

THE LAW OF LARGE NUMBERS FOR BALLISTIC,
MULTI-DIMENSIONAL RANDOM WALKS ON RANDOM
LATTICES WITH CORRELATED SITES

LA LOI DES GRANDS NOMBRES POUR DES MARCHES
ALÉATOIRES MULTIDIMENSIONNELLES AU
COMPORTEMENT BALLISTIQUE SUR DES RÉSEAUX
ALÉATOIRES AVEC DES SITES CORÉLLÉS

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ABSTRACT. – We consider random walk in a random environment on d -dimensional integer lattice \mathbb{Z}^d with a uniform local drift. The environment is described as a stationary field of random vectors $(p_{\mathbf{k}}(e))$, $\mathbf{k}, e \in \mathbb{Z}$, $|e| = 1$ taking values in the standard $2d$ -dimensional simplex with the support of the law of $\sum_{|e|=1} e p_{\mathbf{k}}(e)$ lying on one side of a certain hyperplane (the so-called *non-nestling condition*). We admit that the vectors at different sites may dependent on each other, however their dependence range is finite. The main result of the paper is the law of large numbers for the trajectories of the walk under the probability obtained by averaging the laws of the random walks with respect to the environment (the so-called *annealed probability*). This result is related to the work of Sznitman–Zerner [18] where the law of large numbers has been shown for environments consisting of i.i.d. random vectors and satisfying the so-called Kalikov drift condition.

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RÉSUMÉ. – On considère une marche aléatoire en milieu aléatoire sur un réseau entier \mathbb{Z}^d de dimension d avec une dérive locale uniforme. Le milieu est décrit comme un champ stationnaire de vecteurs aléatoires $(p_{\mathbf{k}}(e))$, $\mathbf{k}, e \in \mathbb{Z}$, $|e| = 1$ à valeurs dans un simplexe standard de dimension $2a$,

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le support de la loi $\sum_{|e|=1} ep_{\mathbf{k}}(e)$ était situé dans un demi espace. On admet que les vecteurs en des sites différents peuvent être dépendants, mais leur domaine de dépendance est supposé fini.

Le résultat principal de cet article est la loi des grands nombres pour les trajectoires de la marche aléatoire sous la probabilité obtenue en moyennant les lois des marches aléatoires par rapport au milieu. Ce résultat est lié au le travail de Sznitman–Zerner où la loi des grands nombres a été démontrée pour des vecteurs aléatoires indépendants, identiquement distribués et vérifiant la condition de dérive de Kalikov.

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1. Introduction

A random walk in a random environment (R.W.R.E.) represents, besides diffusions with random coefficients, a basic model of motion in a highly disorganized medium. Speaking in general terms the model can be described as follows: at each site \mathbf{k} of the d -dimensional integer lattice \mathbb{Z}^d we assign a $2d$ -dimensional random vector $p_{\mathbf{k}} = (p_{\mathbf{k}}(e), e \in \mathfrak{A}_d)$, with $\mathfrak{A}_d := \{e \in \mathbb{Z}^d, |e| = 1\}$, given over a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose components satisfy $p_{\mathbf{k}}(e) \geq 0$, $\sum_{e \in \mathfrak{A}_d} p_{\mathbf{k}}(e) = 1$, for all $e \in \mathfrak{A}_d$, $\mathbf{k} \in \mathbb{Z}^d$. A nearest neighbor random walk $X_n, n \geq 0$, is performed on the lattice with the transition probabilities

$$P_0^\omega[X_{n+1} = X_n + e \mid X_n = \mathbf{k}] = p_{\mathbf{k}}(e), \quad \mathbf{k} \in \mathbb{Z}^d, e \in \mathfrak{A}_d, n \geq 0,$$

$$P_0^\omega[X_0 = \mathbf{0}] = 1.$$

This model has been extensively studied in the literature. In case of $d = 1$ and $\{p_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d\}$ being ergodic, translation invariant random vectors, a number of results concerning the asymptotic behavior of random walk trajectories have been shown under additional assumptions on the environment. These include the law of large numbers and central limit theorem, large deviation principles, see e.g. [17,19] and the references given there.

In the higher dimension, i.e. $d \geq 2$, the situation is more complicated and the scope of available results is considerably more limited. Some special cases were considered in [4,6,8]. More recently, the law of large numbers, central limit theorem and large deviation principles have been established for walks with an imposed drift, see [20,18,15,16].

Sometimes however, it is quite natural to consider lattices with correlated sites. Such a situation may occur when some conservation laws involving several vertices hold. Also, when one treats R.W.R.E. as a discretization of a d -dimensional diffusion with random coefficients, the assumption about independence of $p_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d$, seems unnatural. In this case a quite fruitful approach is to consider an abstract environment process, which roughly speaking describes the environment from the point of view of a random walker. This process is Markovian and allows the application of the ergodic theory provided that we can find an ergodic invariant measure that is absolutely continuous with respect to the probabilistic measure given on the environment. This approach has been taken for example in [11] for diffusions and [6,3,8] for R.W.R.E. case, see also [17,10]. Except for the situation when the invariant measure is explicitly provided, as for example is the case for symmetric or doubly stochastic transition probabilities, see p. 78 of [6], the problem of finding such a measure is usually quite prohibiting and can be solved only in one dimension, see [9,1].

