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A FAMILY OF INTEGRAL REPRESENTATIONS FOR THE BROWNIAN VARIABLES

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ABSTRACT. – The natural filtration of a real Brownian motion and its excursion filtration are sharing a fundamental property: the property of integral representation. As a consequence, every Brownian variable admits two distinct integral representations. We show here that there are other integral representations of the Brownian variables. They make use of a stochastic flow studied by Bass and Burdzy. Our arguments are inspired by Rogers and Walsh's results on stochastic integration with respect to the Brownian local times.

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RÉSUMÉ. – La filtration naturelle d'un mouvement brownien réel et la filtration de ses excursions ont en commun une propriété fondamentale : la propriété de représentation intégrale. Toute variable brownienne admet donc deux représentations intégrales distinctes. Nous montrons ici qu'il existe d'autres représentations intégrales pour les variables browniennes. La construction de ces représentations utilise un flot stochastique qui a été étudié par Bass et Burdzy. Nos arguments s'inspirent du calcul stochastique par rapport aux temps locaux développé par Rogers et Walsh.

1. Introduction

Let $(B_t,\ t \geqslant 0)$ be a one-dimensional Brownian motion starting from 0. Denote by $(\mathcal{B}_t)_{t\geqslant 0}$ its natural filtration. One of the most fundamental properties of the Brownian filtration is that every $\mathcal{L}^2$-bounded $(\mathcal{B}_t)$-martingale can be represented as a stochastic integral with respect to B, and hence is continuous (for a nice application of this representation result, see for example Karatzas et al. [7]). This representation property is in fact equivalent to the integral representation of $\mathcal{L}^2(\mathcal{B}_\infty)$, namely that for every

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$H \in \mathcal{L}^2(\mathcal{B}_\infty)$, there exists a unique process (h_t) which is predictable with respect to $(\mathcal{B}_t)$ such that

$$H = \mathbb{E}(H) + \int_0^\infty h_s \, dB_s, \tag{1.1}$$

and

$$\mathbb{E}\left(\int_0^\infty h_s^2 \, ds \right) < \infty,$$

(see, e.g., Revuz and Yor [9, Chapter V]). The representation (1.1) corresponds in a natural way to the "time variable". Rogers and Walsh [12] have established an analoguous representation result with respect to the "space variable": Let $x \in \mathbb{R}$ and consider the reflecting Brownian motion on $(-\infty, x]$ defined by

$$\widehat{B}_t(x) \stackrel{\text{def}}{=} B_{\rho(t,x)},$$

with $\rho(t,x) \stackrel{\text{def}}{=} \inf\{s > 0\colon \int_0^s du \, \mathbb{1}_{(B_u \leqslant x)} > t\}$. Define $\mathcal{E}_x \stackrel{\text{def}}{=} \sigma\{\widehat{B}_t(x),\ t \geqslant 0\}$. It has been shown that the family $\{\mathcal{E}_x,\ x \in \mathbb{R}\}$ forms a filtration, which is called the Brownian excursion filtration. Williams [17] proved that all $(\mathcal{E}_x)$-martingales are continuous. Rogers and Walsh [12] have given another proof of this result by showing that for any $H \in \mathcal{L}^2(\mathcal{E}_\infty)$, there exists a unique "identifiable" process $(\phi(t,x),\ t \geqslant 0,\ x \in \mathbb{R})$ such that

$$H = \mathbb{E}(H) + \frac{1}{2} \int_{t \geqslant 0} \int_{x \in \mathbb{R}} \phi(t,x) \, dL_B(t,x), \tag{1.2}$$

and

$$\mathbb{E}\left(\int_0^\infty \phi^2(s, B_s) \, ds \right) < \infty,$$

where L_B denotes the local time process related to B. Thanks to Tanaka's formula, we note that

$$H = \mathbb{E}(H) + \int_{t \geqslant 0} \int_{x \in \mathbb{R}} \phi(t,x) \, d\mathbb{M}(t,x), \tag{1.3}$$

where $\mathbb{M}(t,x) \stackrel{\text{def}}{=} \int_0^t \mathbb{1}_{(B_s \leqslant x)} \, dB_s,\ t \geqslant 0,\ x \in \mathbb{R}$.

Since $\mathcal{L}^2(\mathcal{E}_\infty) = \mathcal{L}^2(\mathcal{B}_\infty)$, this gives another representation for every element in $\mathcal{L}^2(\mathcal{B}_\infty)$.

We refer to Williams [17], Walsh [16], McGill [8], Jeulin [6], Rogers and Walsh [10–14], Yor [18] together with their references for detailed studies of Brownian excursion filtration $(\mathcal{E}_x,\ x \in \mathbb{R})$ and related topics.

The purpose of this paper is to put in evidence other integral representations for the Brownian variables. These integral representations will be done with respect to processes involving the local time of the Brownian motion with a generalized drift. More precisely, for β_1 and β_2 two real constants, consider the equation

$$X_t(x) = x + B_t + \beta_1 \int_0^t ds\, \mathbb{1}_{(X_s(x)\leqslant 0)} + \beta_2 \int_0^t ds\, \mathbb{1}_{(X_s(x)>0)}, \quad t \geqslant 0. \tag{1.4}$$

Bass and Burdzy [1] have shown that the solutions of (1.4) for $x \in \mathbb{R}$, form a $\mathcal{C}^1$-diffeomorphism on $\mathbb{R}$. This implies in particular that for any fixed $t \geqslant 0$, $X_t(x)$ is a strictly increasing function of x. We assume that $\beta_1 \geqslant 0 \geqslant \beta_2$. They proved that in that case the solutions are recurrent (i.e., with probability one for each x, $X_\cdot(x)$ visits 0 infinitely often). Define the process $(M(t,x),\ x \in \mathbb{R},\ t \geqslant 0)$ by

$$M(t,x) = \int_0^t \mathbb{1}_{(X_s(x)>0)}\, dB_s, \quad x \in \mathbb{R},\ t \geqslant 0.$$

This process is connected, thanks to Tanaka's formula, to the local times at zero of the semi-martingales $(X_t(x),\ t \geqslant 0)$, $x \in \mathbb{R}$, which can be seen as local times of B along particular random curves.

