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Ordinary Differential Equations

On the normal form of a system of differential equations with nilpotent linear part

Forme normale d'un système d'équations différentielles à partie linéaire nilpotente

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Abstract

We consider prenormal forms associated to generic perturbations of the system $\dot{x} = 2y,\ \dot{y} = 3x^2$. It is known that they have a *formal* normal form $\dot{x} = 2y + 2x\,\Delta^*,\ \dot{y} = 3x^2 + 3y\,\Delta^*$, where $\Delta^* = x + A_0(y^2 - x^3)$ [Differential Equations 158 (1) (1999) 152–173]. We show that the series A_0 and the normalizing transformations are divergent, but 1-summable. ***To cite this article: M. Canalis-Durand, R. Schäfke, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On considère des formes prénormales associées à des perturbations génériques du système $\dot{x} = 2y,\ \ \dot{y} = 3x^2$. Il est connu qu'elles admettent une forme normale formelle $\dot{x} = 2y + 2x\,\Delta^*,\ \ \dot{y} = 3x^2 + 3y\,\Delta^*$, où $\Delta^* = x + A_0(y^2 - x^3)$ [Differential Equations 158 (1) (1999) 152–173]. Nous démontrons que A_0 et les transformations normalisantes sont divergentes, mais 1-sommable. ***Pour citer cet article : M. Canalis-Durand, R. Schäfke, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

On considère le système d'équations différentielles :

$$\begin{cases} \dot{x} = 2y + 2x\,\Delta(x, y), \\ \dot{y} = 3x^2 + 3y\,\Delta(x, y), \end{cases} \tag{Δ}$$

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où $\Delta \in \mathbb{C}\{x,y\}$, $\Delta = x + \cdots$ et $\dot{} = \mathrm{d}/\mathrm{d}z$. Loray [5] a montré que (Δ) peut être transformé en un système dit équivalent, par la substitution $(x \leftarrow xU^2(x,y), y \leftarrow yU^3(x,y),\ \mathrm{d}z \leftarrow U(x,y)^{-1}\mathrm{d}z)$, $U(x,y)$ étant une série convergente ou formelle telle que $U(0,0)=1$. Le système équivalent le plus « simple » est appelé *forme normale* et est noté (Δ^*).

Théorème 0.1 [5]. *Soient* $\Delta \in \mathbb{C}\{x,y\}$, $\Delta = x + \cdots$, *il existe alors une unique série formelle* $U(x,y) = 1 + \cdots$ *transformant* (Δ) *en sa forme normale (formelle)* (Δ^*), $\Delta^* = x + A_0(h)$, *où* $h = y^2 - x^3$ *et* $A_0 \in h\mathbb{C}[[h]]$. *L'*équation de transformation *reliant* Δ, Δ^* *et* U *s'écrit* :

$$(U + 2xU_x + 3yU_y)\Delta^* = \Delta\big(xU^2, yU^3\big) - 2yU_x - 3x^2U_y. \tag{1}$$

Nous étudions le problème de la convergence de cette forme normale Δ^* et de la transformation normalisante U. Dans [2], le calcul des 30 premiers termes de la série $A_0(h)$ dans plusieurs exemples permettaient de conjecturer numériquement son caractère divergent mais Gevrey d'ordre 6. Dans cette Note, nous montrons le résultat suivant :

Théorème 0.2. *Les séries* $U(x,y) = \sum_{k,l} b_{kl}\, x^k y^l$ *et* $A_0(h) = \sum_{m\geqslant 1} A_m h^m$ *du théorème précédent sont Gevrey* 1 *dans le degré homogène ; i.e. il existe* $K, A > 0$ *tels que pour tout* k, l, m

$$|A_m| \leqslant KA^{6m}(6m)!, \qquad |b_{kl}| \leqslant KA^{2k+3l}(2k+3l)!,$$

où le type $A = 0.1844 \pm 0.0001$. *La série* $A_0(t^6)$ *est* 1*-sommable* (*cf.* [7]) *si* $\arg t$ *n'est pas congru à* $\pi/6$ *modulo* $\pi/3$. *Finalement, il existe une fonction analytique non identiquement nulle* Q *telle que* $Q(\Delta) \neq 0$ *implique que* U *and* Δ^* *sont divergentes et que le type est optimal.*

