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Ordinary Differential Equations

Linearization of germs of hyperbolic vector fields

Linéarisation de germes de champs de vecteurs hyperboliques

Patrick Bonckaert [a], Vincent Naudot [b], Jiazhong Yang [c]

[a] *Limburgs Universitair Centrum, 3590 Diepenbeek, Belgium*
[b] *University of Groningen, Department of Mathematics, P.O. Box 800m, 9700 AV Groningen, The Netherlands*
[c] *School of Mathematical Sciences, Peking University, Beijing 100871, PR China*

Abstract

We develop a normal form to express asymptotically a conjugacy between a germ of resonant vector field and its linear part. We show that such an asymptotic expression can be written in terms of functions of the Logarithmic Mourtada type. ***To cite this article: P. Bonckaert et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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Résumé

Nous développons une forme normale pour exprimer asymptotiquement une conjugaison entre un germe de champ de vecteur résonant et sa partie linéaire. Nous montrons qu'une telle conjugaison peut s'ecrire sous en terme de fonctions de fonctions dites LMT. ***Pour citer cet article : P. Bonckaert et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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1. Introduction and main results

Let $\mathcal{X}$ be a smooth (not necessarily analytic) vector field defined in a vicinity of a hyperbolic equilibrium point in an n-dimensional $\mathbb{R}$-manifold, which will be assumed to be $\mathbb{R}^n$, as our discussion is completely local. We also assume that the equilibrium point is fixed at the origin O. This means that in terms of ordinary differential equation $\mathcal{X}$ can be written in the following form.

$$\mathcal{X}: \quad \dot{X} = (A + N) \cdot X + F(X), \tag{1}$$

where $X \in \mathbb{R}^n$, A and N are $n \times n$ matrices, A is diagonal and N nilpotent and they commute, and F is a smooth function that consists of non-linear terms only, i.e., $\|F(X)\| = \mathcal{O}(\|X\|^2)$, where $\|.\|$ represents any norm on $\mathbb{R}^n$.

E-mail addresses: patrick.bonckaert@luc.ac.be (P. Bonckaert), v.naudot@math.rug.nl (V. Naudot), jyang@math.pku.edu.cn (J. Yang).

Consider the associated linear vector field $\mathcal{X}_l$: $\dot{X} = (A + N)X$. We say that $\mathcal{X}_l$ and $\mathcal{X}$ are C^r ($r \geqslant 0$) conjugated if there exists a C^r diffeomorphism which conjugates $\mathcal{X}_l$ with $\mathcal{X}$.

From Poincaré's works, we know how to construct a normal form procedure that enables us to write formally a conjugacy between $\mathcal{X}$ and $\mathcal{X}_l$ as long as the spectrum of A does not possess any resonance. This latter assumption is generic. Sternberg in [12,13] elaborated the Poincaré result in the real case and showed under the same assumption that $\mathcal{X}$ can always be C^∞ linearized. Namely, $\mathcal{X}$ is C^∞ conjugated to $\mathcal{X}_l$. A C^w version of this theorem was established by Siegel in the holomorphic case when the eigenvalues satisfy some Diophantine conditions (see also [6]). In a general context, Hartman and Grobman (see [8]) proved that $\mathcal{X}$ is always C^0 linearizable. The proof of this result, however, is not based on normal form procedure but obtained by showing the existence of a contraction in a Banach space. Therefore, it does not give further information about the transformation itself.

We notice that much work also has already been done for finitely differentiable families of vector field, see [1,2, 4,7,11]. For instance, Hartman in [7] shows that in the 2-dimensional case, one can always find a C^1 linearization. The aim of this paper is to develop a more explicit normal form conjugacy procedure called a "*LMT normal form*" and to present an asymptotic expression of such a linearization. It turns out that the results obtained here essentially generalize the known ones and the methods developed here are ideally suitable to study related conjugacy problems. Moreover, the explicit linearization makes possible in applications, say, in numerical applications or the related fields.

Definition 1.1. Let $\mathcal{U} \subset \mathbb{R}^n$ be a neighbourhood of 0 and f be a continuous function. We say that $f : \mathcal{U} \to \mathbb{R}$ is a Logarithmic Mourtada type function if there exist a positive integer l, a neighbourhood of 0, $\mathcal{V}_l \subset \mathbb{R}^{n(l+1)}$, and a C^∞ function $\mathbf{F} : \mathcal{V}_l \to \mathbb{R}$ such that $f(z) = \mathbf{F}(z, zT, \ldots, zT^l)$, where $T = \log \sum_{k=1}^n a_k z_k^{2m_k}$, and where a_k are positive parameters and m_k are some positive integers.

