PATH DECOMPOSITIONS FOR REAL LEVY PROCESSES

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ABSTRACT. – Let X be a real Lévy process and let $X^{\uparrow}$ be the process conditioned to stay positive. We assume that 0 is regular for $(-\infty, 0)$ and $(0, +\infty)$ with respect to X. Using elementary excursion theory arguments, we provide a simple probabilistic description of the reversed paths of X and $X^{\uparrow}$ at their first hitting time of $(x, +\infty)$ and last passage time of $(-\infty, x]$, on a fixed time interval $[0, t]$, for a positive level x. From these reversion formulas, we derive an extension to general Lévy processes of Williams' decomposition theorems, Bismut's decomposition of the excursion above the infimum and also several relations involving the reversed excursion under the maximum.

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RÉSUMÉ. – Soit X un processus de Lévy et $X^{\uparrow}$ le même processus conditionné à rester positif. On suppose que 0 est régulier pour $(-\infty, 0)$ et $(0, \infty)$ par rapport à X. Par des arguments simples de théorie des excursions, nous décomposons la loi des trajectoires de X et $X^{\uparrow}$ retournées aux temps d'entrée de $(x, +\infty)$ et de sortie de $(-\infty, x]$. De ces formules de reversion, on déduit une extension au cas des processus de Lévy généraux, des théorèmes de décomposition de Williams, du théorème de décomposition de Bismut de l'excursion au dessus du minimum, ainsi que plusieurs relations faisant intervenir l'excursion sous le maximum retournée.

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1. Introduction

Let $(X_t)_{t \geqslant 0}$ be a real Lévy process, that is a real valued process with homogeneous and independent increments. The supremum (respectively infimum) of X on the time interval $[0, t]$ is denoted by S_t (respectively I_t). We assume that 0 is regular for $(0, +\infty)$ and $(-\infty, 0)$ with respect to X. A classical result says that $X - I$ (respectively $X - S$) is a strong Markov process for which 0 is regular (see Bingham [5] or Bertoin [4] Chapter 6 for a proof). Let us denote by L (respectively L^*) the local time at 0 of $X - I$ (respectively $X - S$): they are uniquely defined up to a multiplicative constant and their normalization is specified in Proposition 2.3.

As Rogers noticed in [17], as soon as the Lévy measure charges the positive numbers, $X - S$ may hit zero by X jumping across the level of its previous maximum. The classical

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Itô excursion measure of $S - X$ loses the information about this jump. Let us introduce the relevant definition of the excursion measure under the maximum (respectively above the infimum) denoted by N^* (respectively N). It has the property to record the final jump of the excursion, which represents the amount the excursion overshoots when X attains a new maximum (respectively infimum). Let $(g_i, d_i), i \in \mathcal{I}$ (respectively $(g_j, d_j), j \in \mathcal{I}^*$) the excursion intervals of $X - I$ (respectively $S - X$) above 0. We define the excursions above the infimum and under the supremum by

$$\begin{cases} \omega^i(s) = X_{(g_i+s)\wedge d_i} - X_{g_i}, & i \in \mathcal{I}, \\ \omega^j(s) = X_{(g_j+s)\wedge d_j} - X_{g_j}, & j \in \mathcal{I}^*. \end{cases}$$

Then, the point measures

$$\sum_{i \in \mathcal{I}} \delta_{(L_{g_i}, \omega^i)} \quad \text{and} \quad \sum_{j \in \mathcal{I}^*} \delta_{(L^*_{g_j}, \omega^j)}$$

are distributed respectively as $\mathbf{1}_{\{l \leqslant \eta\}} \mathcal{N}(dl\, d\omega)$ and $\mathbf{1}_{\{l \leqslant \eta^*\}} \mathcal{N}^*(dl\, d\omega)$, where $\mathcal{N}$ and $\mathcal{N}^*$ are Poisson measures with respective intensities $dl\, N(d\omega)$ and $dl\, N^*(d\omega)$, and where

$$\begin{cases} \eta = \inf\{t \geqslant 0 \colon \mathcal{N}([0, t] \times \{\zeta(\omega) = \infty\}) \geqslant 1\}, \\ \eta^* = \inf\{t \geqslant 0 \colon \mathcal{N}^*([0, t] \times \{\zeta(\omega) = \infty\}) \geqslant 1\} \end{cases}$$

($\zeta(\omega)$ being the lifetime of the path ω). The random variables η and η^* have the same law as respectively L_∞ and L^*_∞, that are exponentially distributed or infinite a.s.

In Section 3, Theorem 3.3 provides a decomposition of the law of the excursion under the supremum reversed at its final jump: More precisely, we decompose the law of $(\omega_{\zeta(\omega)} - \omega_{(\zeta(\omega)-s)-}; 0 \leqslant s \leqslant \zeta(\omega))$ under $N^*(\cdot \bigcap \{\omega_\zeta > 0\})$, in terms of the law of X and its Lévy measure. Theorems 3.2 and 3.1 give similar results for $(X_{\tau_x} - X_{(\tau_x-s)-}; 0 \leqslant s \leqslant \tau_x)$ under $\mathbf{P}(\cdot \mid X_{\tau_x} > x)$ and $(X_{\sigma_x(t)} - X_{(\sigma_x(t)-s)-}; 0 \leqslant s \leqslant \sigma_x(t))$ under $\mathbf{P}(\cdot \mid X_{\sigma_x(t)} > x)$, where we have set for any $x, t > 0$:

$$\tau_x = \inf\{s \geqslant 0 \colon X_s > x\} \quad \text{and} \quad \sigma_x(t) = \sup\{s \in [0, t] \colon X_s \leqslant x\}.$$

Williams in [19], and many authors after him, explored the connections between the Brownian motion, the three-dimensional Bessel process and the Brownian excursion (see for instance Pitman [16] and Bismut [6]). Many of these identities in the Brownian case hold in a more general setting for totally asymetric Lévy processes: see Bertoin [2] for a generalized Pitman theorem for spectrally negative Lévy processes and Chaumont [7–9] for Williams' theorems and Bismut's decomposition in the spectrally positive case. Let us mention that Chaumont has also explored the stable case in detail in [10] and [7], providing several path-constructions and identities concerning the stable meander, the normalised excursion and the stable bridge. In these results the role of the three-dimensional Bessel process is played by the Lévy process conditioned to stay positive. This process, denoted by $X^\uparrow$, has been introduced by Bertoin in a general setting (see [3]). Bertoin's construction of $X^\uparrow$ is recalled in Section 2.2. We use it in combination with Theorems 3.2 and 3.1 to get in Section 4.1 the generalized

first Williams' decomposition theorem, then Bismut's decomposition of the excursion above the infimum in Section 4.2 and the second Williams' decomposition theorem in Section 4.3.

Let us explain more precisely these results: For any $t > 0$, we define $U_t^* = X_{L_t^{*-1}}$ if $L_\infty^* > t$ and $U_t^* = +\infty$ if not. The process $(U_t^*; t \geqslant 0)$ is a subordinator (see Bertoin [4], Chapter 6) and its drift coefficient is denoted by d^*. A classical result due to Kesten (see [11]) ensures that $\mathbf{P}(X_{\tau_x} = x) > 0$ iff $d^* > 0$. We assume that $d^* > 0$ and that X does not drift to $-\infty$. Then, we can show that $\sigma_x^\uparrow = \sup\{s \geqslant 0: X_s^\uparrow \leqslant x\}$ is finite a.s. Theorem 4.2 show that

$$\mathbf{P}\big(X_{\sigma_x^\uparrow}^\uparrow = x\big) = \mathbf{P}(X_{\tau_x} = x)$$

and that $(x - X_{(\tau_x - s)-}; 0 \leqslant s \leqslant \tau_x)$ under $\mathbf{P}(\cdot \mid X_{\tau_x} = x)$ has the same law as $(X_s^\uparrow; 0 \leqslant s \leqslant \sigma_x^\uparrow)$ under $\mathbf{P}(\cdot \mid X_{\sigma_x^\uparrow}^\uparrow = x)$.

We also prove in Theorem 4.5 a path decomposition of the excursion above the infimum similar to Bismut's decomposition of the Brownian excursion: we show that for any nonnegative measurable functionals G and D on the space of càdlàg paths with a finite lifetime and for any nonnegative measurable function f,

$$N\left(\int_0^{\zeta(\omega)} dt\, G(\omega_s; 0 \leqslant s \leqslant t) f(\omega_t) D(\omega_{t+s}; 0 \leqslant s \leqslant \zeta(\omega) - t)\right)$$

$$= \int_0^{+\infty} dx\, f(x) u^*(x) \mathbf{E}\big[G(X_s^\uparrow; 0 \leqslant s \leqslant \sigma_x^\uparrow) \mid X_{\sigma_x^\uparrow}^\uparrow = x\big] \mathbf{E}\big[D(X_s; 0 \leqslant s \leqslant \tau_{-x})\big],$$

where $\tau_{-x} = \inf\{s \geqslant 0: X_s < -x\}$ and where u^* is the co-excessive version of the density of the potential measure associated with the subordinator U^*.

Section 4.3 is devoted to the proof of Theorem 4.10 that can be seen as an analogue for general Lévy processes of the second Williams' decomposition theorem that originally concerns the Brownian excursion split at its maximum. Let us describe our result: For any $x > 0$, we set $\tau_x^\uparrow = \inf\{s \geqslant 0: X_s^\uparrow > x\}$. Proposition 4.7 shows that

$$\mathbf{P}\big(X_{\tau_x^\uparrow}^\uparrow = x\big) > 0 \quad \text{iff} \quad d^* > 0.$$

Let us denote by $X^\downarrow$ the process X conditioned to stay negative (that is defined in Section 2.2); we write $\overline{g}(\omega)$ the instant when the excursion ω attains its maximum. Theorem 4.10 shows the law of $\omega_{\overline{g}(\omega)}$ under N admits a density with respect to Lebesgue measure that we specify. Under $N(\cdot \mid \omega_{\overline{g}(\omega)} = x)$, the processes $(\omega_s; 0 \leqslant s \leqslant \overline{g}(\omega))$ and $(\omega_{s+\overline{g}(\omega)}; 0 \leqslant s \leqslant \zeta(\omega) - \overline{g}(\omega))$ are mutually independent. Furthermore,

– the process $(\omega_s; 0 \leqslant s \leqslant \overline{g}(\omega))$ is distributed as $(X_s^\uparrow; 0 \leqslant s \leqslant \tau_x^\uparrow)$ under $\mathbf{P}(\cdot \mid X_{\tau_x^\uparrow}^\uparrow = x)$;

– the law of $(\omega_{s+\overline{g}(\omega)}; 0 \leqslant s \leqslant \zeta(\omega) - \overline{g}(\omega))$ is absolutely continuous with respect to the law of $(X_s^\downarrow; 0 \leqslant s \leqslant \tau_{-x}^\downarrow)$ (with an evident notation for $\tau_{-x}^\downarrow$) and the corresponding density has the form $\varphi(X_{\tau_{-x}^\downarrow}^\downarrow)$, where the function φ is specified.

Let us mention that we provide two other path decompositions that concern the excursion above the infimum (Theorem 4.6) and the process $(X_s^\uparrow; 0 \leqslant s \leqslant \sigma_x^\uparrow)$ when $X_{\sigma_x^\uparrow}^\uparrow > x$ (Theorem 4.1).

2. Preliminary results

2.1. Notation and basic assumptions

In this section we state our notation and the assumptions made at different stages of the paper. We also recall fundamental results of fluctuation theory that are our starting-point and we give some simple facts concerning excursion theory applied to Lévy processes, that is the main tool we use.

We begin with some notations concerning the canonical space. Let Ω be the space of right-continuous functions with left limits from $(0, +\infty)$ to $\mathbb{R}$ (the so-called càdlàg functions space) endowed with the Skorokhod's topology. Let $\mathcal{F}$ stand for its Borel σ-algebra. For any path ω in Ω we define its lifetime $\zeta(\omega)$ by $\inf\{t \geqslant 0: \omega(s) = \omega(t), \forall s \geqslant t\}$, with the usual convention $\inf \emptyset = \infty$. For any time $t \geqslant 0$, we denote the jump of ω at t by $\Delta\omega(t) = \omega(t) - \omega(t-)$; we also define the path respectively stopped at t, stopped just before t, reversed at t and reversed just before t, by

$$\omega(\cdot \wedge t) = \big(\omega(s \wedge t); s \geqslant 0\big), \qquad \omega(\cdot \wedge t-) = \big(\omega(s \wedge t) - \Delta\omega(t)\mathbf{1}_{[t,+\infty)}(s); s \geqslant 0\big),$$

$$\hat{\omega}^t = \big(\omega(t) - \omega((t-s)-); s \geqslant 0\big), \qquad \hat{\omega}^{t-} = \hat{\omega}^t - \Delta\omega(t),$$

with the convention $\omega(s-) = \omega(0)$ for any non-positive real number s. When $\zeta(\omega)$ is finite, $\hat{\omega}^{\zeta(\omega)}$ is well defined and simply denoted by $\hat{\omega}$. We use a non-standard notation for the shifted path at time t defined by

$$\omega \circ \theta_t = \big(\omega(s + t) - \omega(t); s \geqslant 0\big).$$

For any $x > 0$, we denote by $\tau_x(\omega)$ and $\tau_{-x}(\omega)$ the first hitting time of respectively $(x, +\infty)$ and $(-\infty, -x)$:

$$\tau_x(\omega) = \inf\{s > 0: \omega(s) > x\}, \qquad \tau_{-x}(\omega) = \inf\{s > 0: \omega(s) < -x\}$$

(with the usual convention $\inf \emptyset = +\infty$). For any time $t > 0$, we also denote by $\sigma_x(t, \omega)$ and $\sigma_{-x}(t, \omega)$ the last passage time in respectively $(-\infty, x]$ and $[-x, +\infty)$ on the time interval $[0, t]$:

$$\sigma_x(t, \omega) = \sup\{0 \leqslant s \leqslant t: \omega(s) \leqslant x\}, \qquad \sigma_{-x}(t, \omega) = \sup\{0 \leqslant s \leqslant t: \omega(s) \geqslant -x\}$$

(with the convention $\sup \emptyset = +\infty$). We write $\sigma_x(\omega) = \lim_{t \to +\infty} \sigma_x(t, \omega)$, the limit being taken in $[0, +\infty]$. Next, we denote respectively by $\underline{g}_t(\omega)$ and $\overline{g}_t(\omega)$, the last infimum time and the last supremum time of ω before t:

$$\underline{g}_t(\omega) = \sup\Big\{s \in [0, t): \inf_{[0,t]} \omega = \omega(s-) \wedge \omega(s)\Big\}$$

and

$$\overline{g}_t(\omega) = \sup\big\{s \in [0, t): \sup_{[0,t]} \omega = \omega(s-) \vee \omega(s)\big\}.$$

We also write $\underline{g}(\omega) = \lim_{t \to +\infty} \underline{g}_t(\omega)$ and $\overline{g}(\omega) = \lim_{t \to +\infty} \overline{g}_t(\omega)$ (note that these quantities may be infinite).