Nous transformons le système à l'aide du changement de variables noté $(\mathcal{C})$: $x = q(s)\,t^2$, $y = q'(s)t^3$, où q est une fonction elliptique vérifiant $q'^2 = q^3 + 1$, $q(0) = \infty$. L'équation de transformation (1) se simplifie en

$$\big(\widetilde{U} + t\widetilde{U}_t\big)D^* = D\big(s, t\widetilde{U}\big) - 2t\widetilde{U}_s, \tag{2}$$

où $\widetilde{U}$, D et D^* correspondent à U, Δ et Δ^* après le changement de variables $(\mathcal{C})$. Comme on veut que les solutions puissent être exprimées en fonction des variables x et y, on doit imposer des conditions de symétrie.

On applique la *transformation de Borel formelle par rapport à* t, notée $\widehat{\mathcal{B}}$, à toutes les séries précédentes dans l'équation (2). Soit donc $\mathbf{W}(s,\tau) = \widehat{\mathcal{B}}(\frac{1}{t}(\widetilde{U}(s,t)-1))$, $\mathbf{E}^*(\tau) = \widehat{\mathcal{B}}(t^{-2}D^*(s,t) - q(s))$. Éq. (2) s'écrit alors « dans le plan de Borel » :

$$\mathcal{L}(\mathbf{W}, \mathbf{E}^*) := 2\mathbf{W}_s + q(s)\tau\mathbf{W} + \mathbf{E}^* = \mathcal{G}(\mathbf{W}, \mathbf{E}^*) \tag{3}$$

(l'expression de $\mathcal{G}(\mathbf{W}, \mathbf{E}^*)$ est détaillée dans Éq. (10) de la version anglaise).

L'équation linéarisée de (3)

$$\mathcal{L}(\mathbf{W}, \mathbf{E}^*) = 2\mathbf{W}_s + q(s)\tau\mathbf{W} + \mathbf{E}^* = \mathbf{G} \tag{4}$$

admet une solution unique $(\mathbf{W}, \mathbf{E}^*)$ satisfaisant les conditions de symétrie. On obtient en particulier :

$$\mathbf{E}^*(\tau) = \frac{1}{\mathbf{R}(\tau)}\operatorname{res}\big(\mathrm{e}^{\tau I(s)/2}\mathbf{G}(s,\tau),\ s=0\big) \quad \text{avec } \mathbf{R}(\tau) = \operatorname{res}\big(\mathrm{e}^{\tau I(s)/2},\ s=0\big), \tag{5}$$

où I est une primitive de q. On détermine les zéros de la fonction $\mathbf{R}$ de (5) qui introduisent les singularités de $\mathbf{E}^*$ (et $\mathbf{W}$) dans le plan de Borel.

Puis on résout le problème non linéaire en utilisant une méthode de point fixe dans des espaces précisés. Il existe alors une unique solution analytique $(\mathbf{W}, \mathbf{E}^*)$, périodique et symétrique. Donc la solution formelle (U, Δ^*) de (1) est Gevrey 1. De plus, $(\mathbf{W}, \mathbf{E}^*)$ a une croissance au plus exponentielle lorsque $\tau \to \infty$. Ceci prouve la 1-sommabilité de $A_0(t^6)$ du Théorème 0.2 et la 1-sommabilité de $\widetilde{U}(s,t)$ par rapport à t. En utilisant un autre espace de fonctions, on montre que $\mathbf{E}^*(\tau)$ a des singularités en les zéros de $\mathbf{R}$ dans le cas générique ; ceci implique la divergence de la solution formelle dans ce cas.

1. Introduction

We consider the following system of differential equations:

$$\begin{cases} \dot{x} \;=\; 2y + 2x\,\Delta(x,y), \\ \dot{y} \;=\; 3x^2 + 3y\,\Delta(x,y), \end{cases} \tag{Δ}$$

where $\Delta \in \mathbb{C}\{x,y\}$, $\Delta = x + \cdots$ and $\dot{} = \mathrm{d}/\mathrm{d}z$. For simplicity, we denote such a system by (Δ). The above system is a *prenormal* form of a generic perturbation of the system $\dot{x} = 2y,\ \dot{y} = 3x^2$ with Hamiltonian $h = y^2 - x^3$. [3] have shown that this prenormal form can always be reached by a convergent transformation.