In the present paper we consider an environment $p_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^d$, that is statistically stationary, i.e. its statistics does not depend on translations by a fixed vector, with dependent sites, of which we assume that they decorrelate at finite distances, see condition (H3) below for detailed definition. We show, see Theorem 1, that when the environment is biased, see condition (H2) below, the law of large numbers holds in the product (the so-called annealed) measure $\mathbb{P}_0[\cdot] := \mathbb{P}(d\omega) \otimes P_0^\omega[\cdot]$.

Let us briefly describe the strategy of the proof. We assume that the drift imposed on the environment in a certain direction $\mathbf{l} \in \mathbb{R}^d$ (condition (H2)) is sufficiently strong, so we are able to prove the existence of random times τ_k , $k \geq 1$, at which the trajectory of the particle will almost surely perform no backtracking in the direction of \mathbf{l} , see Corollary 1 and the remark afterwards. The proof of this fact is essentially done using the technique of the proof of Proposition 1.2 of [18]. However, the dependence of sites forces us to make a stronger assumption on the drift than Kalikow condition used in *ibid*. In contrast with [18] the dependence of lattice sites prevents us from recovering the renewal structure of the walk at τ_k , $k \geq 1$, that has been shown in the i.i.d. case, which makes impossible a direct application of the argument of Sznitman and Zerner. In fact, the sequence

$$(X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n)_{n \geq 1} \tag{1.1}$$

needs not even be stationary under the annealed measure \mathbb{P}_0 .

The main novelty of the paper lies in the fact that we show, see Theorem 2 below, that there exists a measure $\mathbb{P}_{h_*} \ll \mathbb{P}_0$ such that the random sequence given by (1.1) is stationary and ergodic with respect to \mathbb{P}_{h_*} . The law large numbers is then a consequence of the individual ergodic theorem and can be proven in a manner similar to Theorem 2.3 of [18].

To show the existence of \mathbb{P}_{h_*} we construct, what we call, a transport operator Q on the space of measures absolutely continuous with respect to a certain modification of \mathbb{P} , see Section 4. This operator is in some sense adjoint to the shift

$$\theta((X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n)_{n \geq 1}) := (X_{\tau_{n+2}} - X_{\tau_{n+1}}, \tau_{n+2} - \tau_{n+1})_{n \geq 1},$$

see the proof of stationarity part of Theorem 2 and Lemma 6 in particular (4.11)–(4.13). We use the lower bound technique for probability transport operator associated with a Markov chain, cf. [7], to show that the non-nestling condition (H2) implies the existence of an invariant density for Q , which allows us to define \mathbb{P}_{h_*} .

It should be noted that the law of large numbers for environments with a finite dependence range has also been proven, via a different technique and under a somewhat different set of assumptions, in [19], p. 74. Our argument however does not use any form of ellipticity assumption about the environment, which seems to be needed *ibid.*, and what is probably even more important seems to open the possibility of using the concept of canonical environment process in the investigation of multi-dimensional environments when the invariant measure is a priori unknown. In fact, we construct here a probability measure on the environment space, under which the increments of the walk at no backtracking times, see Theorem 2 below, are stationary. Applying a suitable modification of the definition of the transport operator one should be able to use our

technique to characterize the asymptotic behavior of the particle in strongly mixing, but not necessarily of finite dependence range, environments. This problem is a topic of an ongoing research and shall be presented in a separate paper.

2. Preliminaries and the formulation of the main result

Suppose that (Ω, \mathfrak{d}) is a certain Polish space with a certain probability measure \mathbb{P} on $(\Omega, \mathcal{B}(\Omega))$. Here and in the sequel $\mathcal{B}(\mathcal{M})$ denotes the σ -algebra of Borel sets for a given metric space \mathcal{M} . Let $\overline{\mathcal{B}(\Omega)}$ be the completion of $\mathcal{B}(\Omega)$ under \mathbb{P} and \mathcal{N} be its σ -ideal of \mathbb{P} -null sets. In what follows, unless stated otherwise, we assume that any sub σ -algebra of $\overline{\mathcal{B}(\Omega)}$ contains \mathcal{N} . Also, by \mathbb{E} we denote the mathematical expectation corresponding to \mathbb{P} . We assume that on Ω we are given a group $T_{\mathbf{x}} : \Omega \rightarrow \Omega, \mathbf{x} \in \mathbb{Z}^d$ of measure preserving transformations, i.e. $T_{\mathbf{x}}T_{\mathbf{y}} = T_{\mathbf{x}+\mathbf{y}}, T_{\mathbf{x}}(A) \in \overline{\mathcal{B}(\Omega)}$ for any $A \in \overline{\mathcal{B}(\Omega)}$, $\mathbb{P}T_{\mathbf{x}} = \mathbb{P}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$. Denote by Δ_{2d} the set of all $2d$ -tuples $(p_e)_{e \in \mathfrak{A}_d}$ satisfying

$$p_e \in [0, 1], e \in \mathfrak{A}_d \quad \text{and} \quad \sum_{e \in \mathfrak{A}_d} p_e = 1. \tag{2.1}$$

(H1) *Stationarity of the environment.* Let $p : \Omega \rightarrow \Delta_{2d}$ be a certain random vector.