Under the assumption that $\beta_1 \geqslant 0 \geqslant \beta_2$, we will show that for every variable H of $\mathcal{L}^2(\mathcal{B}_\infty)$, there exists a unique random process ϕ such that

$$H = \mathbb{E}(H) + \int_{t\geqslant 0} \int_{x\in\mathbb{R}} \phi(t,x)\, dM(t,x). \tag{1.5}$$

That way, we will obtain for the variable H a family of integral representations indexed by the parameters β_1 and β_2.

In order to state properly this representation property, we first recall, in Section 2, some results on the flow X and some notations. In Section 3, we construct a stochastic integration with respect to $(M(x,t),\ x \in \mathbb{R},\ t \geqslant 0)$. We show that the arguments of Rogers and Walsh [12] concerning the Brownian local time, can be adjusted to M. But proving that every Brownian variable satisfies (1.5) requires other arguments. This is done in Section 4. Some applications are then presented in Section 5. Indeed (1.5) implies the predictable representation property of a certain filtration $(\mathcal{H}_x)_{x\in\mathbb{R}}$ indexed by the starting points of the solutions of (1.4). As a consequence, all the martingales for this filtration are continuous. This is used to study the intrinsic local time of the flow X.

Now, a natural question arises: what happens if instead of Eq. (1.4), we consider the equation

$$X_t(x) = x + B_t + \int_0^t ds\, b\big(X_s(x)\big) \tag{1.6}$$

with a "smooth" b (for example b in $\mathcal{C}^1$); Could we obtain a similar integral representation for the Brownian variables? Paradoxically, a "smooth" drift makes the situation much more complex. Indeed, in the special case of (1.4), the process of the local times at zero of the semi-martingales $X_{.}(x),\ x\in\mathbb{R}$, actually corresponds to the local time process of a single process $(Y_t)_{t\geqslant 0}$, and the filtration $(\mathcal{H}_x)_{x\in\mathbb{R}}$ mentioned above is the excursion filtration of $(Y_t)_{t\geqslant 0}$. In the case of (1.6), this coincidence does not occur, and one has to deal, in order to answer to the problem of integral representation, with the entire local time process of each semi-martingale $(X_t(x),\ t\geqslant 0)$. The main issue, that we could not solve, is actually to write an analogue of Fact 2.3 (Section 2).

2. Some notation and results on Bass and Burdzy's flow

We keep the notation introduced in Section 1 to present the following facts proved by Hu and Warren in [5]. Some of these facts are extensions of Bass and Burdzy's results in [1].

FACT 2.1. – *With probability one there exists a bicontinuous process $(L_t^x;\ x\in\mathbb{R},\ t\geqslant 0)$ such that for every x the process $(L_t^x;\ t\geqslant 0)$ is the local time at zero for the semi-martingale $(X_t(x),\ t\geqslant 0)$. Moreover*

$$\frac{\partial X_t}{\partial x}(x)=\exp\bigl((\beta_2-\beta_1)L_t^x\bigr). \tag{2.1}$$

Consider Y_t defined as being the unique $x\in\mathbb{R}$ such that $X_t(x)=0$. Almost surely, for every bounded Borel function f,

$$\int_0^t f(Y_s)\,ds=\int_{-\infty}^{\infty} dy\, f(y)\kappa\bigl(L_t^y\bigr), \tag{2.2}$$

where

$$\kappa(\ell)\stackrel{\text{def}}{=}\begin{cases}\dfrac{\mathrm{e}^{(\beta_2-\beta_1)\ell}-1}{\beta_2-\beta_1}, & \text{if } \beta_1\neq\beta_2,\\ \ell, & \text{if } \beta_1=\beta_2,\end{cases}\qquad \ell\geqslant 0. \tag{2.3}$$

Note that the local times process $(L_t^x,\ t\geqslant 0,\ x\in\mathbb{R})$ is indexed by the starting points of the flow $(X_t(x))$. This kind of local times process has been recently studied by Burdzy and Chen [3] for a flow related to skew Brownian motion. The process $(Y_t,\ t\geqslant 0)$ plays an important part in the next sections.

Notation 2.2. –

$$A_t(x)\stackrel{\text{def}}{=}\int_0^t du\,\mathbb{1}_{(X_u(x)>0)},$$

$$\alpha_t(x)\stackrel{\text{def}}{=}\inf\bigl\{s>0\colon\ A_s(x)>t\bigr\},\qquad x\in\mathbb{R},\ t\geqslant 0,$$

$$\mathcal{H}_x\stackrel{\text{def}}{=}\sigma\bigl\{X_{\alpha_t(x)}(x),\ t\geqslant 0\bigr\},\qquad x\in\mathbb{R}.$$

FACT 2.3. – *The family $\{\mathcal{H}_x,\ x\in\mathbb{R}\}$ is an increasing family of σ-fields. Furthermore, for every $x\in\mathbb{R}$ and $H\in\mathcal{L}^2(\mathcal{H}_x)$, there exists a $(\mathcal{B}_t,\ t\geqslant 0)$-predictable process $(h_t,\ t\geqslant 0)$ such that*

$$H = \mathbb{E}H + \int_0^\infty h_s \mathbb{1}_{(X_s(x)>0)}\, dB_s = \mathbb{E}H + \int_0^\infty h_s \mathbb{1}_{(Y_s<x)}\, dB_s$$
$$= \mathbb{E}H + \int_0^\infty h_s\, d_s M(s,x), \tag{2.4}$$

and $\mathbb{E}\int_0^\infty h_s^2 \mathbb{1}_{(Y_s<x)}\, ds < \infty$.

