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Topology

On the degrees of branched coverings over links

Sur les degrés des revêtements ramifiés le long d'entrelacs

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Abstract

Let M and M' be 3-manifolds and L a link in M'. We prove that, under certain conditions, the degree of a branched covering $\pi : M \to (M', L)$ is determined by the topological types of M and (M', L). ***To cite this article: A.M. Salgueiro, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Soient M et M' variétés tridimensionnelles et L un entrelacs dans M'. On prouve que, sous certaines conditions, le degré d'un revêtement ramifié $\pi : M \to (M', L)$ est déterminé par les types topologiques de M et (M', L). ***Pour citer cet article : A.M. Salgueiro, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Une variété tridimensionnelle, compacte, orientable et irréductible M est *géométrisable* s'il existe une famille $\mathcal{T}$ (possiblement vide) de tores incompressibles tel que $M - \mathcal{N}(\mathcal{T})$ est une union disjointe de variétés qui sont soit fibrés de Seifert soit hyperboliques. La *famille JSJ* de M est une telle famille minimale [3,4] et la *décomposition JSJ* de M est la décomposition correspondante à sa famille JSJ. On peut similairement définir la décomposition JSJ d'un orbifold en considérant familles de 2-sous-orbifolds euclidiens orientables et incompressibles [2]. Quand une variété (ou un orbifold) est géométrisable, son *graphe JSJ* est défini comme le graph dual de la décomposition JSJ, c'est-à-dire, les sommets sont en correspondance avec les morceaux de cette décomposition et les arêtes sont en correspondance avec les tores de la famille JSJ.

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Les variétés géométrisables M', excepté celles avec un revêtement du type (surface) $\times \mathbb{S}^1$ ou un fibré en tores sur $\mathbb{S}^1$, ont la propriété $(*)$ de Thurston [6, Problème 3.16], c'est-à-dire, le degré d'un revêtement fini $f : M \to M'$ est déterminé par le type topologique de M [11,12].

On étudie une question analogue pour des revêtements ramifiés le long d'un entrelacs, c'est-à-dire, une sousvariété de dimension 1. Un entrelacs L dans M' est *trivial* s'il est connexe (i.e. s'il est un nœud) et que son *extérieur* $E = M' - \mathcal{N}(L)$ dans M' est soit un tore solide, soit un produit $T^2 \times I$; il est *premier* si toute sphère de M' qui coupe L transversalement en deux points borde une boule qui intersecte L dans un arc non noué. Le résultat principal de cette Note est :

Théorème 0.1. *Soient M et M' des variétés tridimensionnelles compactes et orientables et L un entrelacs premier et non trivial dans M'. Si l'extérieur de L est irréductible et son graphe JSJ est un arbre alors il y a au plus un nombre premier $d > 3$ ($d \geqslant 3$, si M' est irréductible) pour lequel M est un revêtement de M' de degré d ramifié le long de L.*

Comme corollaires, on obtient les résultats suivants concernant des entrelacs dans des sphères d'homologie rationnelle ou entière.

Corollaire 0.2. *Soit M' une sphère d'homologie rationnelle et $L \subset M'$ un entrelacs premier, non trivial et dont l'extérieur est irréductible. Alors deux revêtements ramifiés de M' le long de L ayant des degrés premiers > 3 ($\geqslant 3$, si M' est irréductible) et différents ne sont pas homéomorphes.*

Dans la preuve de Théorème 0.1, on raisonne selon les différentes décompositions géométriques de E, en distinguant trois cas, suivant que E contient un morceau hyperbolique, est une variété de Seifert ou est une variété graphée non triviale. Dans le cas où E est une variété de Seifert on donne des exemples (voir Exemples 1 et 2) où le degré n'est pas déterminé, quand il n'est pas premier. Par contre, quand E contient un morceau hyperbolique, on exige seulement que $\pi : M \to (M', L)$ soit un revêtement ramifié *fortement cyclique*, c'est-à-dire, le groupe des transformations de revêtement est cyclique et l'ordre de ramification est égal au degré de π le long de toute composante de L. On remarque que quand π a un degré premier, il est toujours fortement cyclique.