We denote by X the canonical process on Ω: $X_t(\omega) = \omega(t)$ and we consider the probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ under which X is a Lévy process started at 0, with characteristic exponent ψ:

$$\mathbf{E}\big[e^{i\lambda X_t}\big] = e^{-t\psi(\lambda)}, \quad t \geqslant 0, \ \lambda \in \mathbb{R}.$$

By the Lévy–Khintchine theorem, ψ has the form

$$\psi(\lambda) = ia\lambda + b\lambda^2 + \int \pi(dr)\big(1 - e^{i\lambda r} + i\lambda r\mathbf{1}_{\{|r|<1\}}\big), \quad \lambda \in \mathbb{R},$$

where a is a real number, b is non-negative and the Lévy measure π is a Radon measure on $\mathbb{R}$ not charging 0, which satisfies

$$\int \pi(dr)\big(1 \wedge |r|^2\big) < +\infty.$$

If $J = \{s \geqslant 0: \ \Delta X_s \neq 0\}$, then the point measure $\mathcal{N}(ds\,dr) = \sum_{s \in J} \delta_{(s, \Delta X_s)}$ is a Poisson measure with intensity $ds\,\pi(dr)$.

Let us recall some path-properties of Lévy processes. For any $t \geqslant 0$, we have

$$\widehat{X}^t \overset{\text{(law)}}{=} (X_s; 0 \leqslant s \leqslant t)$$

(see Bertoin [4]). This identity is refered to as the "duality property".

In the whole paper (Section 3 excepted), we make the following assumption:

Assumption (A). – The point 0 is regular for $(0, +\infty)$ and for $(-\infty, 0)$ with respect to X.

(In particular X cannot be a subordinator or a compound Poisson process.) As a consequence of (A), we recall the following result (see Millar [15]): For any $t \geqslant 0$, the Lévy process X reaches its infimum (respectively supremum) on $[0, t]$ at a unique instant that must be $\underline{g}_t(X)$ (respectively $\overline{g}_t(X)$).

For every $t \geqslant 0$, we write

$$S_t = \sup_{s \in [0,t]} X_s, \qquad I_t = \inf_{s \in [0,t]} X_s.$$

It is well-known that $X - S$ and $X - I$ are strong Markov processes (see Bertoin [4], Chapter 6). Assumption (A) implies that 0 is regular for itself with respect to both these processes. Rogers has shown in [17] that this implies

$$\mathbf{P}\big(\exists t \in (0, +\infty): \ X_{t-} = I_{t-} < X_t\big) = 0 \tag{1}$$

and

$$\mathbf{P}\big(\exists t \in (0, +\infty)\colon X_{t-} = S_{t-} < S_t\big) = 0. \tag{2}$$

Let us recall briefly the proof: we only need to show for any $\varepsilon > 0$

$$\mathbf{E}\left[\sum_{s \in J} \mathbf{1}_{\{X_{s-} = I_{s-}\}} \mathbf{1}_{(\varepsilon, +\infty)}(\Delta X_s)\right] = 0.$$

Apply the compensation formula (see Bertoin [4], p. 7) to get

$$\mathbf{E}\left[\sum_{s \in J} \mathbf{1}_{\{X_{s-} = I_{s-}\}} \mathbf{1}_{(\varepsilon, +\infty)}(\Delta X_s)\right] = \pi\big((\varepsilon, +\infty)\big) \int_0^{+\infty} ds\, \mathbf{P}(X_s = I_s).$$

But the duality property implies for any $s > 0$, $\mathbf{P}(X_s = I_s) = \mathbf{P}(S_s = 0) = 0$, because 0 is regular for $(0, +\infty)$. A similar argument proves (2). $\quad\square$

We denote the local times of $X - I$ and $X - S$ at the level 0 by $(L_t)_{t \geqslant 0}$ and $(L_t^*)_{t \geqslant 0}$. They are uniquely determined up to a multiplicative constant specified in a forthcoming lemma. The limit in $[0, +\infty]$ of L_t (respectively L_t^*) when t goes to infinity is denoted by L_∞ (respectively L_∞^*). The quantity L_∞ (respectively L_∞^*) is a.s. finite or a.s. infinite according as X drifts or not to $+\infty$ (respectively $-\infty$). If L_∞ (respectively L_∞^*) is finite a.s., then it is exponentially distributed with parameter denoted by p (respectively p^*).

Eq. (1) and the dual result $\mathbf{P}(\exists t \in (0, +\infty)\colon X_{t-} = S_{t-} > X_t) = 0$ imply that $\mathbf{P}$-a.s. the sets $\{s \geqslant 0\colon X_s > I_s\}$ and $\{s \geqslant 0\colon X_s < S_s\}$ are open sets (we have denoted $(g_i, d_i), i \in \mathcal{I}$ and $(g_j, d_j), j \in \mathcal{I}^*$ their respective connected components). Let m denote the Lebesgue measure on $\mathbb{R}$. The duality property and Assumption (A) imply that $\mathrm{m}(s \geqslant 0\colon X_s = S_s) = \mathrm{m}(s \geqslant 0\colon X_s = I_s) = 0$. Thus,

$$\mathbf{P}\text{-a.s.} \quad \mathrm{m}\left(\mathbb{R} \setminus \bigcup_{i \in \mathcal{I}}(g_i, d_i)\right) = \mathrm{m}\left(\mathbb{R} \setminus \bigcup_{j \in \mathcal{I}^*}(g_j, d_j)\right) = 0. \tag{3}$$

Let N and N^* be the excursion measures of X above its infimum and under its supremum as defined in the first section. Observe that as soon as the Lévy measure π charges $(-\infty, 0)$ (respectively $(0, +\infty)$), the set of excursions ω ending with a negative jump (respectively positive jump) has a positive N-measure (respectively N^*-measure). But thanks to (1) and the dual result, we see that excursions above the infimum and under the supremum leave 0 continuously.

Let $(L_t^{-1})_{t \geqslant 0}$ and $(L_t^{*-1})_{t \geqslant 0}$ be the right-continuous inverses of L and L^*:

$$L_t^{-1} = \inf\{s \geqslant 0\colon L_s > t\}, \qquad L_t^{*-1} = \inf\{s \geqslant 0\colon L_s^* > t\}$$

(with the convention $\inf \emptyset = \infty$). Recall that $\mathbf{P}$-a.s.

$$\bigcup_{s \geqslant 0}(L_{s-}^{-1}, L_s^{-1}) = \bigcup_{i \in \mathcal{I}}(g_i, d_i) \quad \text{and} \quad \bigcup_{s \geqslant 0}(L_{s-}^{*-1}, L_s^{*-1}) = \bigcup_{j \in \mathcal{I}^*}(g_j, d_j). \tag{4}$$

For any $t \geqslant 0$ we define $U_t = -X_{L_t^{-1}}$ if $L_\infty > t$ and $U_t = +\infty$ if not. In a similar way, we define $U_t^* = X_{L_t^{*-1}}$ if $L_\infty^* > t$ and $U_t^* = +\infty$ if not. The processes (L^{-1}, U) and (L^{*-1}, U^*) are called the ladder processes. They are two-dimensional subordinators killed at respective rates p and p^*; their bivariate Laplace exponents are denoted by

$$\kappa(\alpha, \beta) = -\log \mathbf{E}\big[\exp\big(-\alpha L_1^{-1} - \beta U_1\big)\big] \quad \text{and}$$
$$\kappa^*(\alpha, \beta) = -\log \mathbf{E}\big[\exp\big(-\alpha L_1^{*-1} - \beta U_1^*\big)\big]$$

(see Bertoin [4], Chapter 6 for a detailed account). Next, we define the two potential measures $\mathcal{U}$ and $\mathcal{U}^*$ associated with U and U^*:

$$\begin{cases} \int_\mathbb{R} \mathcal{U}(dx) f(x) = \mathbf{E}\big[\int_0^{L_\infty} dv\, f(U_v)\big], \\ \int_\mathbb{R} \mathcal{U}^*(dx) f(x) = \mathbf{E}\big[\int_0^{L_\infty^*} du\, f(U_u^*)\big]. \end{cases}$$

Let d^* be the drift coefficient of the subordinator U^*: $d^* = \lim_{\beta \to +\infty} \kappa^*(0, \beta)/\beta$. We recall the following result, due to Kesten [11] (see also Bertoin [4], Chapter 3, Theorem 5): assume that d^* is positive, and let $u^* : (-\infty, +\infty) \longrightarrow [0, +\infty)$ be the co-excessive version of the density of $\mathcal{U}^*$. Then u^* is continuous and positive on $(0, +\infty)$, $u^*(0^+) = 1/d^*$, and

$$\mathbf{P}(X_{\tau_x} = x) = d^* u^*(x), \quad x > 0,$$

where for convenience we write τ_x instead of $\tau_x(X)$. We prove the following simple lemma that will be used in Section 4.

LEMMA 2.1. – *Assume* (A) *and suppose that d^* is positive. Then for any nonnegative measurable functional F on Ω,*

$$\mathbf{E}\bigg[\int_0^{L_\infty^*} du\, F(X_{\cdot \wedge L_u^{*-1}})\bigg] = \int_0^{+\infty} dx\, u^*(x) \mathbf{E}\big[F(X_{\cdot \wedge \tau_x}) \mid X_{\tau_x} = x\big].$$

Proof. – Set $A = \int_0^{+\infty} dx\, \mathbf{E}[F(X_{\cdot \wedge \tau_x}); X_{\tau_x} = x]$ and for any positive number x define $H_x = L_{\tau_x}^*$. Thanks to (A), we check path by path that

$$\mathbf{P}\text{-a.s.} \quad \text{on} \quad \{\tau_x < \infty\}, \quad L_{H_x}^{*-1} = \tau_x.$$

Thus

$$A = \int_0^{+\infty} dx\, \mathbf{E}\big[F(X_{\cdot \wedge L_{H_x}^{*-1}}); X_{\tau_x} = x\big].$$

Denote by C the random set $\{x \geqslant 0: X_{\tau_x} = x\}$ and define the measures μ and ν by

$$\begin{cases} \mu(dx) = \mathbf{1}_C(x)\mathrm{m}(dx), \\ \int \nu(du) f(u) = \int \mu(dx) f(H_x). \end{cases}$$

Then,

$$A = \mathbf{E}\left[\int_{[0, S_\infty)} \mu(dx) F(X_{\cdot \wedge L_{H_x}^{*-1}}) \right] = \mathbf{E}\left[\int_{[0, L_\infty^*)} \nu(du) F(X_{\cdot \wedge L_u^{*-1}}) \right]. \tag{5}$$

For any positive number a,

$$\nu([0, a]) = \mathrm{m}\big(\{x \geqslant 0 : X_{\tau_x} = x; H_x \leqslant a\}\big)$$
$$= \mathrm{m}\big(\{x \geqslant 0 : X_{\tau_x} = x; \tau_x \leqslant L_a^{*-1}\}\big) = \mathrm{m}(C \cap [0, U_a^*]).$$

Let us first consider the case $U_a^* < \infty$: If there exists some s in $[0, a]$ such that $x \in (U_{s-}^*, U_s^*)$, then, $\tau_x = L_s^{*-1}$ and $U_s^* = X_{\tau_x} > x$. Thus,

$$\bigcup_{0 \leqslant s \leqslant a} (U_{s-}^*, U_s^*) \subset C^c \cap [0, U_a^*].$$

Let x be in $C^c \cap [0, U_a^*]$. Then, $X_{\tau_x} > x$. By (2), it follows that τ_x must be the end-point of some excursion interval of $S - X$ above 0 that is included in $[0, L_a^{*-1}]$. Then, by (4) there exists some s in $[0, a]$ such that

$$L_{s-}^{*-1} < L_s^{*-1} = \tau_x \quad \text{and} \quad S_{\tau_x -} = U_{s-}^* \leqslant x < U_s^* = X_{\tau_x}.$$

Hence,

$$C^c \cap [0, U_a^*] \subset \bigcup_{0 \leqslant s \leqslant a} [U_{s-}^*, U_s^*).$$

By combining this with the previous inclusion we get $\mathrm{m}(C^c \cap [0, U_a^*]) = \sum_{0 \leqslant s \leqslant a} \Delta U_s^*$. But the Lévy–Itô representation of U^* guarantees that $\mathbf{P}$-a.s.

$$U_a^* = d^* a + \sum_{0 \leqslant s \leqslant a} \Delta U_s^*, \quad 0 \leqslant a < L_\infty^*.$$

Then, $\mathbf{P}$-a.s. for every a in $[0, L_\infty^*)$, $\nu([0, a]) = d^* a$. Next, observe that for any $a > L_\infty^*$, $\nu([0, a]) = \mathrm{m}(C) = d^* L_\infty^*$. Thus, $\mathbf{P}$-a.s.