We define $p^\omega(\mathbf{x}, e) := p_e(T_{\mathbf{x}}(\omega)), \mathbf{x} \in \mathbb{Z}^d, e \in \mathfrak{A}_d$.

The nearest neighbor random walk starting from \mathbf{x} in the random environment ω is a Markov chain $(X_n)_{n \geq 0}$ with the state space \mathbb{Z}^d , over the canonical path space $\mathcal{T}_{\mathbf{x}}^\omega := ((\mathbb{Z}^d)^\mathbb{N}, \mathcal{F}, P_{\mathbf{x}}^\omega)$, that satisfies

$$P_{\mathbf{x}}^\omega[X_{n+1} = X_n + e \mid X_0, \dots, X_n] \stackrel{P_{\mathbf{x}}^\omega\text{-a.s.}}{=} p^\omega(X_n, e), \quad n \geq 0, e \in \mathfrak{A}_d, \tag{2.2}$$

$$P_{\mathbf{x}}^\omega[X_0 = \mathbf{x}] = 1.$$

Here \mathcal{F} is the natural product σ -algebra of $(\mathbb{Z}^d)^\mathbb{N}$. Functions $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^d, \sigma(n) := X_n$ will denote elements of the path space.

We shall denote by $\mathbf{M}_{\mathbf{x}}^\omega$ the mathematical expectation corresponding to probability measure $P_{\mathbf{x}}^\omega$ and by \mathcal{F}_n the sub σ -algebra of \mathcal{F} generated by X_0, \dots, X_n . On the path space we can also introduce the shift operators $\theta_k, k \geq 1$, defined by $\theta_k(\sigma)(\cdot) = \sigma(\cdot + k)$.

We define $\mathbb{P}_{\mathbf{x}}$, the “annealed” law of the walk starting from \mathbf{x} , as the semi-direct product on $\mathcal{T}_{\mathbf{x}} := (\Omega \times (\mathbb{Z}^d)^\mathbb{N}, \overline{\mathcal{B}(\Omega)} \otimes \mathcal{F}, \mathbb{P}_{\mathbf{x}})$. In particular, for $A \in \overline{\mathcal{B}(\Omega)}, B \in \mathcal{F}$ we have

$$\mathbb{P}_{\mathbf{x}}(A \times B) := \int_A P_{\mathbf{x}}^\omega(B) \mathbb{P}(d\omega). \tag{2.3}$$

We assume that the walk satisfies the following

(H2) *non-nestling condition*, i.e. (cf. [20]), there exists a nonzero vector $\mathbf{l} = (l_1, \dots, l_d) \in \mathbb{R}^d$, such that the convex hull of the support of the law of the local drift at $\mathbf{0}$

$$d(\omega) := \sum_{e \in \mathfrak{A}_d} p_e(\omega)e$$

is contained in the half space $[\mathbf{x} \in \mathbb{R}^d: \mathbf{l} \cdot \mathbf{x} > 0]$. We also define

$$\delta := 1 \wedge \sup[\tilde{\delta}: \mathbb{P}[d(\omega) \cdot \mathbf{l} \geq \tilde{\delta}] = 1]. \tag{2.4}$$

Let $\mathcal{G}_t, t \in \mathbb{R}$, be a filtration of sub σ -algebras of $\overline{\mathcal{B}(\Omega)}$. We call the filtration *admissible* if

(G1) $p^\omega(\mathbf{x})$, with $\mathbf{x} \cdot \mathbf{l} \leq t$, are \mathcal{G}_t -measurable for all $t \in \mathbb{R}$ and

(G2) $T_{\mathbf{x}}(\mathcal{G}_t) \subseteq \mathcal{G}_{t+\mathbf{x} \cdot \mathbf{l}}$ for all $t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d$.

We shall also use the following notation: $\mathcal{G} := \bigvee_{t=-\infty}^{+\infty} \mathcal{G}_t$, i.e. \mathcal{G} is the smallest σ -algebra containing all $\mathcal{G}_t, t \in \mathbb{R}$, and \mathcal{H}^u to denote the σ -algebra generated by all $p^\omega(\mathbf{x})$, with $\mathbf{x} \cdot \mathbf{l} \geq u, u \in \mathbb{R}$.