3. Construction of the area integral

Following Rogers and Walsh [12], we set the two definitions below:

DEFINITION 3.1. – *A process* $\phi = (\phi(t,x),\ t \geqslant 0, x \in \mathbb{R})$ *is said to be* ($\mathcal{H}$-)*identifiable if*:

(i) *The process* $(\phi(\alpha_t(x), x),\ t \geqslant 0)_{x\in\mathbb{R}}$ *is predictable with respect to* $(\mathcal{H}_x \times \mathcal{B}_\infty)_{x\in\mathbb{R}}$.
(ii) *For all* $0 \leqslant s < t,\ x \in \mathbb{R}$ *such that* $A_s(x) = A_t(x)$, *we have* $\phi(s,x) = \phi(t,x)$.

We denote by $\mathcal{I}$ the σ-field generated by all identifiable processes.

Example. – Let $T > 0$ and Z be two $\mathcal{H}_a$-measurable variables. Let $b > a$. Then one can easily check by using the arguments of Rogers and Walsh [12, p. 461] that the process

$$\phi(t,x) \stackrel{\text{def}}{=} Z\, \mathbb{1}_{(\alpha_T(a),\infty)}(t)\, \mathbb{1}_{(a,b]}(x)$$

is identifiable.

DEFINITION 3.2. – *A process* ϕ *is called elementary identifiable if* ϕ *belongs to the linear span of the family*

$$\left\{Z\mathbb{1}_{(\alpha_S(a),\alpha_T(a)]}(t)\, \mathbb{1}_{(a,b]}(x):\ a<b,\ Z \in \mathcal{L}^\infty(\mathcal{H}_a);\ 0 \leqslant S \leqslant T \in \mathcal{L}^\infty(\mathcal{H}_a)\right\}.$$

For $\phi(t,x) = Z\, \mathbb{1}_{(\alpha_S(a),\alpha_T(a)]}(t)\, \mathbb{1}_{(a,b]}(x)$, *we define*

$$\iint \phi\, dM \stackrel{\text{def}}{=} Z\big(M\big(\alpha_T(a),b\big) - M\big(\alpha_S(a),b\big) - M\big(\alpha_T(a),a\big) + M\big(\alpha_S(a),a\big)\big).$$

We extend the above definition to all elementary identifiable processes by linearity.

Thanks to the arguments developed by Rogers and Walsh [12, proofs of Propositions 2.3 and 2.4], we have the following fact:

FACT 3.1. – *The* σ*-field* $\mathcal{I}$ *is generated by the family of elementary identifiable processes.*

The construction of the area integral with respect to M is given by the following theorem:

THEOREM 3.1. – *For any elementary identifiable process ϕ, we have*

$$\mathbb{E}\left[\left(\iint \phi\, dM\right)^2\right] = \mathbb{E}\left[\int_0^\infty \phi^2(s, Y_s)\, ds\right]. \tag{3.1}$$

Hence we can extend the isometry $\phi \to \iint \phi\, dM$ to all $\phi \in \mathcal{L}^2(\mathcal{I})$, where

$$\mathcal{L}^2(\mathcal{I}) \stackrel{\text{def}}{=} \left\{\phi \text{ identifiable such that } \|\phi\|^2 \stackrel{\text{def}}{=} \mathbb{E}\left[\int_0^\infty \phi^2(s, Y_s)\, ds\right] < \infty\right\}. \tag{3.2}$$

Moreover, for any $\phi \in \mathcal{L}^2(\mathcal{I})$, the process $(\iint \phi(t, y)\, \mathbb{1}_{(y \leqslant x)}\, dM(t, y),\ x \in \mathbb{R})$ is a continuous square-integrable $(\mathcal{H}_x)$-martingale, with increasing process

$$\left(\int_0^\infty \phi^2(s, Y_s)\, \mathbb{1}_{(Y_s \leqslant x)}\, ds,\ x \in \mathbb{R}\right).$$

In order to prove Theorem 3.1, we first establish the next lemma:

LEMMA 3.2. – *Fix $a \in \mathbb{R}$ and let S and T be two $\mathcal{H}_a$-measurable variables such that $0 \leqslant S \leqslant T \leqslant \infty$. The process $(N_x, x \geqslant a) := (\int_{\alpha_S(a)}^{\alpha_T(a)} \mathbb{1}_{(a \leqslant Y_s < x)}\, dB_s,\ x \geqslant a)$ is a square-integrable $(\mathcal{H}_x)$-martingale with increasing process $(\int_{\alpha_S(a)}^{\alpha_T(a)} \mathbb{1}_{(a \leqslant Y_s < x)}\, ds,\ x \geqslant a)$.*

Proof. – We assume $\beta_1 > \beta_2$, the particular case $\beta_1 = \beta_2 = 0$ being exactly the Brownian motion case. Let $x > a$. It follows from (2.2) and (2.3) that

$$\int_0^{\alpha_T(a)} \mathbb{1}_{(a \leqslant Y_s < x)}\, ds = \int_a^x \kappa\left(L^y_{\alpha_T(a)}\right) dy \leqslant \frac{x - a}{\beta_1 - \beta_2},$$

hence N_x is square-integrable. The measurability of N_x with respect to $\mathcal{H}_x$ is immediate. Indeed, for fixed x, the whole process $(\int_0^{\alpha_t(a)} \mathbb{1}_{(a \leqslant Y_s < x)}\, dB_s,\ t \geqslant 0)$ is $\mathcal{H}_x$-mesurable. Consider $y > x \geqslant a$ and $H \in \mathcal{L}^\infty(\mathcal{H}_x)$. We make use of the representation (2.4) for H to obtain:

$$\mathbb{E}\big((N_y - N_x)H\big) = \mathbb{E}\left(\int_0^\infty \mathbb{1}_{(x \leqslant Y_s < y)}\, \mathbb{1}_{(\alpha_S(a) < s \leqslant \alpha_T(a))}\, dB_s \times \int_0^\infty \mathbb{1}_{(Y_s < x)} h_s\, dB_s\right) = 0,$$

proving the martingale property. Thanks to the general result of Bouleau [2], we immediately obtain the formula for the increasing process. □