$$\nu(dx) = d^* \mathbf{1}_{[0, L_\infty^*)}(x) m(dx).$$

The desired result follows from (5) and the identity $\mathbf{P}(X_{\tau_x} = x) = d^* u^*(x)$. $\quad \square$

Let us introduce some notations: for any positive time t and any path ω, we denote the pre-infimum and the post-infimum path on the interval $[0, t]$ by:

$$\begin{cases} \overleftarrow{\omega}^t = \omega(\cdot \wedge \underline{g}_t(\omega)), \\ \underset{\rightarrow}{\omega}^t = (\omega \circ \theta_{\underline{g}_t(\omega)})(\cdot \wedge (t - \underline{g}_t(\omega))). \end{cases}$$

We also denote the pre-supremum and post-supremum processes on $[0, t]$ by $\overleftarrow{\omega}^t$ and $\overrightarrow{\omega}^t$:

$$\begin{cases} \overleftarrow{\omega}^t = \omega(\cdot \wedge \overline{g}_t(\omega)), \\ \overrightarrow{\omega}^t = (\omega \circ \theta_{\overline{g}_t(\omega)})(\cdot \wedge (t - \overline{g}_t(\omega))). \end{cases}$$

We often use the following lemma in Section 4:

LEMMA 2.2. – *Assume* (A). *Let T be independent of X and exponentially distributed with parameter $\alpha > 0$. Then, $\underleftarrow{X}^T$ and $\underrightarrow{X}^T$ are mutually independent and the following identities hold for any nonnegative measurable functional F on Ω:*

(i) $\mathbf{E}[F(\underleftarrow{X}^T)] = \kappa(\alpha, 0)\mathbf{E}[\int_0^{L_\infty} dv\, e^{-\alpha L_v^{-1}} F(X_{\cdot \wedge L_v^{-1}})].$

(ii) $\mathbf{E}[F(\underrightarrow{X}^T)] = \frac{1}{\kappa(\alpha,0)} N(\int_0^{\zeta} ds\, \alpha e^{-\alpha s} F(\omega_{\cdot \wedge s})).$

Proof. – Let G be any nonnegative measurable functional on Ω. We have

$$\mathbf{E}[F(\underleftarrow{X}^T)G(\underrightarrow{X}^T)] = \int_0^\infty dt\, \alpha e^{-\alpha t}\mathbf{E}[F(\underleftarrow{X}^t)G(\underrightarrow{X}^t)].$$

By (3) and by the definition of the excursions above the infimum, we have $\mathbf{P}$-a.s.

$$\int_0^\infty dt\, \alpha e^{-\alpha t} F(\underleftarrow{X}^t)G(\underrightarrow{X}^t) = \sum_{i \in \mathcal{I}} e^{-\alpha g_i} F(X_{\cdot \wedge g_i}) \int_0^{\zeta_i} ds\, \alpha e^{-\alpha s} G(\omega^i(\cdot \wedge s)).$$

Apply the compensation formula to get

$$\mathbf{E}[F(\underleftarrow{X}^T)G(\underrightarrow{X}^T)] = \mathbf{E}\left[\int_0^{L_\infty} dv\, e^{-\alpha L_v^{-1}} F(X_{\cdot \wedge L_v^{-1}})\right] N\left(\int_0^{\zeta} ds\, \alpha e^{-\alpha s} G(\omega_{\cdot \wedge s})\right)$$

and the following identities yield (i) and (ii):

$$N\left(\int_0^{\zeta} ds\, \alpha e^{-\alpha s}\right) = N(1 - e^{-\alpha \zeta}) = \kappa(\alpha, 0)$$

and

$$\mathbf{E}\left[\int_0^{L_\infty} dv\, e^{-\alpha L_v^{-1}}\right] = \frac{1}{\kappa(\alpha, 0)}. \qquad \square$$

We now specify the normalization of L and L^* thanks to the following proposition.

PROPOSITION 2.3. – *Assume* (A). *Fix the normalization of L. Then, the normalization of L^* can be chosen in order to have for any nonnegative measurable functional F on Ω*

$$N\left(\int_0^{\zeta} ds\, F(\hat{\omega}^s)\right) = \mathbf{E}\left[\int_0^{L_\infty^*} du\, F(X_{\cdot \wedge L_u^{*-1}})\right]$$

and the dual identity

$$N^* \left(\int_0^\zeta ds\, F(\hat{\omega}^s) \right) = \mathbf{E} \left[\int_0^{L_\infty} dv\, F(X_{\cdot \wedge L_v^{-1}}) \right].$$

Proof. – We denote by $\overleftarrow{\widehat{X}^T}$ the path $\overleftarrow{X}^T$ reversed at its lifetime $\underline{g}_T$. Observe that

$$\overleftarrow{\widehat{X}^T} = \hat{X}^T \circ \theta_{\overline{g}_T(\hat{X}^T)}$$

(we use the fact that the minimum of X over $[0, T]$ is attained **P**-a.s. at a unique time). We also denote by $\overrightarrow{\widehat{X}^T}$ the path $\overrightarrow{X}^T$ reversed at its lifetime $T - \underline{g}_T$. Similarly, we see that

$$\overrightarrow{\widehat{X}^T} = \hat{X}^T_{\cdot \wedge \overline{g}_T(\hat{X}^T)}.$$

The duality property implies that

$$\left(\overleftarrow{\widehat{X}^T}, \overrightarrow{\widehat{X}^T} \right) \overset{\text{(law)}}{=} \left(\overrightarrow{X}^T, \overleftarrow{X}^T \right).$$

Let G be any nonnegative measurable functional on Ω. Use Lemma 2.2 to get:

$$\mathbf{E}\left[F\left(\overleftarrow{\widehat{X}^T} \right) G\left(\overrightarrow{\widehat{X}^T} \right) \right] = \mathbf{E} \left[\int_0^{L_\infty} dv\, e^{-\alpha L_v^{-1}} F(\hat{X}^{L_v^{-1}}) \right] N\left(\int_0^\zeta ds\, \alpha e^{-\alpha s} G(\hat{\omega}^s) \right).$$

On the other hand, by replacing X with $-X$, we see that Lemma 2.2 also implies

$$\mathbf{E}\left[F\left(\overrightarrow{X}^T \right) G\left(\overleftarrow{X}^T \right) \right] = \mathbf{E} \left[\int_0^{L_\infty^*} du\, e^{-\alpha L_u^{*-1}} G(X_{\cdot \wedge L_u^{*-1}}) \right] N^*\left(\int_0^\zeta ds\, \alpha e^{-\alpha s} F(\omega_{\cdot \wedge s}) \right).$$

Thus, for any $\alpha > 0$

$$\mathbf{E}\left[\int_0^{L_\infty^*} du\, e^{-\alpha L_u^{*-1}} G(X_{\cdot \wedge L_u^{*-1}}) \right] N^*\left(\int_0^\zeta ds\, e^{-\alpha s} F(\omega_{\cdot \wedge s}) \right)$$

$$= \mathbf{E} \left[\int_0^{L_\infty} dv\, e^{-\alpha L_v^{-1}} F(\hat{X}^{L_v^{-1}}) \right] N\left(\int_0^\zeta ds\, e^{-\alpha s} G(\hat{\omega}^s) \right). \qquad (6)$$

By letting α go to 0, we see that the ratio

$$\frac{N^*(\int_0^\zeta ds\, F(\omega_{\cdot \wedge s}))}{\mathbf{E}[\int_0^{L_\infty} dv\, F(\hat{X}^{L_v^{-1}})]}$$

does not depend on F, provided it is well-defined (that is the denominator is positive and finite). Furthermore this ratio coincides with

$$\frac{N(\int_0^\zeta ds\, G(\widehat{\omega}^s))}{\mathbf{E}[\int_0^{L_\infty^*} du\, G(X_{\cdot \wedge L_u^{*-1}})]}$$

for any G such that the denominator is positive and finite. We can choose the normalization of N^*, or equivalently L^*, so that both ratios are equal to 1. $\square$

In the spectrally positive case, the first identity of Proposition 2.3 has been proved by Le Gall and Le Jan in [12] by a different method.

Immediate applications of Proposition 2.3 are the following identities due to Silverstein (see [18]), also mentioned in Rogers' paper [17]:

$$\begin{cases} N\big(\int_0^\zeta ds\, e^{-\alpha s - \beta \omega_s}\big) = 1/\kappa^*(\alpha, \beta), \\ N^*\big(\int_0^\zeta ds\, e^{-\alpha s + \beta \omega_s}\big) = 1/\kappa(\alpha, \beta). \end{cases}$$

We can also derive from Proposition 2.3 the Wiener–Hopf factorisation of the ladder exponents:

$$\kappa(\alpha, i\beta)\kappa^*(\alpha, -i\beta) = \alpha + \psi(\beta).$$

Indeed, (3) gives the following decomposition:

$$\int\limits_0^{+\infty} dt\, e^{-\alpha t + i\beta X_t} = \sum_{i \in \mathcal{I}} \int\limits_{g_i}^{d_i} dt\, e^{-\alpha t + i\beta X_t} = \sum_{i \in \mathcal{I}} e^{-\alpha g_i + i\beta X_{g_i}} \int\limits_0^{\zeta_i} ds\, e^{-\alpha s + i\beta \omega_i(s)}.$$

Taking the expectations and using the compensation formula, we get

$$1/(\alpha + \psi(\beta)) = \mathbf{E}\left[\int\limits_0^{L_\infty} du\, e^{-\alpha L_u^{-1} + i\beta X_{L_u^{-1}}} \right] N\left(\int\limits_0^\zeta ds\, e^{-\alpha s + i\beta \omega_s} \right),$$

which yields the Wiener–Hopf factorization thanks to Proposition 2.3.

2.2. The Lévy process conditioned to stay positive or negative

We introduce now the process conditioned to stay positive, respectively negative, denoted by $X^\uparrow$, respectively $X^\downarrow$. Bertoin in [3] provides a pathwise construction of $X^\uparrow$ and $X^\downarrow$ from concatenation of the excursions of X in $(0, +\infty)$, respectively $(-\infty, 0)$. Let us recall briefly this construction whose details can be found in [3], Section 3.

Although Bertoin's construction holds in a general setting, *we assume* (A). We denote by $(\mathcal{F}_t)_{t \geqslant 0}$ the natural filtration of X completed with the **P**-null sets of $\mathcal{F}$. Then, X is a semimartingale. Its continuous local martingale part is proportional to a standard Brownian motion and is independent of the non-continuous part. Let us denote by ℓ its

semimartingale local time at 0. We consider

$$A_t^+ = \int_0^t ds\, \mathbf{1}_{\{X_s > 0\}} \quad \text{and} \quad A_t^- = \int_0^t ds\, \mathbf{1}_{\{X_s \leqslant 0\}}.$$

Let us denote by α^+, respectively α^-, the right-continuous inverse of A^+, respectively A^-:

$$\alpha_t^+ = \inf\{s \geqslant 0 \colon A_s^+ > t\} \quad \text{and} \quad \alpha_t^- = \inf\{s \geqslant 0 \colon A_s^- > t\}$$

(with the usual convention $\inf \emptyset = \infty$). Let x be a real number. We denote its positive part, respectively negative part, by x_+, respectively x_-. We define a new process $X^\uparrow$ by

$$X_t^\uparrow = X_{\alpha_t^+} + \frac{1}{2}\ell_{\alpha_t^+} + \sum_{0 < s \leqslant \alpha_t^+} \mathbf{1}_{\{X_s \leqslant 0\}}(X_{s-})_+ + \mathbf{1}_{\{X_s > 0\}}(X_{s-})_- \quad \text{if } t < A_\infty^+$$

and by $X_t^\uparrow = +\infty$ if not. When X has no Brownian part, $X^\uparrow$ can be viewed as the concatenation of the excursions of X in $(0, +\infty)$. Similarly, we define $X^\downarrow$ by

$$X_t^\downarrow = X_{\alpha_t^-} - \frac{1}{2}\ell_{\alpha_t^-} - \sum_{0 < s \leqslant \alpha_t^-} \mathbf{1}_{\{X_s \leqslant 0\}}(X_{s-})_+ + \mathbf{1}_{\{X_s > 0\}}(X_{s-})_- \quad \text{if } t < A_\infty^-$$

and by $X_t^\downarrow = -\infty$ if not. The laws of $X^\uparrow$ and $X^\downarrow$ can be recovered by a harmonic transform: Denote by $q_t^+(x, dy)$ and $q_t^-(x, dy)$ the semigroup of the Lévy process killed respectively in $(-\infty, 0]$ and $[0, +\infty)$. One can show (see Silverstein in [18]) that the functions $\mathcal{U}^*([0, x])$ and $\mathcal{U}([0, x])$ are superharmonic respectively for q^+ and for q^- and that the following kernels

$$p_t^+(x, dy) = \frac{\mathcal{U}([0, y])}{\mathcal{U}([0, x]}q_t^+(x, dy), \quad x > 0,$$

and

$$p_t^-(x, dy) = \frac{\mathcal{U}^*([0, -y])}{\mathcal{U}^*([0, -x])}q_t^-(x, dy), \quad x < 0,$$

define two sub-markovian semigroups. Bertoin has shown in [3], Theorem 3.4 that $X^\uparrow$ and $X^\downarrow$ are Markov processes started at 0 with respective semigroups p^+ and p^-. If X does not drift to $-\infty$, respectively $+\infty$, then, p^+, respectively p^-, is markovian and $X^\uparrow$, respectively $X^\downarrow$, has an infinite lifetime. More precisely, if X does not drift to $-\infty$, then, we can show that

$$X_t^\uparrow < \infty, \quad t \geqslant 0 \quad \text{and} \quad \lim_{t \to \infty} X_t^\uparrow = \infty. \tag{7}$$

Proof. – If X does not drift to $-\infty$, it is easy to check that $\lim_{t \to \infty} A_t^+ = \infty$, **P**-a.s. and then, $X_t^\uparrow < \infty$, $t > 0$. If X drifts to $+\infty$, we have $\alpha_{t+A_{\sigma_0}^+}^+ = \sigma_0 + t$; $t \geqslant 0$, where

$\sigma_0 = \sup\{s \geqslant 0: X_s \leqslant 0\} < \infty$. Thus,

$$X^{\uparrow}_{t+A^+_{\sigma_0}} \geqslant X_{\sigma_0+t} \xrightarrow[t \to \infty]{} +\infty.$$

If X oscillates, we must consider to cases: Suppose first that $\pi \neq 0$, then,

$$\lim_{t\to\infty} X^{\uparrow}_t \geqslant \sum_{s>0} \mathbf{1}_{\{X_s \leqslant 0\}}(X_{s-})_+ + \mathbf{1}_{\{X_s > 0\}}(X_{s-})_- = \infty.$$

If π is null, then by assumption (A) there is a Brownian component and $\lim_{t\to\infty} \ell_t = \infty$ that yields the desired result. $\square$

In particular cases, we recover "classical" definitions of the process conditioned to stay positive:
  – In the Brownian case, $\mathcal{U}^*([0, x]) = \mathcal{U}((-x, 0]) = x$ and p^+ is the semigroup of the three-dimensional Bessel process started at 0.
  – In the spectrally positive case and the stable case, Chaumont has shown in [7] and [9] that if the Lévy process does not drift to $-\infty$ and if 0 is regular for $(0, +\infty)$, then, for any bounded $\mathcal{F}_t$ measurable functional F that is continuous for the Skorokhod topology on Ω:

$$\mathbf{E}\big[F(X^{\uparrow})\big] = \lim_{x\to 0} \lim_{T\to+\infty} \mathbf{E}_x\big[F(X) \mid I_T \geqslant 0\big].$$

  – In the spectrally negative case, Bertoin (see [3]) gives another construction of $X^{\uparrow}$ that generalizes Pitman's theorem for Brownian motion (see Pitman [16]).