A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is called a Logarithmic Mourtada type homeomorphism if each component of Φ is a Logarithmic Mourtada type function.

Example 1. The following functions are Logarithmic Mourtada type functions

$$f_1(z) = z^3 \log|z|; \qquad f_2(x, y) = xy^2 \log(|x|^2 + |y|^4).$$

We refer the reader to [4,9] for more examples concerning the presence of logarithmic functions.

We now are in position to state the main theorem of this paper.

Theorem 1.1. *Let $\mathcal{X}$ be a smooth vector field defined in a neighborhood $\mathcal{U} \in \mathbb{R}^n$ of the hyperbolic equilibrium point 0. Suppose that $\mathcal{X}_l$, the linear part of $\mathcal{X}$, at 0 is semi-simple (i.e., $N \equiv 0$). Then up to a C^∞ change of coordinates there exist $\mathcal{V} \subset \mathcal{U}$ and a Logarithmic Mourtada type homeomorphism Φ defined on $\mathcal{V}$ that conjugates $\mathcal{X}_l$ with $\mathcal{X}$.*

2. Some remarks

An interesting consequence of Theorem 1.1 is that we can asymptotically write an expression of the so called Dulac map associated with a vector field near a hyperbolic equilibrium point of saddle type. The Dulac map Δ is the transition between a section transverse to the locally stable manifold and another section transverse to the locally unstable manifold. Roussarie [10] gives such an asymptotic expression of the map Δ in the two-dimensional case. It is shown in [10] that such a map can be expressed in terms of the LMT functions. Namely, there exists a smooth function f such that

$$\Delta(x) = x^r \big(1 + f(x, x \log x)\big),$$

where r is the ratio between the unstable and the stable eigenvalue. Notice that Roussarie's expression also makes sense when the eigenvalues of the vector field at the equilibrium point depend on a parameter ε, and in this case the expression of Δ takes the form

$$\Delta(x) = x^r \big(1 + g\big(x, x\omega_\varepsilon\big(\log|x|\big)\big)\big),$$

where g is a smooth function and

$$\omega_\varepsilon(x) = \frac{|x|^\varepsilon - 1}{\varepsilon} \quad \text{for } \varepsilon \neq 0, \qquad \omega_0(x) = \log|x|.$$

The corresponding 3-dimensional case is studied in [3] where a weaker result is given. We remark that the present theorem does not take families of objects into consideration and it will be the subject of a forthcoming paper.

Theorem 1.1 has an interesting application, as it can be applied to discuss the degree of differentiability of the linearizing conjugacy. More precisely, denote by

$$\{\alpha_i\}_{1 \leqslant i \leqslant q} \quad \text{and} \quad \{-\beta_j\}_{1 \leqslant j \leqslant p},$$

respectively, the set of positive real part eigenvalues and the set of negative real part eigenvalues of $d\mathcal{X}(O)$. Assume that they are ordered as follows.

$$\mathcal{R}e(\beta_1) \leqslant \mathcal{R}e(\beta_2) \leqslant \cdots \leqslant \mathcal{R}e(\beta_p), \qquad \mathcal{R}e(\alpha_1) \leqslant \mathcal{R}e(\alpha_2) \leqslant \cdots \leqslant \mathcal{R}e(\alpha_q).$$

We say that $\mathcal{X}$ satisfies the $P(r)$ condition if, for all resonant conditions of the form

$$\alpha_i = \sum_{k=1}^{q} \alpha_k n_k^+ - \sum_{k=1}^{p} \beta_k m_k^+ \quad \text{or} \quad \beta_j = \sum_{k=1}^{k} \alpha_k n_k^- - \sum_{k=1}^{p} \beta_k m_k^-,$$

where $m^\pm$ and $n^\pm$ are non-negative integers and $|m^\pm| + |n^\pm| \geqslant 2$, the following inequalities hold.