Proof of Theorem 3.1. – The identity (3.1) follows from Lemma 3.2 once we note the following fact: let $0 \leqslant U \leqslant V \leqslant S \leqslant T$ be four $\mathcal{H}_a$-measurable variables. The two martingales

$$\big(M(\alpha_T(a), x) - M(\alpha_T(a), a)\big) - \big(M(\alpha_S(a), x) - M(\alpha_S(a), a)\big)$$

$$= \int_{\alpha_S(a)}^{\alpha_T(a)} \mathbb{1}_{(a \leqslant Y_s < x)}\, dB_s, \quad x \geqslant a,$$

and

$$\bigl(M(\alpha_V(a), x) - M(\alpha_V(a), a)\bigr) - \bigl(M(\alpha_U(a), x) - M(\alpha_U(a), a)\bigr)$$
$$= \int_{\alpha_U(a)}^{\alpha_V(a)} \mathbb{1}_{(a \leqslant Y_s < x)}\, dB_s, \quad x \geqslant a,$$

are orthogonal. Using Fact 3.1, the family of elementary identifiable processes is dense in $\mathcal{L}^2(\mathcal{I})$. Hence we can extend the definition of area integral to all $\phi \in \mathcal{L}^2(\mathcal{I})$.

To finish the proof of Theorem 3.1, it suffices to verify the last assertion for an elementary identifiable process ϕ of the form $Z\,\mathbb{1}_{(\alpha_S(a),\alpha_T(a)]}(t)\,\mathbb{1}_{(a,b]}(x)$ with $Z \in \mathcal{L}^\infty(\mathcal{H}_a)$, $a < b,\ 0 \leqslant S \leqslant T$ and $S, T \in \mathcal{L}^\infty(\mathcal{H}_a)$. Then again by using Lemma 3.2, the process

$$\left(\iint \phi(t, y)\,\mathbb{1}_{(y \leqslant x)}\, dM(t, y),\ x \in \mathbb{R}\right)$$
$$= \left(Z \iint \mathbb{1}_{(\alpha_S(a),\alpha_T(a)]}(t)\mathbb{1}_{(a,x\wedge b]}(y)\, dM(t, y),\ x \in \mathbb{R}\right)$$

is a continuous square-integrable $(\mathcal{H}_x)$-martingale, with increasing process

$$\left(Z^2 \int_{\alpha_S(a)}^{\alpha_T(a)} \mathbb{1}_{(a \leqslant Y_s < x\wedge b)}\, ds,\ x \in \mathbb{R}\right) = \left(\int_0^\infty \phi^2(s, Y_s)\, ds,\ x \in \mathbb{R}\right)$$

completing the whole proof. □

In some special cases, the area integral can be explicitly computed.

PROPOSITION 3.3. – *Let ϕ be an element of $\mathcal{L}^2(\mathcal{I})$ such that for every $x \in \mathbb{R}$, the process $(\phi(\alpha_s(x), x),\ s \geqslant 0)$ is predictable with respect to $(\mathcal{B}_{\alpha_s(x)},\ s \geqslant 0)$. Moreover, we assume that almost surely $\phi(\cdot,\cdot)$ is continuous outside of a set of null Lebesgue measure. Then the process $(\phi(s, Y_s),\ s \geqslant 0)$ is $(\mathcal{B}_s)$-adapted, and we have*

$$\iint_{s\geqslant 0,\ y\in\mathbb{R}} \phi(s, y)\, dM(s, y) = \int_0^\infty \phi(s, Y_s)\, dB_s. \tag{3.3}$$

See Eisenbaum [4] for some related results on double integrals with respect to Brownian local times.

Proof. – We can assume that ϕ has compact support and is bounded. Define the finite sum

$$\phi_n(s, y) \stackrel{\text{def}}{=} \sum_{s_{i+1}-s_i=1/n} \sum_{x_{j+1}-x_j=1/n} \phi\bigl(\alpha_{s_i}(x_j), x_j\bigr)\,\mathbb{1}_{[x_j,x_{j+1})}(y)\,\mathbb{1}_{(\alpha_{s_i}(x_j),\alpha_{s_{i+1}}(x_j)]}(s),$$

for $s \geqslant 0,\ y \in \mathbb{R}$. Since $\phi(\alpha_s(x), x)$ is $\mathcal{H}_x$-adapted, $\phi_n \in \mathcal{L}^2(\mathcal{I})$.

Moreover, the process $(\phi_n(s, Y_s),\ s \geqslant 0)$ is $(\mathcal{B}_s)$-adapted. Indeed, from our assumption, $\phi(\alpha_{s_i}(x_j), x_j)\,\mathbb{1}_{(\alpha_{s_i}(x_j),\alpha_{s_{i+1}}(x_j)]}(s)$ is $\mathcal{B}_s$-measurable, since $\alpha_{s_i}(x_j)$ and $\alpha_{s_{i+1}}(x_j)$ are $(\mathcal{B}_s)$- stopping times.

By definition,

$$\iint\limits_{s\geqslant 0,\ y\in\mathbb{R}} \phi_n(s, y)\, dM(s, y) = \int_0^\infty \phi_n(s, Y_s)\, dB_s. \tag{3.4}$$

Thanks to our assumption, almost surely $\phi_n(s, Y_s) \to \phi(s, Y_s)$ ds-a.s. It follows that

$$\mathbb{E}\left(\int_0^\infty (\phi_n - \phi)^2(s, Y_s)\, ds\right) \to 0, \quad n \to \infty.$$

Consequently, the process $(\phi(s, Y_s), s \geqslant 0)$ is $(\mathcal{B}_s)$-adapted. Hence the two integrals in (3.4) converge in $\mathcal{L}^2$ to the corresponding integrals for ϕ as $n \to \infty$, which completes the proof. □

4. Integral representation

Now that stochastic integration with respect to $(M(x, t),\ x \in \mathbb{R},\ t \geqslant 0)$ has been defined, the following theorem shows that every Brownian variable is an integral with respect to M.