1. Geometric decompositions

A compact orientable irreducible 3-manifold M is *geometrisable* if there is a (possibly empty) family $\mathcal{T}$ of incompressible tori such that $M - \mathcal{N}(\mathcal{T})$ is a disjoint union of manifolds which are either Seifert fibred or hyperbolic. The *JSJ family* of M is a minimal such family [3,4] and the *JSJ decomposition* of M is the decomposition corresponding to its JSJ family. We can similarly define JSJ decompositions of orbifolds by considering families of incompressible Euclidian 2-suborbifolds [2]. When a manifold (or an orbifold) is geometrisable, its *JSJ graph* is defined as the dual graph of the JSJ decomposition, that is, the vertices are in correspondence with the pieces of this decomposition and the edges are in correspondence with the tori of the JSJ family.

It was recently proved [11,12] that all geometrisable manifolds M', except those which are covered by a (surface) $\times \mathbb{S}^1$ or a torus bundle over $\mathbb{S}^1$, have the property $(*)$ of Thurston [6, Problem 3.16], that is, the degree of a finite covering $\pi : M \to M'$ is determined by the topological type of M.

We study an analogous question about branched coverings of a link, that is, a 1-dimensional submanifold. A link L in M' is *trivial* if it is connected (i.e., if it is a knot) and its *exterior* $E = M' - \mathcal{N}(L)$ in M' is either a solid torus or a product $T^2 \times I$; it is *prime* if every sphere of M' that cuts L transversely in two points bounds a ball that intersects L in an unknotted arc. The main result of this Note is:

Theorem 1.1. *Let M and M' be compact orientable* 3*-manifolds and L be a nontrivial prime link in M'. If the exterior of L is irreducible and its JSJ graph is a tree then there is at most one prime number $d > 3$ ($d \geqslant 3$, if M' is irreducible) for which M is a d-fold covering of M' branched along L.*

As corollaries we obtain the following results concerning links in rational or integer homology spheres.

Corollary 1.2. *Let M' be a rational homology sphere and $L \subset M'$ be a nontrivial prime link with irreducible exterior. Then any two branched coverings of (M', L) with different prime degrees > 3 ($\geqslant 3$, if M' is irreducible) are nonhomeomorphic.*

We remark that, since a branched covering $\pi : M \to (M', L)$ restricts to a covering $\pi : \partial M \to \partial M'$, we need only to consider the case where ∂M is an union of tori. By the equivariant sphere theorem [7], the hypothesis concerning the link L and the exterior E imply that M is irreducible. Then Thurston's orbifold theorem [1] asserts that the quotient orbifold $\mathcal{O}_d = M/\mathbb{Z}_d$ is geometrisable. Moreover, the classification of $\mathbb{Z}_d$-actions on $T^2 \times I$ shows that, the JSJ family of $\mathcal{O}_d$ lifts to the JSJ family of M. Since E is a manifold with boundary it is geometrisable [5,8].

In the proof of Theorem 1.1 we will argue on the different geometric decompositions of E, distinguishing three cases, according to whether E contains a hyperbolic piece, is a Seifert manifold or is a nontrivial graph manifold. In the case where E is a Seifert manifold we give examples (see Examples 1 and 2) where the degree is not determined, when it is not prime. On the other hand, when E contains a hyperbolic piece, we only require that $\pi : M \to (M', L)$ be a *strongly cyclic* branched covering, that is, the group of covering transformations is cyclic and the ramification order equals the degree of π along all components of L. Notice that when π has prime degree, it is always strongly cyclic.