Let us denote by $\overleftarrow{X^t}$ the path $\underrightarrow{X}^t$ reversed at its lifetime g_t. We denote by $(X^{\downarrow}_s; 0 \leqslant s < A^-_t)$ (respectively $(X^{\uparrow}_s; 0 \leqslant s < A^+_t)$) the process $X^{\downarrow}$ (respectively $X^{\uparrow}$) stopped at the random time A^-_t (respectively A^+_t). We need the following theorem due to Bertoin that links the process conditioned to stay positive with excursion theory:

THEOREM 2.4 (Bertoin [3], Theorem 3.1). – *For every $t > 0$, the following identity holds*

$$\big(\overleftarrow{X^t}, \underrightarrow{X}^t\big) \overset{(\text{law})}{=} \big((X^{\downarrow}_s)_{0\leqslant s < A^-_t}, (X^{\uparrow}_s)_{0\leqslant s < A^+_t}\big).$$

Remark. – If X drifts to $+\infty$, then, the previous identity holds with $t = \infty$ as to say that $X^{\uparrow}$ has the same law as the post-infimum process (see Millar in [14]).

In Section 4, we use another identity that is proved in [1] (see also [13]). From now on until the end of this section, we assume that X does not drift to $-\infty$. For any $t > 0$ we set

$$J_t = \inf_{s\in[t,\infty)} X^{\uparrow}_s$$

and

$$\bar{d}_t = \inf\{s > t: S_s = X_s\}.$$

LEMMA 2.5 (Bertoin [1], Lemme 4). –

$$(S_{(\overline{d}_t+\overline{g}_t-t)-} - X_{(\overline{d}_t+\overline{g}_t-t)-}, S_{\overline{d}_t})_{t\geqslant 0} \overset{(\text{law})}{=} (X_t^{\uparrow} - J_t, J_t)_{t\geqslant 0}.$$

Proof. – Although Bertoin in Lemma 4 of [1] only considers the case of $X \to +\infty$ (taking $X^{\uparrow}$ as the post-infimum process), the proof can be adapted when X oscillates thanks to Theorem 2.4 and the arguments are exactly the same. □

The process $X^{\uparrow} - J$ is a strong Markov process and 0 is a regular value. We denote by K its local time at 0 normalized in order it is distributed as L^*. Let us denote by (g_i, d_i), $i \in \mathcal{I}^{\uparrow}$ the excursion intervals of $X^{\uparrow} - J$ above 0:

$$\{s \geqslant 0: X_s^{\uparrow} > J_s\} = \bigcup_{i\in\mathcal{I}^{\uparrow}} (g_i, d_i).$$

We define

$$w^i(s) = \Delta J_{g_i} + (X^{\uparrow} - J)_{(s+g_i)\wedge d_i}, \quad s \geqslant 0, \ i \in \mathcal{I}^{\uparrow}.$$

Then, Lemma 2.5 implies that

$$\mathcal{N}^{\uparrow}(dk\,dw) = \sum_{i\in\mathcal{I}^{\uparrow}} \delta_{(K_{g_i}, w^i)} \tag{8}$$

is a Poisson point process with intensity $dk\,\widehat{N}^*(dw)$, where $\widehat{N}^*$ is the law of $\widehat{\omega}^{\zeta}$ under $N^*(d\omega)$. We use this result in Section 4.

3. Reversion formulas

Let x and t be two positive real numbers. We first decompose the law of $\widehat{X}^{\sigma_x(t)}$ on the event $\{X_{\sigma_x(t)} > x\}$ in terms of the law of X, the Lévy measure π and the function Λ that is defined on $(0, +\infty) \times (0, +\infty)$ by $\Lambda(s, a) = \mathbf{P}(I_s \geqslant -a)$. From classical fluctuation identities we have

$$\int\limits_{(0,+\infty)\times(0,+\infty)} ds\,da \exp(-\lambda s - \mu a)\Lambda(s, a) = \frac{\kappa(\lambda, 0)}{\lambda\mu\kappa(\lambda, \mu)}.$$

We also write $\Lambda(a)$ for the limit $\lim_{s\to+\infty} \Lambda(s, a)$ that is positive if and only if X drifts to $+\infty$ (or equivalently $p > 0$). To simplify notations, we write $\mathbf{P}_r$ for the law of the Lévy process started at r. We prove the first reversion formula:

THEOREM 3.1. – *Assume that π charges $(0, +\infty)$. Then for any positive numbers x and t, and for any bounded measurable functional F on Ω,*

$$\mathbf{E}\big[F(\widehat{X}^{\sigma_x(t)}); \sigma_x(t) < \infty; X_{\sigma_x(t)} > x\big]$$

$$= \int\limits_{(0,+\infty)} \pi(dr) \int\limits_0^t du\,\mathbf{E}_r\big[F(X_{\cdot\wedge u})\Lambda(t - u, X_u - x); x < X_u \leqslant x + r\big].$$

Consequences. (i) If X drifts to $+\infty$, then $\Lambda(a) > 0$ for any positive real number a. Thanks to Theorem 3.1 we get

$$\mathbf{E}\left[F\left(\widehat{X}^{\sigma_x(t)}\right)\frac{\Lambda(X_{\sigma_x(t)} - x)}{\Lambda(t - \sigma_x(t), X_{\sigma_x(t)} - x)}; \sigma_x(t) < \infty; X_{\sigma_x(t)} > x\right]$$

$$= \int\limits_{(0,+\infty)} \pi(dr) \int\limits_0^t du\, \mathbf{E}_r\left[F(X_{\cdot \wedge u})\Lambda(X_u - x); x < X_u \leqslant x + r\right]. \tag{9}$$

Observe that $\mathbf{P}$-a.s. $\sigma_x(t) = \sigma_x$ for all t sufficiently large. Thus, $\mathbf{P}$-a.s.

$$F\left(\widehat{X}^{\sigma_x(t)}\right)\frac{\Lambda(X_{\sigma_x(t)} - x)}{\Lambda(t - \sigma_x(t), X_{\sigma_x(t)} - x)}\mathbf{1}_{\{\sigma_x(t)<\infty; X_{\sigma_x(t)}>x\}} \xrightarrow[t\to\infty]{} F\left(\widehat{X}^{\sigma_x}\right)\mathbf{1}_{\{X_{\sigma_x}>x\}}.$$

Since $\Lambda(X_{\sigma_x(t)} - x)/\Lambda(t - \sigma_x(t), X_{\sigma_x(t)} - x)$ is smaller than 1, dominated convergence applies and we deduce from (9) that

$$\mathbf{E}\left[F\left(\widehat{X}^{\sigma_x}\right); X_{\sigma_x} > x\right]$$

$$= \int\limits_{(0,+\infty)} \pi(dr) \int\limits_0^{+\infty} du\, \mathbf{E}_r\left[F(X_{\cdot \wedge u})\Lambda(X_u - x); x < X_u \leqslant x + r\right].$$

(ii) By the duality property applied in the right side of Theorem 3.1, we see that under $\mathbf{P}(\cdot \mid X_{\sigma_x(t)} > x)$

$$X_{\cdot \wedge \sigma_x(t)-} \overset{\text{(law)}}{=} \widehat{X}^{\sigma_x(t)-}.$$

Proof of Theorem 3.1. – Let ε be a positive real number and let $(\sigma_n)_{n\geqslant0}$ be the increasing sequence of the jump times $\{s \geqslant 0: \Delta X_s > \varepsilon\}$. Recall that $\sum_{s\in J} \delta_{(s,\Delta X_s)}$ is a Poisson measure with intensity $dl\,\pi(dr)$. Let f be a bounded function on $\mathbb{R}$. Consider the event $A_\varepsilon = \{\sigma_x(t) < \infty; \inf_{s\in[\sigma_x(t),t]} X_s > \varepsilon + x\}$ and set

$$a(\varepsilon) = \mathbf{E}\left[F\left(\widehat{X}^{\sigma_x(t)-}\right)f(\Delta X_{\sigma_x(t)}); A_\varepsilon\right].$$

Observe that

$$a(\varepsilon) = \sum_{n\geqslant0} \mathbf{E}\left[F\left(\widehat{X}^{\sigma_n-}\right)f(\Delta X_{\sigma_n}); \sigma_n < t; X_{\sigma_n-} < x; X_{\sigma_n} + \inf_{[0,t-\sigma_n]} X \circ \theta_{\sigma_n} > \varepsilon + x\right].$$

Apply the Markov property at σ_n in order to get

$$a(\varepsilon) = \sum_{n\geqslant0} \mathbf{E}\left[F\left(\widehat{X}^{\sigma_n-}\right)f(\Delta X_{\sigma_n})\Lambda(t - \sigma_n, X_{\sigma_n} - \varepsilon - x); \sigma_n < t; X_{\sigma_n-} < x\right].$$

Then,

$$a(\varepsilon) = \mathbf{E}\left[\sum_{s\in J:\, s\leqslant t} \mathbf{1}_{\{\Delta X_s>\varepsilon; X_{s-}<x\}} F\left(\widehat{X}^{s-}\right)f(\Delta X_s)\Lambda(t - s, X_{s-} + \Delta X_s - \varepsilon - x)\right].$$

Apply the compensation formula to get:

$$a(\varepsilon) = \int\limits_{(\varepsilon,+\infty)} \pi(dr) f(r) \int\limits_0^t du\, \mathbf{E}\big[F\big(\widehat{X}^u\big)\Lambda(t-u, X_u + r - \varepsilon - x); x + \varepsilon - r < X_u < x\big]$$

and by duality

$$a(\varepsilon) = \int\limits_{(\varepsilon,+\infty)} \pi(dr) f(r) \int\limits_0^t du\, \mathbf{E}\big[F(X_{\cdot\wedge u})\Lambda(t-u, X_u + r - \varepsilon - x); x + \varepsilon - r < X_u < x\big].$$

(10)

Next, observe that $\mathbf{P}$-a.s.

$$\lim_{\varepsilon \to 0} \mathbf{1}_{A_\varepsilon} = \mathbf{1}_{\{\sigma_x(t) < \infty; X_{\sigma_x(t)} > x\}}$$

and complete the proof by letting ε go to 0 and using dominated convergence in the left side of (10) and monotone convergence in the right side. □

We get a similar result for the reversed path at τ_x on the event $\{X_{\tau_x} > x\}$:

THEOREM 3.2. – *Assume that π charges $(0, +\infty)$. Then, for any positive real number x and for any bounded measurable functional F on Ω,*

$$\mathbf{E}\big[F\big(\widehat{X}^{\tau_x}\big); \tau_x < \infty; X_{\tau_x} > x\big]$$
$$= \int\limits_{(0,+\infty)} \pi(dr) \int\limits_0^{+\infty} du\, \mathbf{E}_r\big[F(X_{\cdot\wedge u}); x < X_u \leqslant x + I_u\big].$$

Remark. – In the subordinator case we get immediately the well-known formula:

$$\mathbf{E}\big[f(X_{\tau_x-}, X_{\tau_x}); X_{\tau_x} > x\big] = \int\limits_{[0,x]} V(da) \int\limits_{(x-a,+\infty)} \pi(dr) f(a, a + r),$$

where V denote the potential measure associated with X.

Proof of Theorem 3.2. – Let f be a bounded measurable function on $\mathbb{R}$. Observe that

$$\mathbf{E}\big[F\big(\widehat{X}^{\tau_x-}\big) f(\Delta X_{\tau_x}); \tau_x < \infty; X_{\tau_x} > x\big]$$
$$= \mathbf{E}\bigg[\sum_{s \geqslant 0} \mathbf{1}_{\{S_{s-} \leqslant x; \Delta X_s + X_{s-} > x\}} F\big(\widehat{X}^{s-}\big) f(\Delta X_s)\bigg].$$

Apply the compensation formula to get

$$\mathbf{E}\big[F\big(\widehat{X}^{\tau_x-}\big) f(\Delta X_{\tau_x}); \tau_x < \infty; X_{\tau_x} > x\big]$$
$$= \int\limits_{(0,+\infty)} \pi(dr) f(r) \int\limits_0^{+\infty} du\, \mathbf{E}\big[F\big(\widehat{X}^u\big); S_u \leqslant x; r + X_u > x\big]$$

and the result follows by the duality property. □

Recall that if the Lévy measure charges $(0, +\infty)$, the excursion under the supremum may end with a jump. We now give a decomposition of the law of the excursion under the supremum reversed at its final jump time.