We say that the filtration $\mathcal{G}_t, t \geq 0$, admits a factorization with respect to \mathcal{G}_0 , see [14] p. 66, if for any $t \geq 0$ there exists a σ -algebra $\mathcal{R}^t, t \geq 0$, such that $\mathcal{G}_t = \mathcal{G}_0 \otimes \mathcal{R}^t$, i.e. \mathcal{G}_0 and \mathcal{R}^t are \mathbb{P} -independent and \mathcal{G}_t is the smallest σ -algebra containing both \mathcal{G}_0 and \mathcal{R}^t . Criteria on factorization of filtrations of σ -algebras can be found in the aforementioned paper by Skorochod, see e.g. Theorem 7 p. 68 and its corollary and also Example 4 below.

We shall suppose that \mathbb{P} is such that the dependence range of sites on \mathbb{Z}^d in the direction of \mathbf{l} is finite. More specifically, we assume the following condition.

(H3) *Finite range dependence in the direction \mathbf{l}* . We suppose that \mathcal{G}_0 factors $\mathcal{G}_t, t \geq 0$ with $\mathcal{G}_t = \mathcal{G}_0 \otimes \mathcal{R}^t$ and there exists $r > 0$ such that $\mathcal{H}^r \subseteq \mathcal{R}$, where $\mathcal{R} := \bigvee_{t=0}^{+\infty} \mathcal{R}^t$.

Remark. – Let us notice that $\mathcal{R}^t, t \geq 0$, form a filtration of σ -algebras. Indeed, any random variable $H(\cdot)$ that is \mathcal{R}^t -measurable is \mathcal{G}_u -measurable for any $u \geq t$ and one can find (from the factorization) a random variable $G(\cdot, \cdot)$ that is $\mathcal{G}_0 \otimes \mathcal{R}^u$ -measurable and $H(\omega) = G(\omega, \omega)$. From the fact that H is independent of \mathcal{G}_0 we immediately conclude that $H(\omega) = \int G(\omega', \omega) \mathbb{P}(d\omega')$, \mathbb{P} -a.s., thus H is \mathcal{R}^u -measurable.

The previous argument also shows that any random vector $p(\mathbf{x})$ with $t := \mathbf{x} \cdot \mathbf{l} \geq r$, with r as in (H3), is \mathcal{R}^t -measurable.

Example 1. – (*The lattices with i.i.d. sites*). In this case the environment is given by i.i.d. $2d$ -dimensional random vectors $(p_e(\mathbf{x}))_{e \in \mathfrak{A}_d}, \mathbf{x} \in \mathbb{Z}^d$, satisfying (2.1). For an arbitrary $t \in \mathbb{R}$ we take as \mathcal{G}_t the σ -algebra generated by \mathcal{N} and $p(\mathbf{x})$, where $\mathbf{x} \cdot \mathbf{l} \leq t$. Condition (H3) is then obviously fulfilled.

Example 2. – (*The lattices with i.i.d. bonds* (see e.g. [13])). Let us denote by \mathbb{B}^d the set of nearest neighbor bonds on \mathbb{Z}^d . The random environment is given by i.i.d. random variables $B(b), b \in \mathbb{B}^d$, over $(\Omega, \overline{\mathcal{B}(\Omega)}, \mathbb{P})$. In this case we take

$$p_e(\mathbf{x}) = f((B(\mathbf{x}, \mathbf{x} + e'))_{e' \in \mathfrak{A}_d}, e), \quad \mathbf{x} \in \mathbb{Z}^d, e \in \mathfrak{A}_d,$$

where $f = (f(\cdot, e), e \in \mathfrak{A}_d)$ is a Δ_{2d} -valued function defined on \mathbb{R}^{2d} . In this case \mathcal{G}_t is defined as the σ -algebra generated by \mathcal{N} and all $B(b)$ with $b = (\mathbf{x}, \mathbf{x} + e)$ satisfying $\mathbf{x} \cdot \mathbf{l} \leq t$. Once again it is easy to see that (H3) is then satisfied.

Example 3. – (Gaussian environments). In this case we assume that $S(\mathbf{x}) \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{Z}^d$, is a n -dimensional homogeneous Gaussian vector field over $(\Omega, \overline{\mathcal{B}(\Omega)}, \mathbb{P})$. We suppose further that the field is centered, i.e. $\mathbb{E} S(\mathbf{0}) = \mathbf{0}$ and

- (U) there exists $R > 0$, for which $S(\mathbf{x}), S(\mathbf{y})$ are uncorrelated when $|\mathbf{x} - \mathbf{y}| > R$, or equivalently $\mathbb{E}[S(\mathbf{x}) \otimes S(\mathbf{0})] = \mathbf{0}$, when $|\mathbf{x}| > R$.