Loray [5] showed that (Δ) can be transformed into a system $(\tilde{\Delta})$ – of the same form, but with Δ replaced by $\tilde{\Delta}$ – by a substitution $(x \leftarrow xU^2(x,y),\ y \leftarrow yU^3(x,y),\ \mathrm{d}z \leftarrow U(x,y)^{-1}\,\mathrm{d}z)$, $U(x,y)$ a convergent or formal series with $U(0,0) = 1$. The *transformation equation* connecting $\Delta,\ \tilde{\Delta}$ and U is

$$(U + 2xU_x + 3yU_y)\tilde{\Delta} = \Delta\big(xU^2, yU^3\big) - 2yU_x - 3x^2U_y, \tag{6}$$

where subscripts $_x$ etc. denote partial derivatives. If U is convergent, we say that (Δ) is *analytically equivalent* to $(\tilde{\Delta})$. If U is only a formal series, we say that (Δ) is *formally equivalent* to $(\tilde{\Delta})$.

It is natural to ask for the “simplest” $(\tilde{\Delta})$ equivalent to a given (Δ), i.e., for a *normal form* which will be denoted by (Δ^*).

Theorem 1.1 [5]. *Given $\Delta \in \mathbb{C}\{x,y\}$, $\Delta = x + \cdots$, there is a unique formal series $U(x,y) = 1 + \cdots$ transforming (Δ) into its (formal) normal form (Δ^*), $\Delta^* = x + A_0(h)$, where $h = y^2 - x^3$ and $A_0 \in h\mathbb{C}[[h]]$. One has*

$$(U + 2xU_x + 3yU_y)\Delta^* = \Delta\big(xU^2, yU^3\big) - 2yU_x - 3x^2U_y. \tag{7}$$

Here, the question of convergence of this normal form and the normalizing transformation arises. [2] calculated the first 30 terms of $A_0(h)$ for several examples and obtained numerical evidence that it is divergent, but of Gevrey order 6 (see below). We will show

Theorem 1.2. *The series $U(x,y) = \sum_{k,l} b_{kl}\ x^k y^l$ and $A_0(h) = \sum_{m\geqslant 1} A_m h^m$ of the preceding theorem are Gevrey 1 in the weighted degree; i.e., there exist $K, A > 0$ such that for all k, l, m*

$$|A_m| \leqslant K A^{6m}(6m)!, \qquad |b_{kl}| \leqslant K A^{2k+3l}(2k+3l)!,$$

where the type $A = 0.1844 \pm 0.0001$. *The series $A_0(t^6)$ is 1-summable (cf. [7]) if $\arg t$ is not congruent to $\pi/6$ modulo $\pi/3$. Finally, there exists a non-zero analytic function*[1] *Q such that $Q(\Delta) \neq 0$ implies that U and Δ^* are divergent and the above type is optimal.*

It can also be shown that $U(x,y)$ is 1-summable in a sense too cumbersome to state here (see below Theorem 4.2). The proof is given in the subsequent sections.

2. Transformation of the coordinates

The solution of the unperturbed system suggests the following coordinate transformation denoted by $(\mathcal{C})$

$$\begin{cases} x \;=\; q(s)\,t^2, \\ y \;=\; q'(s)\,t^3, \end{cases} \tag{8}$$

[1] More precisely, Q is an analytic function of *all* the coefficients of Δ.

where q is an elliptic function[2] verifying $q'^2 = q^3 + 1,\ q(0) = \infty$. Observe that $h = y^2 - x^3 = t^6$. Each point (x, y) outside the cusp $y^2 = x^3$ corresponds to 6 points (s, t) where s is an element of the hexagon $\mathcal{H}$ defined[3] by the first six zeroes of q. Introducing $\widetilde{U}(s,t) = (\mathcal{C}U)(s,t) = U(q(s)t^2, q'(s)t^3)$ and analogously $D = \mathcal{C}\Delta,\ D^*(s,t) = (\mathcal{C}\Delta^*)(s,t) = q(s)t^2 + A_0(t^6)$, the transformation equation (7) is simplified to

$$\big(\widetilde{U} + t\widetilde{U}_t\big)\, D^* = D\big(s, t\widetilde{U}\big) - 2t\widetilde{U}_s. \tag{9}$$