THEOREM 3.3. – *Assume* (A) *and that π charges $(0, +\infty)$. Then, for any nonnegative measurable functional F on Ω,*

$$N^*\big(F(\hat{\omega}^\zeta); \omega_\zeta > 0\big) = \int\limits_{(0,+\infty)} \pi(dr)\mathbf{E}_r\left[\int\limits_0^{L_\infty} dv\, F(X_{\cdot \wedge L_v^{-1}})\mathbf{1}_{\{X_{L_v^{-1}}>0\}}\right].$$

Remark. – In the spectrally positive case, if we take $L = -I$, then, $L_x^{-1} = \tau_{-x}$. Theorem 3.3 shows that under $N^*(\cdot \cap \{\omega_\zeta > 0\})$, the law of $\omega_{\zeta-}$ admits a density with respect to Lebesgue measure that is given by

$$x \to \mathbf{1}_{(-\infty,0)}(x)\mathbf{P}(I_\infty < x)\pi\big((-x, +\infty)\big).$$

Furthermore, under $N^*(\cdot \mid \omega_{\zeta-} = -x)$, the path $\hat{\omega}^{\zeta-}$ is distributed as $X_{\cdot \wedge \tau_{-x}}$ under $\mathbf{P}(\cdot \mid \tau_x < \infty)$. This result has been used by Bertoin in [2] and [1].

Proof of Theorem 3.3. – For any nonnegative measurable function f on $\mathbb{R}$, we have the following decomposition:

$$N^*\big(F(\hat{\omega}^{\zeta-})f(\Delta\omega_\zeta); \omega_\zeta > 0\big) = N^*\left(\sum_{s\geqslant 0}\mathbf{1}_{\{\omega_{s-}+\Delta\omega_s>0\}}F(\hat{\omega}^{s-})f(\Delta\omega_s)\right).$$

Observe that ω under N^* is markovian with the transition kernel of the Lévy process killed in $[0, +\infty)$. We can apply the compensation formula to get

$$N^*\big(F(\hat{\omega}^{\zeta-})f(\Delta\omega_\zeta); \omega_\zeta > 0\big) = N^*\left(\int\limits_0^\zeta ds\, F(\hat{\omega}^s)\int\limits_{(0,+\infty)}\pi(dr)f(r)\mathbf{1}_{\{\omega_s+r>0\}}\right)$$

that yields the theorem thanks to Proposition 2.3.   □

4. Applications

4.1. First Williams' decomposition theorem

From now on until the end of the present article, we assume (A) *and we suppose that X does not drift to $-\infty$.* Let x be a positive real number. Williams has shown in [19] that the standard real Brownian motion reversed at the first hitting time of $(x, +\infty)$ is distributed as the three-dimensional Bessel process up to its last passage time at x. In this section, we extend Williams' result to general Lévy processes, the role of the three-dimensional Bessel process being played by the Lévy process conditioned to stay positive. In order to avoid cumbersome notation, we set for any positive real numbers x and t

$$\underset{\rightarrow}{\sigma}{}^{\,t}_x = \sigma_x(\underset{\rightarrow}{X}{}^t) = \sup\{s \in [0, t - \underline{g}_t]\colon X_{s+\underline{g}_t} - I_t \leqslant x\},$$

$$\sigma^{\uparrow}_x = \sigma_x(X^{\uparrow}) = \sup\{s \geqslant 0\colon X^{\uparrow}_s \leqslant x\},$$

$$\underset{\rightarrow}{\tau}{}^{\,t}_x = \tau_x(\underset{\rightarrow}{X}{}^t) = \inf\{s \in [0, t - \underline{g}_t]\colon X_{s+\underline{g}_t} - I_t > x\},$$

$$\tau^{\uparrow}_x = \tau_x(X^{\uparrow}) = \inf\{s \geqslant 0\colon X^{\uparrow}_s > x\}$$

(with $\inf \emptyset = \sup \emptyset = +\infty$). Observe that $\underset{\rightarrow}{\sigma}{}^{\,t}_x$ may be infinite if $X_t - \inf_{[0,t]} X < x$. Under $\mathbf{P}(\cdot \cap \{\underset{\rightarrow}{\sigma}{}^{\,t}_x < \infty\})$ we define

$$Y^t = \underset{\rightarrow}{X}{}^t \circ \theta_{\underset{\rightarrow}{\sigma}{}^{\,t}_x}.$$

Similarly we denote

$$Y = X^{\uparrow} \circ \theta_{\sigma^{\uparrow}_x}$$

that is well-defined thanks to (7) and our assumptions. Notice that Y^t and Y rely on x although it does not appear in the notations. We recall that $\underset{\leftarrow}{Y}$ and $\underset{\rightarrow}{Y}$ are respectively the pre-infimum process and the post-infimum process of Y. The following theorems describe the law of the path $X^{\uparrow}$ reversed at time $\sigma^{\uparrow}_x$: the first theorem concerns the case of a jump: $\Delta X^{\uparrow}_{\sigma^{\uparrow}_x} > 0$; the second theorem deals with the process leaving continuously level x.

THEOREM 4.1. – *Assume that π charges $(0, +\infty)$. Let $x > 0$.*

(i) $(\widehat{X^{\uparrow}}^{\,\sigma^{\uparrow}_x}, \underset{\leftarrow}{Y})$ *under* $\mathbf{P}(\cdot \mid X^{\uparrow}_{\sigma^{\uparrow}_x} > x) \overset{(\text{law})}{=} (\Delta X_{\tau_x} + X_{\cdot \wedge \overline{g}_{\tau_x}}, \widehat{X}^{\tau_x-}_{\cdot \wedge (\tau_x - \overline{g}_{\tau_x})})$ *under* $\mathbf{P}(\cdot \mid X_{\tau_x} > x)$.

(ii) *Under* $\mathbf{P}(\cdot \mid X^{\uparrow}_{\sigma^{\uparrow}_x} > x)$, $\underset{\rightarrow}{Y}$ *is independent of* $(X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}, \underset{\leftarrow}{Y})$ *and distributed as* $X^{\uparrow}$.

THEOREM 4.2. – *Assume that $d^* > 0$. Then, for any $x > 0$,*

(i) $\mathbf{P}(X^{\uparrow}_{\sigma^{\uparrow}_x} = x) = \mathbf{P}(X_{\tau_x} = x) = d^* u^*(x)$ *and* $\widehat{X}^{\tau_x}$ *under* $\mathbf{P}(\cdot \mid X_{\tau_x} = x)$ *is distributed as* $X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}$ *under* $\mathbf{P}(\cdot \mid X^{\uparrow}_{\sigma^{\uparrow}_x} = x)$.

(ii) *Under* $\mathbf{P}(\cdot \mid X^{\uparrow}_{\sigma^{\uparrow}_x} = x)$, $X^{\uparrow} \circ \theta_{\sigma^{\uparrow}_x}$ *and* $X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}$ *are mutually independent and* $X^{\uparrow} \circ \theta_{\sigma^{\uparrow}_x}$ *is distributed as* $X^{\uparrow}$.

Remark. – We assume that π charges $(-\infty, 0)$. Then, the excursion under the infimum may end with a negative jump. The dual form of the reversion formula of Theorem 3.3 gives

$$N\big(F(\omega_{\cdot \wedge \zeta-}); \omega_{\zeta} < 0\big) = \int_{(-\infty, 0)} \pi(dr)\mathbf{E}\left[\int_0^{L^*_{\infty}} du\, \mathbf{1}_{\{X_{L^{*-1}_u} < -r\}} F\big(\widehat{X}^{L^{*-1}_u}\big)\right]. \quad (11)$$

We assume moreover that $d^* > 0$. By Lemma 2.1, it follows that under $N(\cdot \cap \{\omega_\zeta < 0\})$, $\omega_{\zeta-}$ admits a density with respect to Lebesgue measure given by

$$N(\omega_{\zeta-} \in dx; \omega_\zeta < 0) = u^*(x)\pi\big((-\infty, -x)\big)\, m(dx).$$

By combining (11) with Lemma 2.1, Theorem 4.2 implies that for any $x > 0$,

$$X^\uparrow_{\cdot \wedge \sigma_x^\uparrow} \text{ under } \mathbf{P}(\cdot \mid X^\uparrow_{\sigma_x^\uparrow} = x) \overset{\text{(law)}}{=} \omega_{\cdot \wedge \zeta-} \text{ under } N(\cdot \mid \omega_{\zeta-} = x).$$

This result is due to Chaumont in the stable case (see [7]).

Proof of Theorem 4.1. – First observe that $\tau_x(J) = \sigma_x^\uparrow$ a.s., then by Lemma 2.5

$$\sigma_x^\uparrow \overset{\text{(law)}}{=} \inf\{t > 0 \colon S_{\overline{d}_t} > x\} = \overline{g}_{\tau_x}.$$

Set

$$\gamma = \sigma_x^\uparrow + \underline{g}(Y) = \inf\{t > \sigma_x^\uparrow \colon X_t^\uparrow = J_t\}.$$

Then,

$$\gamma \overset{\text{(law)}}{=} \tau_x. \tag{12}$$

We define the functional Σ by

$$\Sigma_t(X) = (S_{(\overline{d}_t + \overline{g}_t - t)-} - X_{(\overline{d}_t + \overline{g}_t - t)-}, S_{\overline{d}_t}), \quad t \geqslant 0.$$

Deduce from (12) and from the fact that $\overline{d}_{\tau_x} = \tau_x$ that

$$\big((X^\uparrow - J)_{\gamma+t}, J_{\gamma+t} - J_\gamma\big)_{t \geqslant 0} \overset{\text{(law)}}{=} \big(\Sigma_t(X \circ \theta_{\tau_x})\big)_{t \geqslant 0}. \tag{13}$$

Since

$$\underset{\rightarrow}{Y}_t = (X^\uparrow - J)_{\gamma+t} + (J_{\gamma+t} - J_\gamma)$$

we deduce from the Markov property applied at τ_x in the right member of (13) that $\underset{\rightarrow}{Y}$ is independent of $X^\uparrow_{\cdot \wedge \gamma}$ and that $\underset{\rightarrow}{Y}$ has the same distribution as $X^\uparrow$, which proves Theorem 4.1(ii) and also Theorem 4.2(ii) because

$$X^\uparrow \circ \theta_{\sigma_x^\uparrow} = \underset{\rightarrow}{Y} \quad \text{on } \{X^\uparrow_{\sigma_x^\uparrow} = x\}.$$

Next, we denote by K^{-1} the right-continuous inverse of K (recall notation from Section 2.2):

$$K_u^{-1} = \inf\{t \geqslant 0 \colon K_t > u\}, \quad u \geqslant 0.$$

We need the following lemma:

LEMMA 4.3. – *For any $x > 0$*

$$X^{\uparrow}_{\cdot \wedge K_x^{-1}} \overset{(law)}{=} \widehat{X}^{L_x^{*-1}}.$$

Proof. – We first index the excursions of $X^{\uparrow} - J$ above 0 by the corresponding local time: for any $t \geqslant 0$, we set

$$e_t^{\uparrow} = \left((X^{\uparrow} - J)_{(K_{t-}^{-1}+s) \wedge K_t^{-1}} \right)_{s \geqslant 0}$$

(note that $e_t^{\uparrow} = 0$ if $\Delta K_t^{-1} = 0$). We do the same thing for the excursions of X under its supremum and we set

$$e_t = \left((X - S)_{(L_{t-}^{*-1}+s) \wedge L_t^{*-1}} \right)_{s \geqslant 0}.$$

We also write $U_t^{\uparrow} = X^{\uparrow}_{K_t^{-1}}$. We deduce from Lemma 2.5 that $(K^{-1}, U^{\uparrow})$ is a subordinator with the same distribution as (L^{*-1}, U^*). There exists a measurable functional F such that

$$F\left((e_t^{\uparrow}, U_t^{\uparrow})_{0 \leqslant t \leqslant x} \right) = X^{\uparrow}_{\cdot \wedge K_x^{-1}}, \quad \text{a.s.}$$

Let us explain more precisely how to recover $X^{\uparrow}$ from the $e_t^{\uparrow}$ and $U_t^{\uparrow}$, $t \geqslant 0$: For any $s \in [0, K_x^{-1}]$, we define

$$g(s) = \sup\{u \in [0, s]: X_u^{\uparrow} = J_u\}.$$

The Lévy–Ito decomposition for the subordinator K^{-1} implies that

$$K_s = \sup\left\{ a \in [0, x]: d^*a + \sum_{t < a} \zeta(e_t^{\uparrow}) \leqslant s \right\}$$

and

$$g(s) = d^* K_s + \sum_{t < K_s} \zeta(e_t^{\uparrow}).$$

Then,

$$X_s^{\uparrow} = e_{K_s}^{\uparrow}(s - g(s)) + U_{K_s}^{\uparrow}.$$

In order to simplify notations, we set for any $t \geqslant 0$:

$$\hat{e}_t = \hat{e}_t^{\zeta(e_t)-}.$$

Lemma 2.5 implies

$$(e_t^{\uparrow}, U_t^{\uparrow})_{0 \leqslant t \leqslant x} \overset{(law)}{=} (\hat{e}_t, U_t^*)_{0 \leqslant t \leqslant x}. \tag{14}$$

Since $(e_t; t \geqslant 0)$ is a Poisson process and U^* a subordinator, a simple time-reversal argument show that

$$\left(e_{x-t}, \widehat{U^*}_t^x \right)_{0 \leqslant t \leqslant x} \overset{(law)}{=} (e_t, U_t^*)_{0 \leqslant t \leqslant x}.$$