Let A be a finite subset of \mathbb{Z}^d . We set

$$p_e(\mathbf{x}) = f_e((S(\mathbf{y}))_{\mathbf{y} \in A}), \quad \mathbf{x} \in \mathbb{Z}^d, \quad e \in \mathfrak{A}_d, \tag{2.5}$$

where $f(\cdot) := (f_e(\cdot))_{e \in \mathfrak{A}_d}$ is Δ_{2d} -valued function defined on $(\mathbb{R}^n)^k$, with k the cardinality of A . In this case for any $t \in \mathbb{R}$ we take \mathcal{G}_t the σ -algebra generated by \mathcal{N} and $p(\mathbf{x}), \mathbf{x} \cdot \mathbf{1} \leq t$. The filtration defined in that way is admissible thanks to (2.5). In addition \mathcal{G}_0 factors \mathcal{G}_t because we can write $S(\mathbf{x}) = S_0(\mathbf{x}) + S^0(\mathbf{x})$, where S_0 is the orthogonal projection onto $L^2(\Omega, \mathcal{G}_0, \mathbb{P})$ and S^0 its complement. Thanks to the standard theory of Gaussian fields, see e.g. Theorem 10.1, p. 181 of [12], we can define \mathcal{R}^t as the σ -algebra generated by \mathcal{N} and $S^0(\mathbf{x}), \mathbf{x} \cdot \mathbf{1} \leq t, \mathbf{x} \in \mathbb{Z}^d$. Assumption (H3) is then satisfied thanks to condition (U) with any $r > k + R$.

Example 4. – (Lattices with finite dependence range). Suppose that $p(\mathbf{x}), \mathbf{x} \in \mathbb{Z}^d$, is a random field defined as in (H1). We let \mathcal{G}_t be the σ -algebra generated by $p(\mathbf{x}), \mathbf{x} \cdot \mathbf{1} \leq t, \mathcal{H}^u, u \in \mathbb{R}$, be as in (H3) and assume that there exists $r > 0$ such that \mathcal{H}^r is independent of \mathcal{G}_0 . In addition we suppose that the following hypothesis holds.

- (A) For any finite set $I \subseteq \mathbb{Z}^d$ all the probability vectors $p(\mathbf{x}, e), \mathbf{x} \in I, e \in \mathfrak{A}_d$, have absolutely continuous distributions with respect to $\otimes_I m_{\Delta_{2d}}$ -the product of $|I|$ copies of the Lebesgue surface measures on the simplex Δ_{2d} . $|I|$ stands for the cardinality of I .

We define the conditional probability $\mathbb{P}[\cdot | \mathcal{G}_0]$ in the following way. Since $\mathcal{G}_t, \mathcal{G}_0$ are countably generated in the sense of the definition given on p. 63 of [14], there exist random variables ξ_t, ξ_0 , Theorem 1 of *ibid.*, such that $\mathcal{G}_t = \sigma(\xi_t) \vee \mathcal{N}, \mathcal{G}_0 = \sigma(\xi_0) \vee \mathcal{N}$. Let $\Phi_{\xi_t}[\cdot | \xi_0]$ be the conditional law of ξ_t on ξ_0 , we set

$$\mathbb{P}[[\xi_t \in A] \cup N | \mathcal{G}_0](\omega) := \Phi_{\xi_t}[A | \xi_0(\omega)], \quad \text{for any } A \in \mathcal{B}(\mathbb{R}), N \in \mathcal{N}.$$

The following proposition holds.

PROPOSITION 1. – *Under assumption (A) made about the lattice, the conditional probabilities $\mathbb{P}[\cdot | \mathcal{G}_0]$ are atomless on $\mathcal{G}_t, \mathbb{P}$ -a.s. for arbitrary $t \geq 0$.*

We recall that $A \in \Sigma$, where $(\mathcal{X}, \Sigma, \mu)$ is a certain measure space, is called an atom, if for any $B \in \Sigma, B \subseteq A$ and $\mu[B] < \mu[A]$ the equality $\mu[B] = 0$ holds.

Because of rather technical nature of the proof of this proposition and the fact that we wish to stay focused on proving the L.L.N. for the random walk we postpone the argument till Appendix A.

The result established in the proposition implies, according to Theorem 5 of [14], that \mathcal{G}_0 factors \mathcal{G}_t for $t \geq 0$.

The main result of this paper is the following.

THEOREM 1. – Assume that (H1)–(H3) hold. Then, there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^d$ such that

$$\lim_{n \uparrow +\infty} \frac{X_n}{n} = \mathbf{v}, \quad \mathbb{P}_0\text{-a.s.}$$

3. Non-retraction times

For any $\sigma \in (\mathbb{Z}^d)^\mathbb{N}$ we let

$$\begin{aligned} D(\sigma) &:= \inf[n \geq 0: \mathbf{1} \cdot X_n(\sigma) < \mathbf{1} \cdot X_0(\sigma)], \\ U_u(\sigma) &:= \inf[n \geq 0: \mathbf{1} \cdot X_n(\sigma) \geq u], \\ \tilde{U}_u(\sigma) &:= \inf[n \geq 0: \mathbf{1} \cdot X_n(\sigma) \leq u] \end{aligned} \tag{3.1}$$

and

$$M_*(\sigma) := \sup[\mathbf{1} \cdot (X_m(\sigma) - X_0(\sigma)): 0 \leq m \leq D(\sigma)]. \tag{3.2}$$

The last random variable is defined for those σ , for which $D(\sigma) < +\infty$.