Observe that we are not looking for arbitrary solutions of this equation but for solutions with certain symmetries that can be expressed as series in x and y. This can be conveniently written down using the following $\mathbb{C}$-vector spaces. Let H_m the set of all homogeneous polynomials $\sum_{2k+3l=m} a_{kl}x^k y^l$ of weighted degree m and $\mathcal{E}_m$ the set of all functions meromorphic on $\mathbb{C}$ that are p_0- and p_1-periodic,[4] whose only pole in the hexagon $\mathcal{H}$ is 0 and of order $\leqslant m$ and such that $f(\rho s) = \rho^{-m} f(s)$.

The coordinate transformation (8) induces a bijection I between H_m et $\mathcal{E}_m$ by $I(f)(s)t^m = \mathcal{C}(f)(s,t)$. Put $\mathcal{S} = \{\sum_{m=0}^{\infty} f_m(s)t^m / f_m \in \mathcal{E}_m\}$ and $\mathcal{S}_n = \{\sum_{m=n}^{\infty} f_m(s)t^m / f_m \in \mathcal{E}_m\}$.

Using the above notation, given $D \in q(s)t^2 + \mathcal{S}_6$, we are looking for a solution $\widetilde{U} \in 1 + \mathcal{S}_5$, $D^* = q(s)t^2 + A_0(t^6)$ of (9).

We now carry over Eq. (9) to the Borel plane, i.e., we apply the *formal Borel transform with respect to t* denoted by $\widehat{\mathcal{B}}$ and defined by $\widehat{\mathcal{B}}(t^n) = \tau^{n-1}/(n-1)!$ for $n \geqslant 1$, to all the preceding series. Let $D(s,t) = q(s)t^2 + t^2 E(s,t)$, $E(s, t(1+tW))(1+tW)^2 = \sum_{n=0}^{\infty} F_n(s,t)t^n\ W^n$ and $\mathbf{F}_n(s,\tau) = \widehat{\mathcal{B}}(F_n(s,t))$, $\mathbf{W}(s,\tau) = \widehat{\mathcal{B}}(\frac{1}{t}(\widetilde{U}(s,t) - 1))$, $\mathbf{E}^*(\tau) = \widehat{\mathcal{B}}(t^{-2}D^*(s,t) - q(s))$. Using the properties of the Borel transform, in particular $\widehat{\mathcal{B}}(t^2 \frac{\partial}{\partial t} f(t)) = \tau\widehat{\mathcal{B}}(f(t))$, Eq. (9) can be written in the Borel plane

$$\begin{aligned}&\mathcal{L}(\mathbf{W}, \mathbf{E}^*) := 2\mathbf{W}_s + q(s)\tau\mathbf{W} + \mathbf{E}^* = \mathcal{G}(\mathbf{W}, \mathbf{E}^*), \quad \text{where}\\ &\mathcal{G}(\mathbf{W}, \mathbf{E}^*)(s,\tau) := q(s)\tau * \mathbf{W} * \mathbf{W} - 2\mathbf{W} * \mathbf{E}^* - (\tau\mathbf{W}) * \mathbf{E}^* + \mathbf{F}_0 + \sum_{n=1}^{\infty} \mathbf{F}_n * \tau^{n-1} * \mathbf{W}^{*n}/(n-1)!;\end{aligned} \tag{10}$$

here $*$ denotes convolution with respect to τ. The restrictions are now as follows: given are $\tau^3\mathbf{F}_n \in \mathcal{S}_{\max(n,6)}$ and we are looking for $\mathbf{E}^*(\tau) \in \tau^3\mathbb{C}[[\tau^6]]$ and $\tau^2\mathbf{W} \in \mathcal{S}_5$.