THEOREM 4.1. – *Let (β_1, β_2) be a couple of real numbers such that $\beta_1 \geqslant 0 \geqslant \beta_2$. For every $H \in \mathcal{L}^2(\mathcal{B}_\infty)$, there exists a unique $(\mathcal{H}_x)$-identifiable process ϕ such that*

$$H = \mathbb{E}(H) + \int\limits_{t\geqslant 0} \int\limits_{x\in\mathbb{R}} \phi(t, x)\, dM(t, x),$$

and

$$\mathbb{E}\left(\int_0^\infty \phi^2(t, Y_t)\, dt\right) < \infty,$$

where Y_t denotes the unique $x \in \mathbb{R}$ such that $X_t(x) = 0$.

Proof. – Define

$$\mathcal{K} \stackrel{\text{def}}{=} \left\{ H \in \mathcal{L}^2(\mathcal{B}_\infty)\colon\ H = \mathbb{E}(H) + \iint \phi(t, y)\, dM(t, y),\ \text{for some } \phi \in \mathcal{L}^2(\mathcal{I}) \right\}. \tag{4.1}$$

The proof of Theorem 4.1 is equivalent to showing that

$$\mathcal{K} = \mathcal{L}^2(\mathcal{B}_\infty).$$

The proof of the above equality is constructed as follows. We will define a family (D_n) of random variables such that:

Step 4.2. For each $n \geqslant 1$, $D_n \subset \mathcal{K}$. Furthermore, the algebra generated by D_n is itself included in $\mathcal{K}$.

Step 4.3. Let $\mathcal{A}_n = \sigma(D_n)$. We have $\mathcal{L}^2(\mathcal{A}_n) \subset \mathcal{K}$, for all $n \geqslant 1$, and $\mathcal{A}_n$ is increasing with n.

Step 4.4. Define $\mathcal{A}_\infty \stackrel{\text{def}}{=} \bigvee_{n\geqslant 1} \mathcal{A}_n$. We will prove that $\mathcal{A}_\infty = \mathcal{B}_\infty$. Consequently, $\mathcal{K} = \mathcal{L}^2(\mathcal{B}_\infty)$.

In what follows, we exclude the Brownian case ($\beta_1 = \beta_2 = 0$) which has been already treated by Rogers and Walsh [12]. We begin with the choice of D_n. Keeping the notation of Section 2, we define

$$\begin{aligned} &\tau_r(a) \stackrel{\text{def}}{=} \inf\{t > 0\colon\ L_t^a > r\}, \quad r \geqslant 0,\ a \in \mathbb{R},\\ &D_n \stackrel{\text{def}}{=} \{L^{(j+1)/2^n}_{\tau_t(j/2^n)};\ L^{(j+1)/2^n}_{\tau_t(j/2^n)} - L^{(j+1)/2^n}_{\tau_s(j/2^n)}\colon\ 0 \leqslant s < t,\ j \in \mathbb{Z}\}. \end{aligned} \tag{4.5}$$

The main technical result is the following

LEMMA 4.2. – *Fix $a < b \in \mathbb{R}$ and $t > s > 0$. For any $\lambda > 0$,*

$$\mathbb{E}\exp\bigl(\lambda L^b_{\tau_t(a)}\bigr) < \infty. \tag{4.6}$$

For any smooth function $f:\mathbb{R}_+ \to \mathbb{R}$ such that $|f(x)| = \mathrm{o}(\mathrm{e}^{\varepsilon_0 x})$ as $x \to \infty$ for some $\varepsilon_0 > 0$, we have

$$f\bigl(L^b_{\tau_t(a)}\bigr) \in \mathcal{K}, \tag{4.7}$$

$$f\bigl(L^b_{\tau_t(a)} - L^b_{\tau_s(a)}\bigr) \in \mathcal{K}. \tag{4.8}$$

Proof. – Let $x \geqslant a$. Applying Tanaka's formula to the semimartingales $X_\cdot(x)$ and $X_\cdot(a)$, we have

$$L^x_{\tau_r(a)} = r + 2(a^- - x^-) - 2\int_0^{\tau_r(a)} \mathbb{1}_{(a\leqslant Y_s<x)}\, dB_s - 2\beta_1 \int_a^x \kappa(L^y_{\tau_r(a)})\, dy. \tag{4.9}$$

Similarly to Lemma 3.2, one can prove that the process $(\int_0^{\tau_r(a)} \mathbb{1}_{(a\leqslant Y_s<x)}\, dB_s,\ x \geqslant a)$ is a square-integrable $(\mathcal{H}_x)$-martingale, with increasing process $(\int_a^x \kappa(L^y_{\tau_r(a)})\, dy,\ x \geqslant a)$. Consequently there exists a $(\mathcal{H}_x)$-Brownian motion $\widetilde{W}$, independent of $\mathcal{H}_a$ such that:

$$\left(\int_0^{\tau_r(a)} \mathbb{1}_{(a\leqslant Y_s<x)}\, dB_s, x \geqslant a\right) = \left(\int_a^x \sqrt{\kappa\bigl(L^y_{\tau_r(a)}\bigr)}\, d\widetilde{W}_y,\ x \geqslant a\right).$$

Hence, we obtain

$$L^x_{\tau_r(a)} = r + 2(a^- - x^-) - 2\int_a^x \sqrt{\kappa\bigl(L^y_{\tau_r(a)}\bigr)}\, d\widetilde{W}_y - 2\beta_1 \int_a^x \kappa\bigl(L^y_{\tau_r(a)}\bigr)\, dy. \tag{4.10}$$