In analogy with [18] we introduce the sequence of \mathcal{F}_n -stopping times $S_k, k \geq 0, R_k, k \geq 1$, and the sequence of maxima $M_k, k \geq 0$:

$$\begin{aligned} S_0 &= 0, \quad M_0 = \mathbf{1} \cdot X_0, \\ S_1 &= U_{M_0+r+1} \leq +\infty, \quad R_1 = D \circ \theta_{S_1} + S_1 \leq +\infty, \\ M_1 &= \sup[\mathbf{1} \cdot X_m, 0 \leq m \leq R_1] \leq +\infty, \end{aligned} \tag{3.3}$$

where $r > 0$ is as in (H3).

By induction we set for any $k \geq 1$

$$\begin{aligned} S_{k+1} &= U_{M_k+r+1}, \quad R_{k+1} = D \circ \theta_{S_{k+1}} + S_{k+1}, \\ M_{k+1} &= \sup[\mathbf{1} \cdot X_m, 0 \leq m \leq R_{k+1}]. \end{aligned} \tag{3.4}$$

LEMMA 1. – There exists a deterministic constant $\gamma > 0$ such that

$$P_{\mathbf{x}}^\omega[D = +\infty] \geq \gamma, \quad \text{for all } \mathbf{x} \in \mathbb{Z}^d, \mathbb{P}\text{-a.s.} \tag{3.5}$$

Proof. – It suffices to show (3.5) for $\mathbf{x} = \mathbf{0}$, with some constant $\gamma > 0$ whose choice, as it becomes apparent in the course of the proof, is independent of the choice of the starting point of the walk. For any $M > 0$ we denote

$$S_M^+ := [\mathbf{x} \in \mathbb{Z}^d: 0 \leq \mathbf{1} \cdot \mathbf{x} \leq M] \tag{3.6}$$

and $T_{S_M^+}$ the exit time from the strip. Since

$$P_{\mathbf{0}}^\omega[D = +\infty] = \lim_{M \uparrow +\infty} P_{\mathbf{0}}^\omega[T_{S_M^+} < +\infty, \mathbf{1} \cdot X_{S_M^+} > M]. \tag{3.7}$$

Inequality (3.5) will be proven once we show that there exists a constant $c > 0$, which bounds the right hand side of (3.7) from below. Let

$$\mathcal{M}_n := X_n - \sum_{k=0}^{n-1} d(X_k), \tag{3.8}$$

with $d(\mathbf{x}, \omega) := d(T_{\mathbf{x}}(\omega))$. It is an \mathcal{F}_n -martingale under P_0^ω . Take any connected and finite set $V \subseteq \mathcal{S}_M^+$. From the optional sampling theorem we conclude that

$$\mathbf{M}_0^\omega[X_{n \wedge T_V} \cdot \mathbf{1}] - \mathbf{M}_0^\omega \left[\sum_{k=0}^{n \wedge T_V - 1} d(X_k) \cdot \mathbf{1} \right] = 0. \tag{3.9}$$

From the non-nestling condition (H2) we have

$$d(\mathbf{x}) \cdot \mathbf{1} \geq \delta > 0 \tag{3.10}$$

for all $\mathbf{x} \in \mathbb{Z}^d$ and some deterministic $1 \geq \delta > 0$. Since $0 \leq X_{n \wedge T_V} \cdot \mathbf{1} \leq M + \|\mathbf{l}\|$, with $\|\mathbf{l}\|$ the Euclidean norm of a vector, (3.9) implies that $\mathbf{M}_0^\omega T_V \leq (M + \|\mathbf{l}\|)/\delta$, thus also $\mathbf{M}_0^\omega T_{\mathcal{S}_M^+} \leq (M + \|\mathbf{l}\|)/\delta$ and in consequence

$$P_0^\omega[T_{\mathcal{S}_M^+} < +\infty] = 1. \tag{3.11}$$

To finish the proof of (3.7) we find $\theta_0, K > 0$ such that $K\theta_0 \leq \delta$ and $|e^{-\theta u} - 1 + \theta u| \leq K\theta^2$, for $|u| \leq \|\mathbf{l}\|, |\theta| \leq \theta_0$. Then from the Markov property of $P_0^\omega[\cdot]$ we get

$$\mathbf{M}_0^\omega[\exp\{-\theta \mathbf{l} \cdot X_{n+1}\} | \mathcal{F}_n] = \exp\{-\theta \mathbf{l} \cdot X_n\} [1 - \theta \mathbf{l} \cdot d(X_n) + R], \tag{3.12}$$

where $|R| \leq K\theta^2$. Recalling (3.10) we conclude that, for any $0 < \theta \leq \theta_0$

$$\mathbf{M}_0^\omega[\exp\{-\theta \mathbf{l} \cdot X_{n+1}\} | \mathcal{F}_n] \leq \exp\{-\theta \mathbf{l} \cdot X_n\},$$

which shows that $\exp\{-\theta \mathbf{l} \cdot X_n\}, n \geq 0$, is a (\mathcal{F}_n) -supermartingale. From the non-nestling condition (H2) we immediately conclude that

$$\sum_{e: \mathbf{l} \cdot e > \delta/2} e \cdot \mathbf{l} P_0^\omega[X_1 = e] \geq \delta/2. \tag{3.13}$$