3. The linearized problem

Now consider the linearization of (10), i.e., given $\tau^3\mathbf{G} \in \mathcal{S}_6$ find $\tau^2\mathbf{W} \in \mathcal{S}_5$, $\mathbf{E}^*(\tau) \in \tau^3\mathbb{C}[[\tau^6]]$ with

$$\mathcal{L}(\mathbf{W}, \mathbf{E}^*) = 2\mathbf{W}_s + q(s)\tau\mathbf{W} + \mathbf{E}^* = \mathbf{G}. \tag{11}$$

This linear ordinary differential equation can be solved by variation of constants. The "constant" of integration and $\mathbf{E}^*$ are uniquely determined by the conditions (in particular $\mathbf{W}$ has to be a single valued function of s). We find

$$\mathbf{W}(s,\tau) = \mathrm{e}^{-\tau I(s)/2} \int_{\infty}^{s} \mathrm{e}^{\tau I(\sigma)/2}\big(\mathbf{G}(\sigma,\tau) - \mathbf{E}^*(\tau)\big)\,\mathrm{d}\sigma, \tag{12}$$

where I is a primitive[5] of q, and

$$\mathbf{E}^*(\tau) = \frac{1}{\mathbf{R}(\tau)} \operatorname{res}\big(\mathrm{e}^{\tau I(\sigma)/2}\mathbf{G}(\sigma,\tau),\ \sigma = 0\big) \quad \text{with } \mathbf{R}(\tau) = \operatorname{res}\big(\mathrm{e}^{\tau I(\sigma)/2},\ \sigma = 0\big). \tag{13}$$

---

[2] Precisely, $q = 4\mathcal{P}$ where $\mathcal{P}$ is the Weierstrass $\mathcal{P}$-function with parameters $g_2 = 0$ and $g_3 = -1/16$ [1].

[3] $\mathcal{H}$ is the hexagon with vertices $2\rho^j b,\ j = 0,\dots,5$, where $b = \int_0^1 (1-t^3)^{-1/2}\,\mathrm{d}t \simeq 1.40218$, $\rho = \mathrm{e}^{(\pi/3)\mathrm{i}}$.

[4] $p_0 = 2\mathrm{i}a$ and $p_1 = 2\mathrm{i}\rho a$, where $a = \int_1^{\infty} (t^3 - 1)^{-1/2}\,\mathrm{d}t \simeq 2.42865$, are fundamental periods of q.

[5] Precisely, $I = -4\zeta$, where ζ is the Weierstrass' zeta function with parameters $g_2 = 0$ and $g_3 = -1/16$.

In (12), the path of integration is from ∞ to s avoiding the poles of q and such that $\mathrm{Re}(\tau I(\sigma))$ tends to $-\infty$ as $\sigma \to \infty$.

As (11) has a unique solution $(\mathbf{W}, \mathbf{E}^*)$ satisfying the restrictions, we obtain two linear operators $\mathcal{W}(\mathbf{G}) := \mathbf{W}$, $\mathcal{E}(\mathbf{G}) := \mathbf{E}^*$.

The zeroes of the function $\mathbf{R}$ of (13) are important because they introduce the singularities of $\mathbf{E}^*$ (and $\mathbf{W}$) in the Borel plane. Using Laplace's method with error bounds [6] for large τ and Rouché's theorem, we prove

Theorem 3.1. *The function* $\mathbf{R}$ *has exponential growth as* $|\tau| \to \infty$. *Its zeroes* (*other than* $\tau = 0$) *are on the rays* $\arg\tau = (2l+1)\frac{\pi}{6}$, $l = 0, \dots, 5$, *and form six sequences* $m_k\, \mathrm{e}^{(2l+1)\pi \mathrm{i}/6}$, $k \in \mathbb{N}$, $l = 0, \dots, 5$. *One has* $|m_0 - 5.4204| < 0.0002$ *and* $|m_k - t_k| < 12 t_k^{-3}$ *with* $t_k = \frac{\pi}{I(-2b)}(3 + 4k)$ *if* $k \geqslant 1$; *here* $I(-2b) = 2\int_0^1 t(1 - t^3)^{-1/2}\,\mathrm{d}t \simeq 1.72474$.

