to a Morita equivariant groupoid [2]. There are many different choices for such a groupoid. Basically, one needs to choose a surjective submersion $f: M'\to G$ such that the pullback three form $f^*\Omega$ is exact on M'. Then $\Gamma': M' \times_{G,\alpha} \Gamma \times_{\beta,G} M' \rightrightarrows M'$ becomes a groupoid, and the natural projection from $\Gamma' \rightrightarrows M'$ to $\Gamma \rightrightarrows G$ is a Morita morphism [2]. For instance, one choice is to take a good open cover $\{U_i\}$ of G. Another choice, which is the one that we will pursue in this Note, is the infinite dimensional manifold $L\mathfrak{g}$ while f is the holonomy map $\mathrm{Hol}: L\mathfrak{g}\to G$, i.e., the time-1 map of the differential equation: $\mathrm{Hol}_s(X)^{-1}\frac{\partial}{\partial s}\mathrm{Hol}_s(X) = X$, $\mathrm{Hol}_0(X) = e$. Then we have $\mathrm{Hol}^*\,\Omega = d\mu$, where μ is the two-from on $L\mathfrak{g}$: $\mu = \frac{1}{2}\int_0^1 (\mathrm{Hol}_s^*\bar\theta, \frac{\partial}{\partial s}\mathrm{Hol}_s^*\bar\theta)\,\mathrm{d}s$ [1].

Proposition 3.2.

(1) *We have a Morita morphism f of Lie groupoids from $LG\times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ to $G\times G\rightrightarrows G$, which is given by $f(g,r) = (g(0),\mathrm{Hol}(r))$ on the space of morphisms and by $f(r) = \mathrm{Hol}(r)$ on the space of objects, $\forall g\in LG$, $r\in L\mathfrak{g}$.*

(2) *Under the induced isomorphism $f^*: H^3_G(G) \xrightarrow{\sim} H^3_{LG}(L\mathfrak{g})$, $[\omega+\Omega]$ goes to $[\omega_{LG\times L\mathfrak{g}}]$. Indeed we have*

$$\omega_{LG\times L\mathfrak{g}} - f^*(\omega+\Omega) = \delta\mu.$$

As a consequence, we have

Theorem 3.3. *Let G be a Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Assume that $\lambda \in \wedge^2(L\mathfrak{g}^*)$ as in Eq.* (3) *satisfies the integrability condition. Then the* 3*-cocycle $\omega + \Omega$ corresponds to an integer class in $H^3_G(G)$. The S^1-central extension $\widetilde{LG} \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ of $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ corresponds to an S^1-gerbe over the stack G/G with the Dixmier–Douady class $[\omega + \Omega] \in H^3_G(G)$.*

Remark 3. If there is a Morita morphism from Lie groupoid $\Gamma' \rightrightarrows M'$ to $\Gamma \rightrightarrows M$, then these two groupoids are also Morita equivalent in the sense of [10], which means that there is a bimodule. Indeed, these two notions of Morita equivalence are equivalent [3]. Morita equivalence via bimodules is particularly useful in constructing S^1-central extensions. It allows one to construct the S^1-central extension of one groupoid in terms of an S^1-central extension of the other together with a prequantization of the bimodule. See [3] for the details. This, for instance, will lead to a construction of an S^1-central extension of the Morita equivalent Lie groupoid $\Gamma' \rightrightarrows M'$ when $M' = \bigcup U_i$ is an open covering as in [4,6].

We end the paper with the following proposition which explicitly describes the equivalence bimodule between the groupoids $G \times G \rightrightarrows G$ and $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$.

Proposition 3.4. *The groupoids $G \times G \rightrightarrows G$ and $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ are Morita equivalent in the sense of Definition* 2.1 *in* [10], *where the bimodule X can be taken as $G \times L\mathfrak{g}$, and $\rho : X \to G$ and $\sigma : X \to L\mathfrak{g}$ are given, respectively by $\rho(g,r) = g\,\mathrm{Hol}(r)g^{-1}$ and $\sigma(g,r) = r$, $\forall (g,r) \in G \times L\mathfrak{g}$.*

The groupoid $G \times G \rightrightarrows G$ acts on X from the left by: $(g_1,g_2)\cdot(g,r) = (g_1 g, r)$, $\forall (g_1,g_2) \in G \times G$, $(g,r) \in G \times L\mathfrak{g}$, such that $g_2 = g\,\mathrm{Hol}(r)g^{-1}$; while $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ acts from right: $(g,r)\cdot(g(s),r') = (gg(0),r')$, $\forall (g,r) \in G \times L\mathfrak{g}$, $(g(s),r') \in LG \times L\mathfrak{g}$, such that $r = g(s)\cdot r'$.

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