2. E contains a hyperbolic piece

Proposition 2.1. *If E contains a hyperbolic piece, then there is at most one number $d > 3$ ($\geqslant 3$, if M' is irreducible) for which M is a strongly cyclic branched covering of M' over L with degree d.*

Proof. Since $d > 3$, the JSJ family of $\mathcal{O}_d$ contains only tori and thus coincides with the JSJ family of E (if M' is irreducible, this happens for $d \geqslant 3$, since in this case there are no spheres that cut L transversely in exactly 3 points). Moreover, the proof of Thurston's orbifold theorem shows that $\mathcal{O}_d$ must also have some hyperbolic piece [1]. The hyperbolic pieces of $\mathcal{O}_d$ lift to the hyperbolic pieces of M and

$$0 < \sum_j \mathrm{vol}\big(\mathcal{O}_d^j\big) = \frac{\sum_i \mathrm{vol}(M^i)}{d} < \infty, \tag{1}$$

where $\mathcal{O}_d^j$ and M^i are the hyperbolic pieces of the decompositions of $\mathcal{O}_d$ and M. We may regard each $\mathcal{O}_d^j$ that touches L as a hyperbolic cone-manifold with cone angles $2\pi/d$. Again by [1], when we increase the cone angles from zero, no degeneration occurs, and the Schläfli formula shows that the hyperbolic volume of $\mathcal{O}_d^j$ either decreases strictly or remains constant according to whether it contains a component of L or not. Then the left-hand member of (1) is a non-decreasing function of d. Since the right-hand member is a decreasing function of d, this shows that (1) can be verified by at most one value of d. □

3. E is a Seifert fibred space

Proposition 3.1. *If E is a Seifert fibred space then there is at most one prime number $d \geqslant 1$ for which M is a branched covering of M' over L with degree d.*

A *Seifert fibration* of (M', L) is a Seifert fibration of M' such that L is an union of fibres. Since L is a prime link, a Seifert fibration of E extends to a Seifert fibration of (M', L). By lifting this fibration to M we obtain a Seifert fibration preserved by the covering transformations. Therefore $\pi: M \to M'$ induces an orbifold covering $\varphi: \mathcal{B}M \to \mathcal{B}\mathcal{O}_d$, where $\mathcal{B}X$ is the base orbifold of the Seifert fibration of X. Let u be the degree of this covering and v the degree of the restriction of π to a general regular fibre of M. In these conditions we say that the covering has *type* (u, v) and we have $d = uv$. Since d is prime, we have only to consider two types of coverings, namely $(u, v) = (1, d)$ and $(u, v) = (d, 1)$.

Denote the fibration of M by $(g, n|e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$, where n is the number of boundary components of the underlying surface F of $\mathcal{B}M$, g is the genus of the closed surface obtained by gluing discs to those components (we consider $g < 0$ for non-orientable surfaces), $e_0 \in \mathbb{Q}$ is the Euler number of the Seifert fibration, and $\beta_i/\alpha_i \in \mathbb{Q}/\mathbb{Z}$ are the Seifert invariants of the fibres of M. If $n = 0$ we omit from the notation; if $n \neq 0$, then $e_0 = 0$. The orbifold $\mathcal{B}M$ will be represented by $F(\alpha_1, \ldots, \alpha_m)$.

Consider a fibred torus T in M and its image T' in M'. Let β/α and β'/α' be the Seifert invariants of the central fibres of T and T'. A generalisation of a calculus of Seifert [9] shows that $\beta'/\alpha' \equiv (v/u_T)(\beta/\alpha)(\bmod 1)$, where u_T is the degree of $\varphi_{|\mathcal{B}T}$. This relation shows that when d is not prime uniqueness may fail, even when the manifolds are closed and the Euler numbers of their Seifert fibrations are nonzero (note that the branched coverings in the following examples are not strongly cyclic).

Example 1. Let $M \cong (0|0; 1/3, 1/3, 1/3)$ and $(M', L) \cong (\mathbb{S}^2 \times \mathbb{S}^1, \{x, y, z\} \times \mathbb{S}^1)$. Then M is a $(1, 3n)$-cyclic branched cover of M' over L, for every $n \in \mathbb{N}$.