4. The nonlinear problem

In order to find a solution of (10), it is sufficient to solve the fixed point equation $\mathbf{G} = \mathcal{G}(\mathcal{W}(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ in $\tau^{-3}\,\mathcal{S}_6$, i.e., to solve

$$
\begin{aligned}
&\mathbf{G} = \mathcal{F}(\mathbf{G}), \quad \text{where} \\
&\mathcal{F}(\mathbf{G}) = q(s)\tau * \mathcal{W}(\mathbf{G}) * \mathcal{W}(\mathbf{G}) - 2\mathcal{W}(\mathbf{G}) * \mathcal{E}(\mathbf{G}) - (\tau \mathcal{W}(\mathbf{G})) * \mathcal{E}(\mathbf{G}) \\
&\qquad + \mathbf{F}_0 + \sum_{n=1}^{\infty} \mathbf{F}_n * \tau^{n-1} * \mathcal{W}(\mathbf{G})^{*n}/(n-1)!.
\end{aligned} \tag{14}
$$

Let $\mathcal{D} = \mathbb{C} \setminus \bigcup_{\mu=0}^{5} \mathrm{e}^{\frac{\pi}{6}\mathrm{i}}\, \mathrm{e}^{\mu\frac{\pi}{3}\mathrm{i}}\, [m_0, \infty[$.

Theorem 4.1. *There exists a unique analytic solution* $\mathbf{G} : (\mathbb{C} \setminus \mathcal{R}) \times \mathcal{D} \to \mathbb{C}$ *of* (14). *It is* $\mathcal{R}$*-periodic,*[6] *symmetric*: $\mathbf{G}(\rho s, \rho\tau) = \rho^{-3}\mathbf{G}(s, \tau) = -\mathbf{G}(s, \tau)$ *and* $\mathbf{G}(s, \tau) \in \tau^{-3}\,\mathcal{S}_6$.

As a consequence, $\mathbf{G}$ and thus also $\mathcal{W}(\mathbf{G}) = \mathbf{W}$, $\mathcal{E}(\mathbf{G}) = \mathbf{E}^*$ have convergent series representations at $\tau = 0$, hence by Section 3, the formal solution (Δ^*, U) of (9) is Gevrey 1. We prove this theorem by showing that $\mathcal{F}$ defines a contraction on some subset of the following Banach space: for small $\delta > 0$ and large $M > 0$, let $\mathcal{D}_\delta^M$ the set of all τ with $|\tau| < M$ such that $\arg\tau$ is not in one of the intervals $|\arg\tau - j\frac{\pi}{3} - \frac{\pi}{6}| \leqslant \delta$, $j = 0, \dots, 5$, if $|\tau| \geqslant m_0 - \delta$ and $\mathcal{C}_\delta$ the set of all $s \in \mathbb{C}$ with $\mathrm{dist}(s, \mathcal{R}) > \delta$. Then $\mathcal{O} = \{V : \mathcal{D}_\delta^M \times \mathcal{C}_\delta \to \mathbb{C} \mid V$ holomorphic, bounded, $\mathcal{R}$-periodic, $\tau^{-3}V$ holomorphic at $\tau = 0\}$ is a Banach space equipped with the family of equivalent norms

$$
\|V\|_L = \sup_{s \in \mathcal{C}_\delta,\ \tau \in \mathcal{D}_\delta^M} \bigl|V(s, \tau)\bigr|\, \mathrm{e}^{-L|\tau|}, \quad L > 0.
$$

First, it is shown that $\mathcal{W}$ and $\mathcal{E}$ are bounded linear operators on $\mathcal{O}$ with norms independent of L. Then it is shown that $\mathcal{F}$ defines a contraction on some neighborhood of 0 in $\mathcal{O}$ if L is chosen sufficiently large. As δ and M are arbitrary, the theorem follows.

Theorem 4.2. *Consider* $\theta \in \mathbb{R} \setminus (\frac{\pi}{6} + \frac{\pi}{3}\mathbb{Z})$. *For sufficiently small* $\delta > 0$, *there exists* $K, M > 0$ *such that* $|\mathbf{G}(s, \tau)| \leqslant M \exp(K|\tau|)$ *for all* $s, \tau \in \mathbb{C}$ *with* $\mathrm{dist}(s, \mathcal{R}) > \delta$ *and* $|\arg\tau - \theta| < \delta$.