Hence,

$$\max[P_0^\omega[X_1 = e]: e \cdot \mathbf{l} > \delta/2] \geq \delta/(2dc_*), \tag{3.14}$$

with

$$c_* := 1 + \max_{1 \leq i \leq d} |l_i|. \tag{3.15}$$

From the optional sampling theorem for supermartingales we obtain immediately that

$$\begin{aligned} \exp\{-\theta\delta/2\} \sum_{e \cdot \mathbf{1} > \delta/2} P_0^\omega[X_1 = e] &\geq \sum_{e \cdot \mathbf{1} > \delta/2} \mathbf{M}_0^\omega[\exp\{-\theta \mathbf{1} \cdot e\}, X_1 = e] \\ &\geq \sum_{e \cdot \mathbf{1} > \delta/2} \mathbf{M}_0^\omega[\exp\{-\theta \mathbf{1} \cdot X_{T_{S_M^+} \wedge n}\}, X_1 = e] \\ &\geq \sum_{e \cdot \mathbf{1} > \delta/2} P_0^\omega[\mathbf{1} \cdot X_{T_{S_M^+} \wedge n} < 0, X_1 = e] \end{aligned}$$

for $n \in \mathbb{N}$. Thus, in consequence of the above estimate and (3.11) we conclude that

$$\begin{aligned} P_0^\omega[\mathbf{1} \cdot X_{T_{S_M^+}} > M] &\geq \sum_{e \cdot \mathbf{1} > \delta/2} P_0^\omega[\mathbf{1} \cdot X_{T_{S_M^+}} > M, X_1 = e] \\ &\geq \sum_{e \cdot \mathbf{1} > \delta/2} P_0^\omega[X_1 = e](1 - \exp\{-\delta\theta/2\}) \\ &\geq \frac{\delta}{2dc_*}(1 - \exp\{-\delta\theta/2\}). \quad \square \end{aligned} \tag{3.16}$$

LEMMA 2. – *There exist deterministic constants $\gamma_1, \gamma_2 > 0$ such that*

$$P_{\mathbf{x}}^\omega[\tilde{U}_{\mathbf{x} \cdot \mathbf{1} - M} < U_{\mathbf{x} \cdot \mathbf{1} + M}] \leq \gamma_2 \exp\{-\gamma_1 M\}, \quad \text{for all } M \geq 0, \mathbf{x} \in \mathbb{Z}^d, \mathbb{P}\text{-a.s.} \tag{3.17}$$

Proof. – As in the proof of the previous lemma we assume, with no loss of generality, that $\mathbf{x} = \mathbf{0}$. We can write then that the left hand side of (3.17) is less than or equal to

$$P_0^\omega[T_{S_M} > t_M] + P_0^\omega[T_{S_M} \leq t_M, \mathbf{1} \cdot X_{T_{S_M}} < -M]. \tag{3.18}$$

Here $t_M := [2M\delta^{-1}]$, $S_M := [\mathbf{x} \in \mathbb{Z}^d: -M \leq \mathbf{1} \cdot \mathbf{x} \leq M]$ and T_{S_M} denotes the exit time from the strip. Parameter δ is given by (2.4). Using the notation of (3.8) we can write that on the event $[T_{S_M} > t_M]$

$$|\mathcal{M}_{t_M}| = \left| X_{t_M} - \sum_{k=0}^{t_M-1} d(X_k) \right| \geq \frac{M}{c_*}.$$

Indeed, $X_{t_M} \in S_M$, thus $X_{t_M} \cdot \mathbf{1} \leq M$. On the other hand $\sum_{k=0}^{t_M-1} d(X_k) \cdot \mathbf{1} \geq \delta \cdot t_M \geq 2M$. Therefore, $|X_{t_M} \cdot \mathbf{1} - \sum_{k=0}^{t_M-1} d(X_k) \cdot \mathbf{1}| \geq M$ and the above estimate follows.