Thus

$$(e_t^\uparrow, U_t^\uparrow)_{0 \leqslant t \leqslant x} \overset{\text{(law)}}{=} (\hat{e}_{x-t}, \widehat{U^*}_t^x)_{0 \leqslant t \leqslant x}. \tag{15}$$

Applying the Lévy–Ito decomposition for the subordinator L^{*-1} reversed at time x, it is easy to check that

$$F\big((\hat{e}_{x-t}, \widehat{U^*}_t^x)_{0 \leqslant t \leqslant x}\big) = \widehat{X}^{L_x^{*-1}}, \quad \text{a.s.}$$

and we conclude thanks to (15). $\quad \square$

Let us prove now Theorem 4.1(i): Let F and G be two nonnegative measurable functionals and f be a nonnegative measurable function. Set

$$\alpha = \mathbf{E}\big[F(X^\uparrow_{\cdot \wedge \sigma_x^\uparrow -}) f(\Delta X^\uparrow_{\sigma_x^\uparrow}) G(\overleftarrow{Y}); X^\uparrow_{\sigma_x^\uparrow} > x\big].$$

It is sufficient to show that

$$\alpha = \mathbf{E}\big[F(\widehat{X}^{\overline{g}_{\tau_x}}) f(\Delta X_{\tau_x}) G(\widehat{X}^{\tau_x -}_{\cdot \wedge (\tau_x - \overline{g}_{\tau_x})}); X_{\tau_x} > x\big]. \tag{16}$$

First observe that

$$\alpha = \mathbf{E}\bigg[\sum_{i \in \mathcal{I}^\uparrow} \mathbf{1}_{\{X^\uparrow_{g_i} \leqslant x < X^\uparrow_{g_i} + w^i(0)\}} F(X^\uparrow_{\cdot \wedge g_i}) f(w^i(0)) G(w^i - w^i(0))\bigg].$$

Then by (8), we get

$$\alpha = \int_0^\infty du \, \mathbf{E}\big[F(X^\uparrow_{\cdot \wedge K_u^{-1}}) \widehat{N}^*(f(w(0)) G(w - w(0)); U_u^\uparrow \leqslant x < U_u^\uparrow + w(0))\big].$$

The previous lemma implies that

$$\alpha = \int_0^\infty du \, \mathbf{E}\big[F(\widehat{X}^{L_u^{*-1}}) N^*(f(\Delta\omega_\zeta) G(\widehat{\omega}^{\zeta -}); U_u^* \leqslant x < U_u^* + \omega(\zeta))\big]. \tag{17}$$

But we have a.s.

$$\mathbf{1}_{\{X_{\tau_x} > x\}} F(\widehat{X}^{\overline{g}_{\tau_x}}) f(\Delta X_{\tau_x}) G(\widehat{X}^{\tau_x -}_{\cdot \wedge (\tau_x - \overline{g}_{\tau_x})})$$
$$= \sum_{i \in \mathcal{I}^*} \mathbf{1}_{\{X_{g_i} \leqslant x < X_{g_i} + \omega^i(\zeta_i)\}} F(\widehat{X}^{g_i}) f(\Delta\omega^i_{\zeta_i}) G(\widehat{\omega^i}^{\zeta_i -}).$$

Then, the compensation formula combined with (17) achieve the proof of (16). $\quad \square$

Proof of Theorem 4.2. – We only need to show (i). From Theorem 4.1, we deduce that

$$\mathbf{P}(X^\uparrow_{\sigma_x^\uparrow} > x) = \mathbf{P}(X_{\tau_x} > x).$$

So, we have for any $x > 0$:

$$d^* u^*(x) = \mathbb{P}(X^\uparrow_{\sigma^\uparrow_x} = x). \tag{18}$$

We need the following lemma:

LEMMA 4.4. – *Under the assumptions of Theorem* 4.2, *we have for any nonnegative measurable functional* F

$$\mathbf{E}\left[\int_0^\infty du\, F(X^\uparrow_{\cdot \wedge K_u^{-1}}) \right] = \int_0^\infty dx\, u^*(x) \mathbf{E}\left[F(X^\uparrow_{\cdot \wedge \sigma^\uparrow_x}) \mid X^\uparrow_{\sigma^\uparrow_x} = x \right].$$

Proof. – We argue exactly as in Lemma 2.1 replacing, X by $X^\uparrow$, L^* by K and τ_x by $\sigma^\uparrow_x$. ☐

Lemmas 2.1 and 4.4 imply that for any nonnegative measurable functional F, the set

$$\lambda_F = \left\{ x > 0\colon \mathbf{E}\left[F(X^\uparrow_{\cdot \wedge \sigma^\uparrow_x}); X^\uparrow_{\sigma^\uparrow_x} = x \right] = \mathbf{E}\left[F(\widehat{X}^{\tau_x}); X_{\tau_x} = x \right] \right\}$$

is of full Lebesgue measure. We have to show that actually $\lambda_F = (0, +\infty)$: Let $x_0 > 0$, let G be such that for any $x > 0$:

$$G(X^\uparrow_{\cdot \wedge \sigma^\uparrow_x}) = \mathbf{1}_{(x_0, \infty)}(x) F(X^\uparrow_{\cdot \wedge \sigma^\uparrow_{x_0}}) \mathbf{1}_{\{X^\uparrow_{\sigma^\uparrow_{x_0}} = x_0\}}.$$

Observe that on $\{X^\uparrow_{\sigma^\uparrow_{x_0}} = x_0\}$, we have for any $x > x_0$

$$\sigma^\uparrow_x = \sigma_{x-x_0}(X^\uparrow \circ \theta_{\sigma^\uparrow_{x_0}}) + \sigma^\uparrow_{x_0}.$$

Then, Theorem 4.1(ii) (already proved) implies that

$$\mathbf{E}\left[G(X^\uparrow_{\cdot \wedge \sigma^\uparrow_x}); X^\uparrow_{\sigma^\uparrow_x} = x \right] = d^* u^*(x - x_0) \mathbf{E}\left[F(X^\uparrow_{\cdot \wedge \sigma^\uparrow_{x_0}}); X^\uparrow_{\sigma^\uparrow_{x_0}} = x_0 \right].$$

Since λ_G is a set of full Lebesgue measure we can assume that $x > x_0$ is in λ_G and consequently

$$\mathbf{E}\left[G(X^\uparrow_{\cdot \wedge \sigma^\uparrow_x}); X^\uparrow_{\sigma^\uparrow_x} = x \right] = \mathbf{E}\left[G(\widehat{X}^{\tau_x}); X_{\tau_x} = x \right].$$

But

$$\mathbf{1}_{\{X_{\tau_x} = x\}} G(\widehat{X}^{\tau_x}) = F(\widehat{Z}^{\tau_{x_0}(Z)}) \mathbf{1}_{\{Z_{\tau_{x_0}(Z)} = x_0 \text{ and } X_{\tau_{x-x_0}} = x - x_0\}},$$

where $Z = X \circ \theta_{\tau_{x-x_0}}$. Applying the Markov property at time τ_{x-x_0}, we get that

$$\mathbf{E}\left[G(X^\uparrow_{\cdot \wedge \sigma^\uparrow_x}); X^\uparrow_{\sigma^\uparrow_x} = x \right] = d^* u^*(x - x_0) \mathbf{E}\left[F(\widehat{X}^{\tau_{x_0}}); X_{\tau_{x_0}} = x_0 \right]$$

which implies the desired result. ☐

4.2. Bismut's decomposition

As a consequence of Theorem 4.2 and Lemma 2.1, we extend to real Lévy processes Bismut's decomposition of the excursion above the infimum.

THEOREM 4.5. – *Assume that d^* is positive. Then for any nonnegative measurable functionals G and D on Ω and any nonnegative measurable function f,*

$$N\left(\int_0^\zeta ds\, G(\omega_{\cdot \wedge s}) f(\omega_s) D(\omega \circ \theta_s) \right)$$
$$= \int_0^{+\infty} dx\, f(x) u^*(x) \mathbf{E}\big[G(X^\uparrow_{\cdot \wedge \sigma^\uparrow_x}) \mid X^\uparrow_{\sigma^\uparrow_x} = x \big] \mathbf{E}\big[D(X_{\cdot \wedge \tau_{-x}}) \big].$$

Remark. – The spectrally positive case is due to Chaumont (see [9]).

Proof. – Apply Markov property under N in order to get:

$$N\left(\int_0^\zeta ds\, G(\omega_{\cdot \wedge s}) f(\omega_s) D(\omega \circ \theta_s) \right) = N\left(\int_0^\zeta ds\, G(\omega_{\cdot \wedge s}) f(\omega_s) d(\omega_s) \right),$$

where, for any positive number x, $d(x)$ stands for $\mathbf{E}[D(X_{\cdot \wedge \tau_{-x}})]$. Then, by Proposition 2.3, we have

$$N\left(\int_0^\zeta ds\, G(\omega_{\cdot \wedge s}) f(\omega_s) D(\omega \circ \theta_s) \right)$$
$$= \mathbf{E}\left[\int_0^{L^*_\infty} du\, G(\widehat{X}^{L^{*-1}_u}) f(X_{L^{*-1}_u}) d(X_{L^{*-1}_u}) \right]$$

and we use Lemma 2.1 and Theorem 4.2 to complete the proof. □

We have seen in Section 3 that the excursion under the supremum may end with a jump if π charges $(0, +\infty)$. Theorem 3.3 provides a reversion formula for the excursion under the supremum at its final jump time. If we assume that d^* is positive, then, the excursion may end continuously, as to say $\omega_\zeta = 0$. More precisely it is clear that $N^*(\omega_\zeta = 0) = 0$ if $d^* = 0$; let us show that $N^*(\omega_\zeta = 0) = +\infty$ if $d^* > 0$:

$$N^*\big(1 - e^{-\lambda \zeta}; \omega_\zeta = 0\big) = \lambda \int_0^{+\infty} ds\, e^{-\lambda s} N^*(\zeta > s; \omega_\zeta = 0)$$
$$= \lambda \int_0^{+\infty} ds\, e^{-\lambda s} N^*\big(\big[\mathbf{P}(X_{\tau_a} = a)\big]_{a = -\omega_s}; \zeta > s \big)$$
$$= \lambda \int_0^{+\infty} ds\, e^{-\lambda s} N^*\big(d^* u^*(-\omega_s); \zeta > s \big).$$

By a change of variable, we have

$$\lambda \int_0^{+\infty} ds\, e^{-\lambda s} N^*\big(d^* u^*(-\omega_s); \zeta > s\big) = \int_0^{+\infty} ds\, e^{-s} N^*\big(d^* u^*(-\omega_{s/\lambda}); \zeta > s/\lambda\big).$$

Then for any $\lambda > 0$, we have

$$N^*\big(1 - e^{-\lambda \zeta}; \omega_\zeta = 0\big) \geqslant \frac{d^*}{e} \inf_{x \in (0,1]} u^*(x) N^*\Big(\sup_{s \in [0,1/\lambda]} (-\omega_s) \leqslant 1; \zeta > 1/\lambda\Big).$$

But

$$\lim_{\lambda \to +\infty} N^*\big(1 - e^{-\lambda \zeta}; \omega_\zeta = 0\big)$$

$$= N^*(\omega_\zeta = 0) \geqslant \frac{d^*}{e} \inf_{x \in (0,1]} u^*(x) \lim_{\lambda \to +\infty} N^*\Big(\sup_{s \in [0,1/\lambda]} (-\omega_s) \leqslant 1; \zeta > 1/\lambda\Big) = +\infty.$$

The following theorem complements Theorem 3.3 by providing a reversion identity for the excursion under the supremum ending continuously.

THEOREM 4.6. – *Assume that $dd^* > 0$. Then, for any nonnegative measurable functional F on Ω,*

$$dN^*\big(F(\hat\omega^\zeta); \omega_\zeta = 0\big) = d^* N\big(F(\omega); \omega_\zeta = 0\big).$$

Remark. – The theorem remains true if $d^*d = 0$: in that case, it just means that either $N^*(\omega_\zeta = 0) = 0$ or $N(\omega_\zeta = 0) = 0$.

Proof. – We prove the following identity:

$$dN^*\bigg(\int_0^\zeta ds\, F(\hat\omega^\zeta_{\cdot \wedge s}); \omega_\zeta = 0\bigg) = d^* N\bigg(\int_0^\zeta ds\, F(\omega_{\cdot \wedge s}); \omega_\zeta = 0\bigg), \qquad (19)$$

which easily leads to the statement of the theorem. First, observe that

$$\hat\omega^\zeta_{\cdot \wedge s} = \widehat{\omega \circ \theta_{\zeta-s}}.$$

After the change of variable $s \to \zeta - s$, the Markov property under N^* combined with the latter observation give

$$dN^*\bigg(\int_0^\zeta ds\, F(\hat\omega^\zeta_{\cdot \wedge s}); \omega_\zeta = 0\bigg) = dN^*\bigg(\int_0^\zeta ds\, \mathbf{E}\big[F(\widehat{X}^{\tau - \omega_s}); X_{\tau_{-\omega_s}} = -\omega_s\big]\bigg).$$

By Proposition 2.3 and the dual version of Lemma 2.1, it follows that

$$dN^*\bigg(\int_0^\zeta ds\, F(\hat\omega^\zeta_{\cdot \wedge s}); \omega_\zeta = 0\bigg) = d \int_0^{+\infty} dx\, u(-x)\mathbf{E}\big[F(\widehat{X}^{\tau_x}); X_{\tau_x} = x\big]$$

$$= dd^* \int\limits_0^{+\infty} dx \, u^*(x)u(-x)\mathbf{E}\big[F\big(\widehat{X}^{\tau_x}\big) \mid X_{\tau_x} = x\big].$$

Use Proposition 2.3 and Lemma 2.1 to get

$$dd^* \int\limits_0^{+\infty} dx \, u^*(x)u(-x)\mathbf{E}\big[F\big(\widehat{X}^{\tau_x}\big) \mid X_{\tau_x} = x\big] = d^* N\left(\int\limits_0^{\zeta} ds \, F(\omega_{\cdot \wedge s}); \omega_\zeta = 0\right)$$

that is the desired result.   □

4.3. The second Williams' decomposition theorem

Williams has shown in [19] that the Brownian excursion splits at its maximum in two three-dimensional Bessel processes stopped at a certain hitting time. In this section, we extend this result to general Lévy processes. To simplify, we set

$$Z^T = \underset{\rightarrow}{X}{}^T \circ \theta_{\underset{\rightarrow}{\tau}{}_x^T} \text{ under } \mathbf{P}\big(\cdot \cap \{\underset{\rightarrow}{\tau}{}_x^T < \infty\}\big) \text{ and } Z = X^\uparrow \circ \theta_{\tau_x^\uparrow}.$$

We first prove the following proposition.