This shows that $(L(\tau_r(a), a+t),\ t \geqslant 0)$ is an inhomogeneous Markov process (this fact has been already noticed in [5]). Since $\beta_1 > \beta_2,\ \kappa(x) \leqslant 1/(\beta_1 - \beta_2)$, it follows from (4.10) and the Dubins–Schwarz representation theorem (see [9, Chapter V]) that

$$\exp\bigl(\lambda L^b_{\tau_t(a)}\bigr) \leqslant C \exp\Bigl(2\lambda \gamma_{\int_a^b \kappa(L^y_{\tau_t(a)})\,dy}\Bigr) \leqslant C \exp\Bigl(2\lambda \sup_{0 \leqslant v \leqslant (b-a)/(\beta_1-\beta_2)} |\gamma_s|\Bigr), \tag{4.11}$$

where $C \stackrel{\text{def}}{=} \mathrm{e}^{\lambda(t+2a^- + 2(b-a)|\beta_1|/(\beta_1-\beta_2))}$ and γ is a one-dimensional Brownian motion. It is well known that the rhs of the above inequalities is integrable, hence (4.6) follows.

The conclusions (4.7) and (4.8) follow from a martingale projection argument. Let us first show (4.7). Using the Markov property of $(L^{a+x}_{\tau_t(a)},\ x \geqslant 0)$, we obtain

$$\mathbb{E}\bigl(f\bigl(L^b_{\tau_t(a)}\bigr) \mid \mathcal{H}_x\bigr) = u_b\bigl(x, L^x_{\tau_t(a)}\bigr), \quad a < x < b,$$

where $u_b(x,l) \stackrel{\text{def}}{=} \mathbb{E}(f(L^b_{\tau_t(a)}) \mid L^x_{\tau_t(a)} = l)$. It can be checked that the function $u_b(\cdot,\cdot)$ belongs to $\mathcal{C}^{1,2}([a,b) \times \mathbb{R}_+)$ and u_b is continuous on $[a,b] \times \mathbb{R}_+$ (see Stroock and Varadhan [15, Theorem 6.3.4]). Applying Itô's formula to the martingale $(\mathbb{E}(f(L^b_{\tau_t(a)}) \mid \mathcal{H}_x), x \geqslant a)$, we obtain:

$$\begin{aligned}
\mathbb{E}\bigl(f\bigl(L^b_{\tau_t(a)}\bigr) \mid \mathcal{H}_x\bigr) &= u_b\bigl(x, L^x_{\tau_t(a)}\bigr) \\
&= u_b(a,t) - 2\int_a^x \frac{\partial u_b}{\partial l}\bigl(y, L^y_{\tau_t(a)}\bigr)\sqrt{\kappa\bigl(L^y_{\tau_t(a)}\bigr)}\, d\widetilde{W}_y \\
&= u_b(a,t) - 2\iint \phi_b(s,y)\mathbb{1}_{(y \leqslant x)}\, dM(s,y),
\end{aligned}$$

where $\phi_b(s,y) \stackrel{\text{def}}{=} \frac{\partial u_b}{\partial l}(y, L^y_{\tau_t(a)})\sqrt{\kappa(L^y_{\tau_t(a)})}\,\mathbb{1}_{(a,b]}(y)\mathbb{1}_{(0,\tau_t(a)]}(s)$ is an identifiable process. Hence for each $x < b$, $\mathbb{E}(f(L^b_{\tau_t(a)}) \mid \mathcal{H}_x) \in \mathcal{K}$. The continuity of the function $u_b(\cdot,\cdot)$ implies that when x tends to b,

$$\mathbb{E}\bigl(f\bigl(L^b_{\tau_t(a)}\bigr) \mid \mathcal{H}_x\bigr) \equiv u_b\bigl(x, L^x_{\tau_t(a)}\bigr) \xrightarrow{\text{a.s.}} u_b\bigl(b, L^b_{\tau_t(a)}\bigr) = f\bigl(L^b_{\tau_t(a)}\bigr).$$

This convergence also holds in $\mathcal{L}^2$, by the fact that $\sup_{a \leqslant x \leqslant b} \mathbb{E}(L^x_{\tau_t(a)})^2 < \infty$ (cf. (4.11)). Hence $f(L^b_{\tau_t(a)}) \in \mathcal{K}$, since $\mathcal{K}$ is closed in $\mathcal{L}^2$.

To show (4.8), we consider the stopping time $\tau_s(a)$ and define the new flow

$$\widehat{X}_u(x) \stackrel{\text{def}}{=} X_{u+\tau_s(a)}\bigl(X^{-1}_{\tau_s(a)}(x)\bigr), \quad u \geqslant 0,\ x \in \mathbb{R},$$

where $X^{-1}_{\tau_s(a)}(x)$ denotes the unique $y \in \mathbb{R}$ such that $X_{\tau_s(a)}(y) = x$. By using the strong Markov property, the flow $\widehat{X}$ has the same law as X, and is independent of $\mathcal{B}_{\tau_s(a)}$. Define $\hat{L}^x_t$ and $\hat{\tau}$ relative to $\widehat{X}$ in the same way as L and τ are defined relative to X. We have $\tau_t(a) - \tau_s(a) = \hat{\tau}_{t-s}(0)$ and

$$L^{a+x}_{\tau_t(a)} - L^{a+x}_{\tau_s(a)} = \hat{L}^{X_{\tau_s(a)}(a+x)}_{\hat{\tau}_{t-s}(0)},\ x \in \mathbb{R},$$

and by using Fact 2.1,

$$X_{\tau_s(a)}(a+x) = \int_0^x dy \, \exp\bigl((\beta_2 - \beta_1) L^{a+y}_{\tau_s(a)}\bigr).$$