[6] $\mathcal{R} = \mathbb{Z}p_0 + \mathbb{Z}p_1$ is the lattice of periods of q.

Observe that this theorem implies that also $\mathbf{E}^* = \mathcal{E}(\mathbf{G})$ is at most of exponential growth as $\tau \to \infty$. This proves the 1-summability of $A_0(t^6)$ stated in Theorem 2.2. We obtain also the 1-summability of $\widetilde{U}(s,t) = (\mathcal{C}U)(s,t)$ with respect to t, but it is too cumbersome here to state this in terms of x, y.

Here, we can only give a rough idea of the proof of Theorem 4.2. We introduce a certain subset $\mathcal{A}_\theta$ of $\mathbb{C}$ of s that can be joined to ∞ by a path $\gamma(\sigma)$, $-\infty < \sigma \leqslant 0$ on which $\mathrm{Re}(\mathrm{e}^{\theta \mathrm{i}} I(\sigma))$ increases. Then we obtain an inequality containing convolutions for $f(r) = \sup\{|\mathbf{G}(s,t)| \mid s \in \mathcal{A}_\theta,\ |\arg\tau - \theta| < \delta, |\tau| \leqslant r\}$. Following an idea of B.L.J. Braaksma and W. Walter, we show, using the Laplace transform, that the corresponding equation has a solution having at most exponential growth and that it is a majorant of $f(r)$. Finally we discuss $s \in \mathbb{C} \setminus \mathcal{A}_\theta$.

In order to show the divergence of the formal solution in the generic case, it is sufficient to show that the Borel transform $\mathbf{E}^*(\tau)$ has a singularity at $\tau = \tau_j = m_0 \exp(\frac{\pi}{6}\mathrm{i} + j\frac{\pi}{3}\mathrm{i})$, $j = 0, \ldots, 5$. This cas be done in a way similar to the preceding proof, but working with a different function space (a very simplified version of the resurgent functions of Ecalle [4]).

So we consider the subset $\widetilde{D}$ of the universal covering $\widehat{D}$ of $\{\tau \in \mathbb{C} \mid |\tau| < m_0 + 1\} \setminus \{\tau_0, \ldots, \tau_5\}$ of all points that can either be joined (in $\widehat{D}$) to 0 by a segment or whose distance from one of the τ_j, $j = 0, \ldots, 5$, is smaller than 1. We write this decomposition $\widetilde{D} = \check{D} \cup \bigcup_{j=0}^{5} B_j$. Let $\mathcal{O}$ the Banach space of all holomorphic functions $\mathbf{G} : \mathcal{C}_\delta \times \widetilde{D} \to \mathbb{C}$ bounded on $\mathcal{C}_\delta \times \check{D}$ that can be written as

$$\mathbf{G}(s,\tau) = \alpha_j^{\mathbf{G}}(s,\tau) + \beta_j^{\mathbf{G}}(s,\tau)(\tau - \tau_j)\log(\tau - \tau_j), \quad 0 < |\tau - \tau_j| < 1,\ j = 0, \ldots, 5,$$

where $\alpha_j^{\mathbf{G}}$ and $\beta_j^{\mathbf{G}}$ are bounded holomorphic functions on $\mathcal{C}_\delta \times \{|\tau - \tau_j| < 1\}$.

We show that (14) has a unique solution $\mathbf{G}$ in the closed subspace $\mathcal{O}_{6,6}$ of all $\mathbf{H} \in \tau^6\mathcal{O}$ such that $(\tau - \tau_j)^{-6}\beta_j^{\mathbf{H}}(s,\tau)$ are also bounded for $j = 0, \ldots, 5$. By (13), there exists a constant $a_0 \in \mathbb{C}$ such that $\mathbf{E}^* - \sum_{j=0}^{5} a_0 \rho^{-2j}/(\tau - \tau_j) \in \mathcal{O}_{6,5}$. We show that a_0 depends analytically upon the coefficients of Δ and, by considering the example $\Delta = x + \varepsilon y$, $\varepsilon \neq 0$ small, we show that it is a nontrivial function. This proves the divergence of D^* (and hence of Δ^* and consequently U) in the generic case and yields the function Q of the theorem.

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