$$r\mathcal{R}e(\alpha_q) < \mathcal{R}e(\alpha_1)n_1^+ + \cdots + \mathcal{R}e(\alpha_q)n_q^+ \quad \text{or} \quad r\mathcal{R}e(\beta_p) < \mathcal{R}e(\beta_1)m_1^- + \cdots + \mathcal{R}e(\beta_p)m_p^-.$$

Proposition 2.1. *Assume $\mathcal{X}$ satisfies the $P(r)$ condition. Then the LMT conjugacy Φ betweeen $\mathcal{X}_l$ and $\mathcal{X}$ given by Theorem* 1.1 *is C^r.*

The so-called $S(k)$ condition given by Samovol in [11] is weaker than $P(r)$ condition. He proves that if $\mathcal{X}$ satisfies the $S(r)$ condition, then $\mathcal{X}$ admits C^r linearizing conjugacy. For this particular conjugacy, however, it is not yet known whether it is of LMT type or not.

3. Sketch of proof of the theorem

We shall only give a sketch of the proof of Theorem 1.1 in the two-dimensional case, where we take the linear part of $\mathcal{X}$ to be $x\partial/\partial x - y\partial/\partial y$, i.e., $\mathcal{X}_l = x\partial/\partial x - y\partial/\partial y$. The idea of the proof in general cases is essentially the same.

Let $\mathcal{X}$ be given by the following ordinary differential equations

$$\dot{x} = x + f_1(x, y), \qquad \dot{y} = -y + g_1(x, y), \tag{2}$$

where we assume that the nonlinear part of (2) is already reduced to the Poincaré Dulac normal form. In other words, both functions f_1 and g_1 contain resonant monomials only, i.e.,

$$f_1(x, y) = \sum_{k=1}^{\infty} a_{i,1} x^{k+1} y^k, \qquad g_1(x, y) = \sum_{k=1}^{\infty} b_{i,1} x^k y^{k+1}, \tag{3}$$

where $a_{k,f}$'s and $b_{k,f}$'s are real numbers. Since the original system (1) is assumed to be C^∞ smooth, therefore, according to the Chen theorem [5] which states that two formally equivalent hyperbolic vector fields are always smoothly equivalent, we can simply ignore the possible existence of flat terms in (2).

It is easy to show that there exist two sequences of smooth resonant functions $\{f_i\}_{i\geqslant 1}$ and $\{g_i\}_{i\geqslant 1}$ such that

$$\mathcal{X}(f_i)(x,y) - f_i(x,y) = f_{i+1}(x,y), \qquad \mathcal{X}(g_i)(x,y) + g_i(x,y) = g_{i+1}(x,y), \tag{4}$$

where f_i and g_i have the form

$$f_i(x,y) = \sum_{k=i}^{\infty} a_{k,i} x^{k+1} y^k, \qquad g_i(x,y) = \sum_{k=i}^{\infty} b_{k,i} x^k y^{k+1}. \tag{5}$$

We now introduce a vector field $\tilde{\mathcal{X}} = \mathcal{X} + \partial/\partial t + \partial/\partial s$ (thus we have $\mathcal{X}_l(t) = \mathcal{X}_l(s) = 1$). Let

$$\bar{x} = x + \sum_{k=1}^{\infty} \frac{(-1)^k s^k}{k!} f_k, \qquad \bar{y} = y + \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{k!} g_k. \tag{6}$$

Then we are ready to check the following equalities.

$$\dot{\bar{x}} = \bar{x} + h_1(\bar{x}, \bar{y}, s, t), \qquad \dot{\bar{y}} = -\bar{y} + h_2(\bar{x}, \bar{y}, s, t), \tag{7}$$

where h_1 and h_2 are smoothly flat functions. Putting $t = -\log|\bar{y}|$, $s = \log|\bar{x}|$, we then have $h_1(\bar{x}, \bar{y}, s, t) = H_1(\bar{x}, \bar{y})$, $h_2(\bar{x}, \bar{y}, s, t) = H_2(\bar{x}, \bar{y})$, where H_1 and H_2 remain smoothly flat. Thus we have

$$\dot{\bar{x}} = \bar{x} + H_1(\bar{x}, \bar{y}), \qquad \dot{\bar{y}} = -\bar{y} + H_2(\bar{x}, \bar{y}). \tag{8}$$

Notice that formally (8) is already a linear system, therefore again by the Chen theorem, it is smoothly linearizable. We end the sketch of the proof of the theorem.

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