Example 2. Consider the manifolds $M \cong (0| - 4/3; 1/3, 1/3, 1/3, 1/3)$ and $M' \cong L_{4,1}$ and the link L made of three fibres of the fibration $(0| - 4; -)$ of M'. Then M is a $(2, 6)$- and a $(4, 12)$-cyclic branched cover of M' over L.

When the Euler number of the Seifert fibration of M is nonzero and the degree is supposed to be prime, its uniqueness is immediate, since $e_0(M')/e_0(M) = v/u = d^{\pm 1}$. If we do not assume that the degree is prime, we can show, by reasoning with the Seifert invariants, that uniqueness still holds for strongly cyclic branched coverings.

For the general case, we restrict ourselves to prime degrees and we consider the Euler characteristics of the bases. First observe that $\chi(\mathcal{B}M) = u\chi(\mathcal{B}\mathcal{O}_d)$. The orbifold $\mathcal{B}\mathcal{O}_d$ can be obtained from $\mathcal{B}M'$ by multiplying by d the multiplicities of certain points (given by the components of L). If the multiplicities of these points of $\mathcal{B}M'$ are $a_1, a_2, \ldots, a_n$, then, by setting $r = \sum_{i=1}^{n} 1/a_i$, the Euler characteristic of the base of $\mathcal{O}_d$ is given by

$$\chi(\mathcal{B}\mathcal{O}_d) = \chi(\mathcal{B}M') - r + r/d. \tag{2}$$

The manifold M has an unique Seifert fibration except when it is a lens space, a prism space, a solid torus, a twisted I-bundle over the Klein bottle or its double [10]. These last two manifolds have however only two Seifert fibrations, which have bases of equal (null) Euler characteristic. On the other hand, (M', L) has an unique Seifert fibration except when it is a lens space and L is composed by one or both axis of M', a prism space with L a fibre such that E is the twisted I-bundle over the Klein bottle, or a solid torus with L composed only by the axis of M'.

Claim. *If (M', L) is one of exceptional cases above, then M is either a lens space, a prism space or a solid torus.*

Proposition 3.2. *Proposition* 3.1 *is true when M is not a lens space, a solid torus or a prism space.*

Proof. By the claim, (M', L) admits an unique fibration and $\chi(\mathcal{B}M)$ and $\chi(\mathcal{B}M')$ are well defined. Then from (2) we obtain $\chi(\mathcal{B}M) = u(\chi(\mathcal{B}M') - r + r/d)$ which shows that given u there is at most one possible value for d. Since $u = 1$ or d there are at most two solutions (if $\chi(\mathcal{B}M) = 0$ there is only one), each corresponding to one type of covering. We denote these two solutions by d_1 and d_2, according to $u = 1$ or $u = d$. Suppose that there is a $(1, d_1)$-

and a $(d_2, 1)$-covering. Since the orbifolds $\mathcal{B}M$ and $\mathcal{B}\mathcal{O}_{d_1}$ coincide, they have the same topological type. On the other hand, there is a covering $\mathcal{B}M \xrightarrow{d_2} \mathcal{B}\mathcal{O}_{d_2}$, where the number over the arrow stands for the degree of the covering. This covering induces a cyclic branched covering $|\mathcal{B}M| \xrightarrow{d_2} |\mathcal{B}M|$. The only surfaces that have a cyclic branched covering over itself are the sphere (with two branch points), the projective plane and the disc (both with one branch point). Since $\mathcal{B}M$ and $\mathcal{B}\mathcal{O}_{d_1}$ have the same number of singular points, the possibilities for $\mathcal{B}M \xrightarrow{d_2} \mathcal{B}\mathcal{O}_{d_2}$ are $\mathbb{S}^2(a,a,b,b) \xrightarrow{2} \mathbb{S}^2(a,b,2,2)$, $\mathbb{S}^2(a,a,a) \xrightarrow{3} \mathbb{S}^2(a,3,3)$, $\mathbb{S}^2(a,b,b) \xrightarrow{2} \mathbb{S}^2(2a,b,2)$ and $D(a,a) \xrightarrow{2} D(a,2)$. We conclude the proof by getting a contradiction in each case, by using Seifert invariants. □