Since $\hat{L}$ and $\hat{\tau}$ are independent of L and τ, it follows from (4.10) that the process $(L^{a+x}_{\tau_t(a)} - L^{a+x}_{\tau_s(a)},\ x \geqslant 0)$ is an (inhomogeneous) Markov process, and (4.8) follows by the same method. $\square$

Step 4.2. The first assertion $D_n \subset \mathcal{K}$ follows from Lemma 4.2. To show the second assertion, we first note the following fact: let $\phi_1, \phi_2 \in \mathcal{L}^2(\mathcal{I})$ such that the supports of ϕ_1 and ϕ_2 are disjoint. Define

$$H_i \stackrel{\text{def}}{=} \iint \phi_i(s, y) \, dM(s, y) \in \mathcal{K}, \quad i = 1, 2.$$

We will show that if $H_1 \in \mathcal{L}^4$ and $H_2 \in \mathcal{L}^4$ (so that $H_1 H_2 \in \mathcal{L}^2$), then

$$H_1 H_2 \in \mathcal{K}. \tag{4.12}$$

To this end, we use the projections $(i = 1, 2)$

$$H_i(x) \stackrel{\text{def}}{=} \mathbb{E}\bigl(H_i \mid \mathcal{H}_x\bigr) = \mathbb{E}(H_i) + \iint \phi_i(s, y) \, \mathbb{1}_{(y \leqslant x)} \, dM(s, y).$$

Thanks to Theorem 3.1, we know that $(H_1(x))$ and $(H_2(x))$ are two continuous martingales. Note that $\int H_1(y) \, dH_2(y) = \iint H_1(y) \phi_2(s, y) \, dM(s, y)$ (which can be proven by approximating ϕ_2 by elementary identifiable processes). Itô's formula yields to

$$H_1 H_2 = \mathbb{E}(H_1 H_2) + \iint \bigl(H_1(y)\phi_2(s, y) + H_2(y)\phi_1(s, y)\bigr) \, dM(s, y) \in \mathcal{K},$$

because the finite variation term vanishes thanks to the assumption of disjoint supports of ϕ_1 and ϕ_2. Moreover we note directly from Definition 3.1 that the product $H\psi$ of an $(\mathcal{H}_x)_{x \in \mathbb{R}}$-predictable process $(H(x))$ and an $\mathcal{H}$-identifiable process $(\psi(x, t),\ x \in \mathbb{R},\ t \geqslant 0)$ is still $\mathcal{H}$-identifiable. Hence, (4.12) is proved.

Thanks to (4.12) and Lemma 4.2 applied to $f(x) = x^k$ for $k \geqslant 0$, Step 4.2 is established. $\square$

Step 4.3. Using Step 4.2 and (4.6) and applying a result due to Rogers [10, Lemma 3], the inclusion $\mathcal{L}^2(\mathcal{A}_n) \subset \mathcal{K}$ follows. Let us show that

$$\mathcal{A}_n \subset \mathcal{A}_{n+1}. \tag{4.13}$$

Let $x < y \in \mathbb{R}$. Denote by $v \stackrel{\text{def}}{=} L^y_{\tau_t(x)}$. Since on $[\tau_t(x), \tau_v(y))$, $X^{-1}(0) < y$, we have

$$L^z_{\tau_t(x)} = L^z_{\tau_v(y)}, \quad \forall z \geqslant y.$$

Write $a_j^n \stackrel{\text{def}}{=} j2^{-n}$ for $j \in \mathbb{Z}$ and $n \geqslant 1$. Applying the above observation to $x = a_j^n = a_{2j}^{n+1}$, $y = a_{2j+1}^{n+1}$ and $z = a_{j+1}^n = a_{2j+2}^{n+1}$, we obtain

$$L_{\tau_t(a_j^n)}^{a_{j+1}^n} = L_{\tau_t(x)}^z = L_{\tau_v(y)}^z = L_{\tau_v(a_{2j+1}^{n+1})}^{a_{2j+2}^{n+1}},$$

with $v \stackrel{\text{def}}{=} L_{\tau_v(a_{2j}^{n+1})}^{a_{2j+1}^{n+1}}$. This shows that $L_{\tau_t(a_j^n)}^{a_{j+1}^n}$ is $\mathcal{A}_{n+1}$-measurable, as desired. $\square$

Step 4.4. For $x \in \mathbb{R}$, we will prove that $(L_{\tau_t(x)}^y,\ y \in \mathbb{R})$ is $\mathcal{A}_\infty$-measurable. In fact, if $y > x$, we have that $L_{\tau_t(x)}^y = \lim_{n\to\infty, j2^{-n}\to x, (j+k)2^{-n}\to y} L_{\tau_t(j2^{-n})}^{(j+k)2^{-n}}$ is $\mathcal{A}_\infty$-measurable. Now consider $y < x$, observe that $L_{\tau_t(x)}^y = \inf\{u > 0\colon\ L_{\tau_u(y)}^x > t\}$ is $\mathcal{A}_\infty$-measurable.

Using Fact 2.1, for all t, x, $\tau_t(x) = \int_{\mathbb{R}} \kappa(L_{\tau_t(x)}^y)\,dy$ is $\mathcal{A}_\infty$-measurable. We obtain that the process $L_\cdot^\cdot$ is $\mathcal{A}_\infty$-measurable. Hence the same holds for $\kappa(L_\cdot^\cdot)$, which implies that $(Y_t,\ t \geqslant 0)$ is $\mathcal{A}_\infty$-measurable. Applying (1.4) with $x = Y_t$, we have

$$B_t = -Y_t - \beta_1 \int_0^t \mathbb{1}_{(Y_s \geqslant Y_t)}\,ds - \beta_2 \int_0^t \mathbb{1}_{(Y_s < Y_t)}\,ds, \quad t \geqslant 0.$$

Consequently $(B_t,\ t \geqslant 0)$ is $\mathcal{A}_\infty$-measurable.