PROPOSITION 4.7. – (i) *For any bounded measurable functional F on Ω,*

$$\mathbf{E}\big[F\big(X^\uparrow_{\cdot \wedge \tau_x^\uparrow}\big)\big] = N\big(F(\omega_{\cdot \wedge \tau_x(\omega)})\mathcal{U}\big((-\omega_{\tau_x(\omega)}, 0]\big); \tau_x(\omega) < \infty\big).$$

Thus, $\mathbf{P}(X^\uparrow_{\tau_x^\uparrow} = x) > 0$ if and only if $d^* > 0$.

(ii) *If $d^* > 0$, then, under $\mathbf{P}(\cdot \mid X^\uparrow_{\tau_x^\uparrow} = x)$, $\underset{\leftarrow}{Z}$ and $\underset{\rightarrow}{Z}$ are mutually independent; the process $\underset{\rightarrow}{Z}$ has the same law as $X^\uparrow$ and the law of $\underset{\leftarrow}{Z}$ is characterized by the following identity that holds for any nonnegative measurable functional F on Ω:*

$$\mathbf{E}\big[F(\underset{\leftarrow}{Z})\big] = \frac{1}{\mathcal{U}((-x, 0])}\mathbf{E}\left[\int\limits_0^{L_\infty} dv \, F(X_{\cdot \wedge L_v^{-1}})\mathbf{1}_{\{X_{L_v^{-1}} > -x\}}\right].$$

Remark. – Recall from Section 2.1, that under $\mathbf{P}(\cdot \mid X^\uparrow_{\tau_x^\uparrow} = x)$, the process $x + Z$ is markovian with a transition kernel given by

$$p_t^+(y, dz) = \frac{\mathcal{U}((-z, 0])}{\mathcal{U}((-y, 0])}q_t^+(y, dz), \quad y \geqslant 0,$$

where q_t^+ stands for the semigroup of the Lévy process killed in $(-\infty, 0]$. Then, the latter proposition combined with Lemma 2.1 gives the following corollary.

COROLLARY 4.8. – *Assume that $dd^* > 0$. Let $X^\uparrow(x)$ denote the Lévy process started at $x > 0$ and conditioned to stay positive. The path $X^\uparrow(x)$ has the following decomposition at its infimum:*

(i) *The pre-infimum process $\underleftarrow{X}^{\uparrow}(x)$ and the post-infimum process $\underrightarrow{X}^{\uparrow}(x)$ are mutually independent and $\underrightarrow{X}^{\uparrow}(x)$ is distributed as the Lévy process conditioned to stay positive started at* 0.

(ii) *The law of the infimum of $X^{\uparrow}(x)$ admits a density with respect to Lebesgue measure that is given by*:

$$y \to \mathbf{1}_{[0,x]}(y)\frac{u(y-x)}{\mathcal{U}((-x,0])}$$

and under $\mathbf{P}(\cdot \mid \inf X^{\uparrow}(x) = y)$, $\underleftarrow{X}^{\uparrow}(x)$ *is distributed as $x + X_{\cdot \wedge \tau_{y-x}}$ under $\mathbf{P}(\cdot \mid X_{\tau_{y-x}} = y - x)$.*

Remarks. – (i) We can actually show that the corollary remains true even if $d^* = 0$.

(ii) When X does not drift to $-\infty$ the result is Theorem 5 of Chaumont [9] (see also [7] and [8]).

Proof of Proposition 4.7. – Let T be independent of X and exponentially distributed with parameter α. Recall from Lemma 2.2 that for any nonnegative measurable functional H defined on Ω,

$$\mathbf{E}\big[H\big(\underrightarrow{X}^{T}\big)\big] = \frac{\alpha}{\kappa(\alpha,0)}N\bigg(\int_0^{\zeta} ds\, e^{-\alpha s}H(\omega_{\cdot \wedge s})\bigg).$$

Let G and D be nonnegative measurable functionals on Ω. Take

$$H(\omega) = \mathbf{1}_{\{\tau_x(\omega)<\infty\}}G\big(\omega(\cdot \wedge \tau_x(\omega))\big)D(\omega \circ \theta_{\tau_x(\omega)}), \quad \omega \in \Omega.$$

Then,

$$\mathbf{E}\big[G\big(\underrightarrow{X}^{T}_{\cdot \wedge \underrightarrow{\tau}^{T}_x}\big)D\big(\underrightarrow{X}^{T} \circ \theta_{\underrightarrow{\tau}^{T}_x}\big); \underrightarrow{\tau}^{T}_x < \infty\big]$$

$$= \frac{\alpha}{\kappa(\alpha,0)}N\bigg(\mathbf{1}_{\{\tau_x(\omega)<\infty\}}G(\omega_{\cdot \wedge \tau_x(\omega)})e^{-\alpha \tau_x(\omega)}\int_0^{\zeta - \tau_x(\omega)} ds\, e^{-\alpha s}D\big((\omega \circ \theta_{\tau_x(\omega)})_{\cdot \wedge s}\big)\bigg).$$

Apply the Markov property under N in order to get

$$\mathbf{E}\big[G\big(\underrightarrow{X}^{T}_{\cdot \wedge \underrightarrow{\tau}^{T}_x}\big)D\big(\underrightarrow{X}^{T} \circ \theta_{\underrightarrow{\tau}^{T}_x}\big); \underrightarrow{\tau}^{T}_x < \infty\big]$$

$$= N\big(\mathbf{1}_{\{\tau_x(\omega)<\infty\}}e^{-\alpha \tau_x(\omega)}G(\omega_{\cdot \wedge \tau_x(\omega)})d_\alpha(\omega_{\tau_x(\omega)})\big), \tag{20}$$

where for any $a > 0$,

$$d_\alpha(a) = \frac{1}{\kappa(\alpha,0)}\mathbf{E}\big[D(X_{\cdot \wedge T}); I_T > -a\big].$$

Take $D = 1$ in (20). We get

$$\mathbf{E}\big[G\big(\underrightarrow{X}^{\,T}_{\cdot \wedge \underrightarrow{\tau}^{\,T}_x}\big)\big] = N\left(\mathbf{1}_{\{\tau_x(\omega) < \infty\}} e^{-\alpha \tau_x(\omega)} G(\omega_{\cdot \wedge \tau_x(\omega)}) \frac{\mathbf{P}(I_T > -\omega_{\tau_x(\omega)})}{\kappa(\alpha, 0)}\right).$$

Next we need the following lemma:

LEMMA 4.9. – *Let $\alpha > 0$ and T_α be independent of X and exponentially distributed with parameter α. Let F, G and K be three bounded measurable functionals on Ω. Under the same assumptions as Theorem* 4.1, *we have*

$$\lim_{\alpha \to 0} \mathbf{E}\big[F\big(\underrightarrow{X}^{\,T_\alpha}_{\cdot \wedge \underrightarrow{\sigma}^{\,T_\alpha}_x}\big) G\big(\underleftarrow{Y}^{\,T_\alpha}\big) K\big(\underrightarrow{Y}^{\,T_\alpha}\big); \underrightarrow{\sigma}^{\,T_\alpha}_x < \infty\big] = \mathbf{E}\big[F(X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}) G(\underleftarrow{Y}) K(\underrightarrow{Y})\big].$$

Proof. – It is sufficient to show the limit

$$\lim_{s \to +\infty} \mathbf{E}\big[F\big(\underrightarrow{X}^{\,s}_{\cdot \wedge \underrightarrow{\sigma}^{\,s}_x}\big) G\big(\underleftarrow{Y}^{\,s}\big) K\big(\underrightarrow{Y}^{\,s}\big); \underrightarrow{\sigma}^{\,s}_x < \infty\big] = \mathbf{E}\big[F(X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}) G(\underleftarrow{Y}) K(\underrightarrow{Y})\big].$$

We use the notation of Section 2.2. Recall that $A^+_s = \int_0^s du\, \mathbf{1}_{\{X_u > 0\}}$. From Theorem 2.4, we have

$$(\underrightarrow{X}^{\,s}_u)_{0 \leqslant u < s - \underline{g}_s} \overset{(\text{law})}{=} (X^{\uparrow}_u)_{0 \leqslant u < A^+_s}. \tag{21}$$

Set $\beta = \sigma^{\uparrow}_x + \underline{g}(Y)$. By (7), $\beta < \infty$. In order to avoid cumbersome notations, we denote by W and W' respectively the pre-infimum process and the post-infimum process of $X^{\uparrow} \circ \theta_{\sigma_x(X^{\uparrow}, A^+_s)}$. Let $M > 0$ be an upper bound for F, G and K. Observe that on $\{A^+_s > \beta\}$, we have $\sigma_x(X^{\uparrow}, A^+_s) = \sigma^{\uparrow}_x$ and

$$F\big(X^{\uparrow}_{\cdot \wedge \sigma_x(X^{\uparrow}, A^+_s)}\big) G(W) K(W') = F\big(X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}\big) G(\underleftarrow{Y}) K(\underrightarrow{Y}). \tag{22}$$

Then by (21) and (22), we have

$$\mathbf{E}\big[F\big(\underrightarrow{X}^{\,s}_{\cdot \wedge \underrightarrow{\sigma}^{\,s}_x}\big) G\big(\underleftarrow{Y}^{\,s}\big) K\big(\underrightarrow{Y}^{\,s}\big)\big] = \mathbf{E}\big[F\big(X^{\uparrow}_{\cdot \wedge \sigma_x(X^{\uparrow}, A^+_s)}\big) G(W) K(W'); A^+_s \leqslant \beta\big]$$

$$+ \mathbf{E}\big[F\big(X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}\big) G(\underleftarrow{Y}) K(\underrightarrow{Y}); A^+_s > \beta\big].$$

Consequently,

$$\big|\mathbf{E}\big[F\big(\underrightarrow{X}^{\,s}_{\cdot \wedge \underrightarrow{\sigma}^{\,s}_x}\big) G\big(\underleftarrow{Y}^{\,s}\big) K\big(\underrightarrow{Y}^{\,s}\big)\big] - \mathbf{E}\big[F\big(X^{\uparrow}_{\cdot \wedge \sigma^{\uparrow}_x}\big) G(\underleftarrow{Y}) K(\underrightarrow{Y})\big]\big| \leqslant 2M^3 \mathbf{P}(\beta \geqslant A^+_s).$$

But $\lim_{s \to \infty} A^+_s = \infty$. So $\lim_{s \to \infty} \mathbf{P}(\beta \geqslant A^+_s) = 0$ which yields the lemma. □

Let us achieve the proof of the proposition: Lemma 4.9 implies

$$\lim_{\alpha \to 0} \mathbf{E}\big[G\big(\underrightarrow{X}^{\,T}_{\cdot \wedge \underrightarrow{\tau}^{\,T}_x}\big)\big] = \mathbf{E}\big[G(X^{\uparrow}_{\cdot \wedge \tau^{\uparrow}_x})\big].$$

Next, we deduce from Lemma 2.2 that for any $a > 0$

$$\lim_{\alpha \to 0} \frac{\mathbf{P}(I_T > -a)}{\kappa(\alpha, 0)} = \mathbf{E}\left[\int_0^{L^\infty} dv \, \mathbf{1}_{\{X_{L_v^{-1}} > -a\}}\right] = \mathcal{U}((-a, 0]),$$

which yields (i) by dominated convergence.

Assume now that the drift coefficient d^* is positive. Let D_1 and D_2 be two nonnegative measurable functionals on Ω. Take,

$$D(\omega) = D_1(\underleftarrow{\omega})D_2(\underrightarrow{\omega}) \quad \text{and} \quad G(\omega) = \mathbf{1}_{\{\tau_x(\omega)<\infty;\, \omega(\tau_x(\omega))=x\}},$$

in (20). From Lemma 2.2, we note that

$$d_\alpha(a) = \frac{1}{\kappa(\alpha, 0)}\mathbf{E}\big[D_1(\underleftarrow{X}^T); I_T > -a\big]\mathbf{E}\big[D_2(\underrightarrow{X}^T)\big]$$

$$= \mathbf{E}\left[\int_0^{L_\infty} dv \, e^{-\alpha L_v^{-1}} D_1(X_{\cdot \wedge L_v^{-1}})\mathbf{1}_{\{X_{L_v^{-1}} > -a\}}\right]\mathbf{E}\big[D_2(\underrightarrow{X}^T)\big].$$

Thus, (20) gives

$$\mathbf{E}\big[F(\underrightarrow{X}^T_{\cdot \wedge \underrightarrow{\tau}^T_x})D(\underrightarrow{X}^T \circ \theta_{\underrightarrow{\tau}^T_x});\ \underrightarrow{\tau}^T_x < \infty;\ \underrightarrow{X}^T_{\underrightarrow{\tau}^T_x} = x\big]$$

$$= \mathbf{E}\big[D_2(\underrightarrow{X}^T)\big]N\big(e^{-\alpha \tau_x(\omega)};\ \tau_x(\omega) < \infty;\ \omega_{\tau_x(\omega)} = x\big)$$

$$\times \mathbf{E}\left[\int_0^{L_\infty} dv \, e^{-\alpha L_v^{-1}} D_1(X_{\cdot \wedge L_v^{-1}})\mathbf{1}_{\{X_{L_v^{-1}} > -x\}}\right]. \tag{23}$$

To get (ii), pass to the limit $\alpha \to 0$ in (23) using Lemma 4.9 to write:

$$\lim_{\alpha \to 0} \mathbf{E}\big[D(\underrightarrow{X}^T \circ \theta_{\underrightarrow{\tau}^T_x});\ \underrightarrow{\tau}^T_x < \infty;\ \underrightarrow{X}^T_{\underrightarrow{\tau}^T_x} = x\big]$$

$$= \mathbf{E}\big[F(X^\uparrow_{\cdot \wedge \tau_x^\uparrow})D_1(\underleftarrow{Z})D_2(\underrightarrow{Z});\ X^\uparrow_{\tau_x^\uparrow} = x\big].$$

and

$$\lim_{\alpha \to 0} \mathbf{E}\big[D_2(\underrightarrow{X}^T)\big] = \mathbf{E}\big[D_2(X^\uparrow)\big]. \qquad \square$$

We are now able to state the second Williams' decomposition theorem.