When M is a solid torus and $\pi : M \to (M', L)$ is a branched covering, M' is also a solid torus and L is the axis of M'. Then L is a trivial link and in fact π can have any degree. When M is a lens space or a prism space, then M' is also a lens space or a prism space, and Proposition 3.1 is proved for these remaining cases by an analysis of the Seifert invariants of the different fibrations of M and (M', L). Note that when L is an axis of a lens space $L_{a,b'}$, it is a trivial link, and in fact $L_{a,b}$ is a branched covering of degree d over $(L_{a,b'}, L)$, when $db' \equiv b (\mathrm{mod}\, a)$.

4. E is a nontrivial graph manifold

Proposition 4.1. *If E is a nontrivial graph manifold whose JSJ graph is a tree, then there is at most one prime number $d \geqslant 1$ for which M is a branched covering of M' over L with degree d.*

We explain the proof when $d \geqslant 3$. Since L is a prime link, $\mathcal{O}_d$ must be a nontrivial graph orbifold with the same JSJ graph as E. The manifold M is also a nontrivial graph manifold. Consider the JSJ graphs Γ and Γ' of M and $\mathcal{O}_d$, respectively. Suppose that there are two branched coverings π_p and π_q with degrees p and q from M to M', corresponding to the actions f_p and f_q. When $d \geqslant 3$, they induce maps $\Gamma \to \Gamma'$ that we will also denote by π_i. The proof that $p = q$ is combinatorial, and it uses the following lemma, which is proven by induction over the distances to v.

Lemma 4.2. *If a vertex v of Γ is p-fixed, then $d(v,w) = d(\pi_p(v), \pi_p(w))$, for every vertex w of Γ.*

A vertex v of Γ is *p-fixed* if $\pi_p^{-1}(\pi_p(v)) = \{v\}$. We call the vertices of valence 1 *terminal* and the remaining ones *interior*. Denote by $\mathcal{I}[\mathcal{T}]$ the subgraph determined by the interior [terminal] vertices of Γ and the edges joining them ($\mathcal{I}$ is always connected and $\mathcal{T} = \Gamma - \mathcal{I}$ is totally disconnected except when $\Gamma = K_2$). The notations for the vertices of Γ' are analogous to those for the vertices of Γ.

Proposition 4.3. *Proposition* 4.1 *is true when Γ is a tree.*

Proof. Clearly $\pi_p(\mathcal{T}) \subseteq \mathcal{T}'$ and $\pi_q(\mathcal{T}) \subseteq \mathcal{T}'$. If $\pi_p^{-1}(\mathcal{T}')$ and $\pi_q^{-1}(\mathcal{T}')$ are both contained in $\mathcal{T}$, we may restrict the original coverings to coverings $\pi_i : M_0 \to M_0'$, where M_0 and M_0' are the submanifolds of M and M' determined by $\mathcal{I}$ and $\mathcal{I}'$. By induction, we have the uniqueness of p (the initial cases for $\mathcal{I}'$ are easily proved).