Finally, using Steps 4.2 and 4.3, we obtain

$$\mathcal{L}^2(\mathcal{A}_\infty) \subset \mathcal{K} \subset \mathcal{L}^2(\mathcal{B}_\infty) = \mathcal{L}^2(\mathcal{A}_\infty),$$

implying the desired result and completing the whole proof of Theorem 4.1. $\square$

5. Applications

Since $\mathcal{L}^2(\mathcal{H}_\infty) = \mathcal{L}^2(\mathcal{B}_\infty)$ (cf. Step 4.4), by standard arguments, the following corollary follows immediatly from Theorem 4.1.

COROLLARY 5.1. – *Every $\mathcal{L}^2$-bounded $(\mathcal{H}_x)$-martingale $(N_x,\ x \in \mathbb{R})$ admits a continuous version with the following representation*

$$N_x = \mathbb{E}(N_0) + \int_{t\geqslant 0} \int_{y\in\mathbb{R}} \psi(t, y)\,\mathbb{1}_{(y\leqslant x)}\,dM(t, y),$$

for some identifiable process ψ.

Define now

$$\hat{L}_t^x \stackrel{\text{def}}{=} L_{\alpha_t(x)}^x, \quad t > 0, x \in \mathbb{R}, \tag{5.1}$$

which is called the intrinsic local time in the Brownian motion case ($\beta_1 = \beta_2 = 0$). In the Brownian motion case, McGill [8] showed the important result that for a fixed $t > 0$, the process $\hat{L}_t^\cdot$ is a continuous semimartingale in the excursion filtration, and gave an explicit decomposition of this semimartingale. With different approaches, Rogers and Walsh [13] and [12, Theorem 4.1] gave the canonical decomposition of $\hat{L}_t^{\cdot}$ and interpreted the martingale part as an area integral with respect to local times. Here, we ask the same question for the flow. Note that $x \to \hat{L}_t^x$ is continuous. Define

$$\psi_r(t) \stackrel{\text{def}}{=} \inf\{x \in \mathbb{R}\colon\ A_r(x) > t\}, \quad r > t, \tag{5.2}$$

and $\psi_r(t) \stackrel{\text{def}}{=} \infty$ for $r \leqslant t$. Observe that $\psi_\cdot(t)$ is nonincreasing and continuous. Using (1.4) with $x = \psi_r(t)$, we obtain that for $r > t$:

$$X_r\big(\psi_r(t)\big) = \psi_r(t) + B_r + \beta_1 r + (\beta_2 - \beta_1)t, \tag{5.3}$$

since $A_r(\psi_r(t)) = t$. We denote by $(\lambda_r(t),\ r > t)$ the local time at 0 of the continuous semimartingale $(X_r(\psi_r(t)),\ r > t)$.

THEOREM 5.1 ($\beta_1 \geqslant 0 \geqslant \beta_2$). – *Fix $t > 0$ and $a \in \mathbb{R}$. The process $(\hat{L}_t^x, x \geqslant a)$ is an $(\mathcal{H}_x)$-continuous semimartingale with the following canonical decomposition*:

$$\hat{L}_t^x = N_x + V_x, \quad x \geqslant a,$$

where

$$N_x \stackrel{\text{def}}{=} -2 \iint \mathbb{1}_{(a<y\leqslant x\wedge\psi_s(t))}\, dM(s,y), \quad x \geqslant a,$$

is an $(\mathcal{H}_x)$-continuous martingale, with increasing process $(\int_0^\infty \mathbb{1}_{(a<Y_s\leqslant x\wedge\psi_s(t))}\, ds$, $x \geqslant a)$, and

$$V_x = \hat{L}_t^a - 2(x^- - a^-) - 2\beta_1 \int_{\alpha_t(x)}^{\alpha_t(a)} \mathbb{1}_{(Y_s\leqslant\psi_s(t))}\, ds + \big(\lambda_{\alpha_t(x)}(t) - \lambda_{\alpha_t(a)}(t)\big), \quad x \geqslant a,$$

is a bounded variation process adapted to $(\mathcal{H}_x)$.

Proof. – Notice that $0 \leqslant X_{\alpha_t(x)}(x) = X_{\alpha_t(x)}(\psi_{\alpha_t(x)}(t))$. By applying Tanaka's formula to the continuous semimartingale $X_\cdot(\psi_\cdot(t))$ and to the continuous semimartingale $X_\cdot(x)$ at times $\alpha_t(x)$ and $\alpha_t(a)$, we obtain in the same way as Rogers and Walsh in [13, proof of Theorem 2]:

$$\begin{aligned}\hat{L}_t^x - \hat{L}_t^a &= -2(x^- - a^-) - 2\beta_1 \int_{\alpha_t(x)}^{\alpha_t(a)} \mathbb{1}_{(Y_s\leqslant\psi_s(t))}\, ds + \big(\lambda_{\alpha_t(x)}(t) - \lambda_{\alpha_t(a)}(t)\big)\\ &\quad - 2\int_0^\infty \mathbb{1}_{(a<Y_s\leqslant x\wedge\psi_s(t))}\, dB_s.\end{aligned}$$

Therefore the proof of Theorem 5.1 will be completed once we show that

$$\iint \phi(s,y)\, dM(s,y) = \int_0^\infty \phi(s,Y_s)\, dB_s, \tag{5.4}$$

for the identifiable process $\phi(s,y) \stackrel{\text{def}}{=} \mathbb{1}_{(a<y\leqslant x\wedge\psi_s(t))}$ $(x,\ t$ are fixed). Observe that $\{\alpha_t(y)\leqslant s\}=\{t\leqslant A_s(y)\}=\{\psi_s(t)\leqslant y\}$. Hence $\phi(s,y)=\mathbb{1}_{(s<\alpha_t(y))}\mathbb{1}_{(a<y\leqslant x)}$. Thanks to Proposition 3.3, (5.4) follows. □

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