THEOREM 4.10. – *Assume that π charges $(0, +\infty)$. Suppose also that $d^* > 0$ and that X oscillates.*

(i) *The law of $\omega_{\overline{g}(\omega)}$ under N admits a density with respect to Lebesgue measure that is given by*

$$x \to \frac{1}{d^*}N^*\big(\tau_{-x}(\omega) < \infty\big)N\big(\tau_x(\omega) < \infty;\ \omega_{\tau_x(\omega)} = x\big).$$

(ii) *Under* $N(\cdot \mid \omega_{\overline{g}(\omega)} = x)$, *the processes* $\omega_{\cdot \wedge \overline{g}(\omega)}$ *and* $\omega \circ \theta_{\overline{g}(\omega)}$ *are mutually independent. Furthermore,*

– *the process* $\omega_{\cdot \wedge \overline{g}(\omega)}$ *is distributed as* $X^{\uparrow}_{\cdot \wedge \tau^{\uparrow}_x}$ *under* $\mathbf{P}(\cdot \mid X^{\uparrow}_{\tau^{\uparrow}_x} = x)$;

– *the law of* $\omega \circ \theta_{\overline{g}(\omega)}$ *is absolutely continuous with respect to the law of* $X^{\downarrow}_{\cdot \wedge \tau^{\downarrow}_{-x}}$ *and the corresponding density is* $\varphi(X^{\downarrow}_{\tau^{\downarrow}_{-x}})$, *where*

$$\frac{1}{\varphi(y)} = \mathcal{U}^*([0, -y)) N^*(\tau_{-x}(\omega) < \infty), \quad y \in (-\infty, 0).$$

Remark. – The spectrally positive case is due to Chaumont in [9] or [7].

Proof. – Let G and D be two nonnegative measurable functionals on Ω. From Proposition 4.7(i) and the corresponding dual equality, we get for any $x > 0$,

$$\mathbf{E}\big[G\big(X^{\uparrow}_{\cdot \wedge \tau^{\uparrow}_x}\big)\big] = N\big(G(\omega_{\cdot \wedge \tau_x(\omega)})\mathcal{U}((-\omega_{\tau_x(\omega)}, 0]); \tau_x(\omega) < \infty\big)$$

and

$$\mathbf{E}\big[D\big(X^{\downarrow}_{\cdot \wedge \tau^{\downarrow}_{-x}}\big)\big] = N^*\big(D(\omega_{\cdot \wedge \tau_{-x}(\omega)})\mathcal{U}^*([0, -\omega_{\tau_{-x}(\omega)})); \tau_{-x}(\omega) < \infty\big).$$

The various assertions of the theorem then follow from the identity

$$N\big(G(\omega_{\cdot \wedge \overline{g}(\omega)})D(\omega \circ \theta_{\overline{g}(\omega)})\big)$$
$$= \frac{1}{d^*} \int\limits_0^{+\infty} dx \; N\big(G(\omega_{\cdot \wedge \tau_x(\omega)}); \omega_{\tau_x(\omega)} = x\big) N^*\big(D(\omega_{\cdot \wedge \tau_{-x}(\omega)}); \tau_{-x}(\omega) < \infty\big), \quad (24)$$

which we now prove.

Let a and b be two positive real numbers. Observe that

$$\{a \leqslant \omega_{\overline{g}(\omega)} < a + b\} = \{\tau_a(\omega) < \infty\} \cap \{\sup \omega \circ \theta_{\tau_a(\omega)} < b + a - \omega_{\tau_a(\omega)}\}.$$

On this event, $\omega \circ \theta_{\overline{g}(\omega)}$ is the post-supremum process of $\omega \circ \theta_{\tau_a(\omega)}$. Hence, by the Markov property under N at $\tau_a(\omega)$, we have

$$N\big(G(\omega_{\cdot \wedge \tau_a(\omega)})D(\omega \circ \theta_{\overline{g}(\omega)}); a \leqslant \omega_{\overline{g}(\omega)} < a + b\big)$$
$$= N\big(G(\omega_{\cdot \wedge \tau_a(\omega)})d(\omega_{\tau_a(\omega)}); \tau_a(\omega) < \infty\big),$$

where for any positive number x,

$$d(x) = \mathbf{E}\big[D\big(\overrightarrow{X}^{\tau_{-x}}\big); S_{\tau_{-x}} < b + a - x\big].$$

Let us write $d(x)$ in a more suitable form. First, observe that

$$d(x) = \mathbf{E}\left[\sum_{j \in \mathcal{I}^*} \mathbf{1}_{\{X_{g_j} < b+a-x; I_{g_j} > -x\}} D\big(\omega^j_{\cdot \wedge \tau_{-x-X_{g_j}}(\omega^j)}\big) \mathbf{1}_{\{\tau_{-x-X_{g_j}}(\omega^j) < \infty\}}\right].$$

Then, apply the compensation formula to get

$$d(x) = \mathbf{E}\left[\int\limits_0^\infty du\, \mathbf{1}_{\{X_{L_u^{*-1}} < b+a-x;\, I_{L_u^{*-1}} > -x\}} N^*\big(D(\omega_{.\wedge \tau_{-x-X_{L_u^{*-1}}}}(\omega))\mathbf{1}_{\{\tau_{-x-X_{L_u^{*-1}}}(\omega) < \infty\}}\big)\right].$$

Set for any real numbers x and y,

$$v(x, y) = \mathbf{1}_{[0,+\infty)}(x)\mathbf{1}_{[0,+\infty)}(y)u^*(y)\mathbf{P}(I_{\tau_y} > -x \mid X_{\tau_y} = y).$$

Then, for any $a \leqslant x < a + b$, Lemma 2.1 implies

$$d(x) = \int\limits_0^{a+b-x} dy\, u^*(y)\mathbf{E}\big[\mathbf{1}_{\{I_{\tau_y} > -x\}} N^*\big(D(\omega_{.\wedge \tau_{-x-y}}(\omega)); \tau_{-x-y}(\omega) < \infty\big) \mid X_{\tau_y} = y\big]$$

$$= \int\limits_a^{a+b} dy\, v(x, y-x)N^*\big(D(\omega_{.\wedge \tau_{-y}}(\omega)); \tau_{-y}(\omega) < \infty\big).$$

Thus,

$$N\big(G(\omega_{.\wedge \tau_a(\omega)})D(\omega \circ \theta_{\overline{g}(\omega)}); a \leqslant \omega_{\overline{g}(\omega)} < a+b\big)$$

$$= \int\limits_a^{a+b} dy\, N\big(\mathbf{1}_{\{\tau_a(\omega) < \infty\}} v(\omega_{\tau_a(\omega)}, y - \omega_{\tau_a(\omega)})G(\omega_{.\wedge \tau_a(\omega)})\big)$$

$$\times N^*\big(D(\omega_{.\wedge \tau_{-y}(\omega)}); \tau_{-y}(\omega) < \infty\big). \tag{25}$$

Next, set for any positive integer n,

$$m_n = \frac{[2^n \omega_{\overline{g}(\omega)}]}{2^n} \quad \text{and} \quad y_n = \frac{[2^n y]}{2^n}, \quad y \geqslant 0.$$

We apply (25) with $a = i2^{-n}$ and $b = 2^{-n}$ for every integer $i \geqslant 0$, and we sum over i. It follows that

$$N\big(G(\omega_{.\wedge \tau_{m_n}(\omega)})D(\omega \circ \theta_{\overline{g}(\omega)})\big)$$

$$= \int\limits_0^{+\infty} dy\, N\big(\mathbf{1}_{\{\tau_{y_n}(\omega) < \infty\}} v(\omega_{\tau_{y_n}(\omega)}, y - \omega_{\tau_{y_n}(\omega)})G(\omega_{.\wedge \tau_{y_n}(\omega)})\big)$$

$$\times N^*\big(D(\omega_{.\wedge \tau_{-y}(\omega)}); \tau_{-y}(\omega) < \infty\big). \tag{26}$$

Let $\varepsilon, A > 0$. It is sufficient to prove (24) for

$$G(\omega) = \mathbf{1}_{\{\zeta(\omega) > \varepsilon;\, \sup \omega \leqslant A\}} F(\omega) \quad \text{and} \quad D(\omega) = \mathbf{1}_{\{\zeta(\omega) > \varepsilon\}} K(\omega)$$

where $F(\omega)$ and $K(\omega)$ depend continuously on the values of ω at some finitely many positive times. Assumption (A) implies that X attains continuously its supremum on any

finite time interval (see Millar [15]). So does the excursion above the infimum. Thus, $\tau_{m_n}(\omega)$ increases to $\overline{g}(\omega)$ when n goes to infinity N-almost everywhere. Then,

$$\lim_{n \to \infty} \mathbf{1}_{\{\tau_{m_n}(\omega) > \varepsilon; m_n \leqslant A\}} = \mathbf{1}_{\{\overline{g}(\omega) > \varepsilon; \sup \omega \leqslant A\}}, \quad N\text{-a.e.}$$

Since $N(\overline{g}(\omega) > \varepsilon; \sup \omega \leqslant A) \leqslant N(\zeta(\omega) > \varepsilon) < \infty$, dominated convergence applies in the left side of (26) and we get:

$$\lim_{n \to \infty} N\big(G(\omega_{\cdot \wedge \tau_{m_n}(\omega)}) D(\omega \circ \theta_{\overline{g}(\omega)})\big)$$
$$= N\big(F(\omega_{\cdot \wedge \overline{g}(\omega)}) K(\omega \circ \theta_{\overline{g}(\omega)}); \sup \omega \leqslant A; (\zeta - \overline{g}(\omega)) \wedge \overline{g}(\omega) > \varepsilon\big). \quad (27)$$

We now turn to the limit of the right hand side of (26): Recall (2) from Section 2.1

$$\mathbf{P}\big(\exists t \in (0, +\infty): S_{t-} = X_{t-} < X_t\big) = 0.$$

It implies that for any positive number y, $N(\omega_{\tau_y(\omega)-} = y < \omega_{\tau_y(\omega)}) = 0$. We also recall that $\omega(0) = 0$, N-a.e. Thus,

$$N\text{-a.e.} \quad \lim_{n \to \infty} \mathbf{1}_{\{\omega_{\tau_{y_n}(\omega)} \leqslant y\}} = \mathbf{1}_{\{\omega_{\tau_y(\omega)} = y\}}. \quad (28)$$

Next, for any $x > 0$, $\lim_{\varepsilon \to 0} v(x, \varepsilon) = u^*(0+) = \frac{1}{d^*}$ because

$$0 \leqslant d^* u^*(\varepsilon) - d^* v(x, \varepsilon) = d^* u^*(\varepsilon) \mathbf{P}(I_{\tau_\varepsilon} \leqslant -x \mid X_{\tau_\varepsilon} = \varepsilon) \leqslant \mathbf{P}(I_{\tau_\varepsilon} \leqslant -x) \xrightarrow[\varepsilon \to 0]{} 0.$$

Thanks to (28) we get N-a.e.

$$\lim_{n \to \infty} \mathbf{1}_{\{\tau_{y_n}(\omega) < \infty\}} v(\omega_{\tau_{y_n}(\omega)}, y - \omega_{\tau_{y_n}(\omega)}) = u^*(0+) \mathbf{1}_{\{\omega_{\tau_y(\omega)} = y\}} = \frac{1}{d^*} \mathbf{1}_{\{\omega_{\tau_y(\omega)} = y\}}.$$

Observe that N-a.e. on $\{\omega_{\tau_y(\omega)} = y\}$, $\tau_{y_n}(\omega)$ increases towards $\tau_y(\omega)$. Therefore, by dominated convergence

$$\lim_{n \to \infty} N\big(\mathbf{1}_{\{\tau_{y_n}(\omega) < \infty\}} v(\omega_{\tau_{y_n}(\omega)}, y - \omega_{\tau_{y_n}(\omega)}) G(\omega_{\cdot \wedge \tau_{y_n}(\omega)})\big)$$
$$= \frac{1}{d^*} \mathbf{1}_{(0,A]}(y) N\big(F(\omega_{\cdot \wedge \tau_y(\omega)}); \varepsilon < \tau_y(\omega) < \infty; \omega_{\tau_y(\omega)} = y\big).$$

However, for any $y > 0$, we have

$$N\big(\mathbf{1}_{\{\tau_{y_n}(\omega) < \infty\}} v(\omega_{\tau_{y_n}(\omega)}, y - \omega_{\tau_{y_n}(\omega)}) G(\omega_{\cdot \wedge \tau_{y_n}(\omega)})\big) N^*\big(D(\omega_{\cdot \wedge \tau_{-y}(\omega)}; \tau_{-y}(\omega) < \infty)\big)$$
$$\leqslant M^2 \mathbf{1}_{(0,A]}(y) \sup_{x \in (0, 2^{-n}]} u^*(x) N(\zeta > \varepsilon) N^*(\zeta > \varepsilon),$$

where M is a bounding constant of F and K. Then, dominated convergence applies in the right side of (26), which yields the desired identity (24) thanks to (25). $\quad \square$

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