Suppose now that there is a vertex v' of $\mathcal{T}'$ such that $v_p = \pi_p^{-1}(v') \in \mathcal{I}$. Since Γ has no cycles, $\pi_p^{-1}(v')$ has only one vertex from $\mathcal{I}$ and the only p-fixed vertex from Γ is v_p. Therefore, the only vertex of Γ' that contains components of L is v' and $\Gamma - v_p$ is composed by p copies of $\Gamma' - v'$. Denote by l the maximal distance from v' to the remaining vertices of Γ'. By Lemma 4.2, the distances to v_p and $v_q = \pi_q^{-1}(v')$ are preserved by π_p and π_q, respectively. If $v_q \neq v_p$, there would exist vertices from Γ at a distance from v_q greater than l, namely the vertices at a distance l from v_p, but in a component of $\Gamma - v_p$ that does not contain v_q, which is absurd. Thus $v_q = v_p$ which by Proposition 3.1 completes the proof. □

Lemma 4.4. *If Γ is not a tree and it has a vertex v which is fixed for both actions, then $p = q$.*

Proposition 4.5. *Proposition* 4.1 *is true when Γ is not a tree.*

Proof. Let c be a minimal cycle in Γ. This cycle projects under π_p to a subtree of Γ', therefore there must be a vertex v_1 of c such that the two vertices x and y of c adjacent to v_1 have the same image (if there was an edge of c connecting two vertices with the same image, this edge would project to a cycle). If v_1 were not p-fixed, then $\pi_p(xv_1)$ and $\pi_p(v_1y)$ are two different edges connecting $\pi_p(v_1)$ and $\pi_p(x)$. Therefore they define a cycle in Γ' (absurd). This shows that v_1 is p-fixed.

Since c is minimal, c has exactly one more p-fixed vertex v_2 and c is the union of two of the p minimal paths connecting v_1 to v_2. By the same reasoning applied to the covering of degree q we conclude that there are exactly two q-fixed vertices w_1 and w_2 in c and that c is the union of two of the q minimal paths from w_1 to w_2. By Lemma 4.4, we can suppose that v_1 and v_2 are not q-fixed and that w_1 and w_2 are not p-fixed.

Suppose that $\pi_q(v_1) \neq \pi_q(v_2)$ and consider a minimal path connecting v_1 and v_2, not contained in c. Since v_1 is not q-fixed, there is one vertex $x \neq v_1$ in this path such that $\pi_q(x) = \pi_q(v_1)$. Suppose without loss of generality that v_1 is closer to w_1 than to w_2. Then $\text{length}(v_1xw_1v_1) < \text{length}(v_1xv_2) + \text{length}(v_2w_1v_1) = 2\,\text{length}(v_1v_2) = \text{length}(c)$, which contradicts minimality of c. This shows that $\pi_q(v_1) = \pi_q(v_2)$. Similarly $\pi_p(w_1) = \pi_p(w_2)$. We can thus set $\pi_q^{-1}(\pi_q(v_1)) = \{v_i\}_{i=1,\dots,q}$ and $\pi_p^{-1}(\pi_p(w_1)) = \{w_j\}_{j=1,\dots,p}$.

Let $C = \bigcup_{l,k} f_q^l(f_p^k(c))$. For all x in C consider the union Γ_x of all components of $\Gamma - x$ that do not contain $C - x$. Suppose without loss of generality that x is not p-fixed. Then $f_p(\Gamma_x) = \Gamma_{f_p(x)}$, since C is invariant under $\langle f_p \rangle$. Then Γ_x is a (possibly disconnected and possibly empty) tree, because Γ' is a tree.

Suppose that there are paths of Γ, not contained in C, connecting two vertices x and y of C, and consider a minimal one. Then, since Γ' has no cycles, we have $\pi_p(x) = \pi_p(y)$ and $\pi_q(x) = \pi_q(y)$, which contradicts the construction. Therefore every vertex of $\Gamma - C$ is in some Γ_x.

Now remove all terminal vertices from Γ' and iterate this process. Then, for every $x \in \Gamma$, $\pi_p(\Gamma_x)$ and $\pi_q(\Gamma_x)$ are exhausted at the same time, since the distances of the vertices of Γ_x to x are preserved by π_p and π_q. This shows that the last vertex to be removed is $\pi_p(w_i) = \pi_q(v_j)$. Since the valence of this vertex at the last step is both p and q (which can be seen by considering both projections $\Gamma \to \Gamma'$), this concludes the proof that $p = q$. □

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