From Azuma’s inequality for martingales, see [2] p. 85, we obtain

$$P_0^\omega[T_{S_M} > t_M] \leq P_0^\omega\left[|\mathcal{M}_{t_M}| \geq \frac{M}{c_*}\right] \leq 2d \exp\left\{-\frac{\delta M}{4c_*^2}\right\}.$$

On the other hand

$$P_0^\omega[T_{S_M} \leq t_M, \mathbf{1} \cdot X_{T_{S_M}} < -M] \leq P_0^\omega\left[\sup_{0 \leq n \leq t_M} |\mathcal{M}_n| \geq \frac{M}{c_*}\right]. \tag{3.19}$$

Using again Azuma’s inequality, we estimate the right hand side of (3.19) by $2d(t_M + 1) \exp\{-\delta M/(4c_*^2)\}$ and (3.17) follows. \square

LEMMA 3. – *There exists a deterministic constant $\gamma_3 > 0$ such that*

$$\mathbf{M}_{\mathbf{x}}^\omega[M_*, D < +\infty] \leq \gamma_3, \quad \text{for all } \mathbf{x} \in \mathbb{Z}^d, \mathbb{P}\text{-a.s.} \tag{3.20}$$

Proof. – Again with no loss of generality we shall assume that $\mathbf{x} = \mathbf{0}$. Then, for any integer $m \geq 1$

$$\begin{aligned} & P_{\mathbf{0}}^\omega [2^m \leq M_* < 2^{m+1}, D < +\infty] \\ & \leq P_{\mathbf{0}}^\omega \left[\left| X_{U_{2^m}} - 2^m \frac{\mathbf{1}}{\|\mathbf{1}\|^2} \right| \geq \frac{2^{m+1}}{\delta} \right] \\ & \quad + P_{\mathbf{0}}^\omega \left[\left| X_{U_{2^m}} - 2^m \frac{\mathbf{1}}{\|\mathbf{1}\|^2} \right| < \frac{2^{m+1}}{\delta}, \tilde{U}_0 \circ \theta_{U_{2^m}} < U_{2^{m+1}} \circ \theta_{U_{2^m}} \right]. \end{aligned} \tag{3.21}$$

Let

$$C := \left[\mathbf{x} \in \mathbb{Z}^d : \left| \mathbf{x} - \frac{\mathbf{1} \cdot \mathbf{x}}{\|\mathbf{1}\|^2} \mathbf{1} \right| \leq \frac{\mathbf{1} \cdot \mathbf{x}}{\delta} \right].$$

C is a cone containing the support of the law of d – the local drift at zero. Therefore $\sum_{k=0}^{n-1} d(X_k) \in C$, for all n . On the other hand there exists $c_1 > 0$ such that for all $m \geq 1$ if

$$\left| \mathbf{x} - 2^m \frac{\mathbf{1}}{\|\mathbf{1}\|^2} \right| \geq \frac{2^{m+1}}{\delta}, \quad \mathbf{x} \cdot \mathbf{1} \leq 2^m + \|\mathbf{1}\|,$$

then $\text{dist}(\mathbf{x}, C) > c_1 2^m$. The first term on the right hand side of (3.21) can be therefore estimated by

$$\begin{aligned} & P_{\mathbf{0}}^\omega \left[\left| X_{U_{2^m}} - 2^m \frac{\mathbf{1}}{\|\mathbf{1}\|^2} \right| \geq \frac{2^{m+1}}{\delta}, U_{2^m} \leq \frac{2^{m+1}}{\delta} \right] + P_{\mathbf{0}}^\omega \left[U_{2^m} > \frac{2^{m+1}}{\delta} \right] \\ & \leq P_{\mathbf{0}}^\omega [|\mathcal{M}_{U_{2^m}}| \geq c_1 2^m, U_{2^m} \leq 2^{m+1} \delta^{-1}] + P_{\mathbf{0}}^\omega \left[|\mathcal{M}_{\lfloor 2^{m+1} \delta^{-1} \rfloor}| \geq \frac{2^m}{c_*} \right]. \end{aligned} \tag{3.22}$$

Using Azuma’s inequality we bound the right hand side of (3.22) from above by $c_2 \exp\{-c_3 2^m\}$ for some deterministic constants $c_2, c_3 > 0$ independent of m .

The second term on the right hand side of (3.21) can be estimated by

$$\begin{aligned} c_4 2^{m(d-1)} P_{\mathbf{0}}^\omega [\tilde{U}_0 \circ \theta_{U_{2^m}} < U_{2^{m+1}} \circ \theta_{U_{2^m}}] &= c_4 2^{m(d-1)} \mathbf{M}_{\mathbf{0}}^\omega P_{X_{U_{2^m}}}^\omega [\tilde{U}_0 < U_{2^{m+1}}] \\ &\leq c_5 2^{m(d-1)} \exp\{-\gamma_1 2^m\} \end{aligned}$$

by virtue of Lemma 2. We have therefore shown that

$$P_{\mathbf{0}}^\omega [2^m \leq M_* < 2^{m+1}, D < +\infty] \leq c_6 \exp\{-c_7 2^m\} \tag{3.23}$$

for some deterministic constants $c_6, c_7 > 0$ independent of m and (3.20) follows. \square

LEMMA 4. –

$$\liminf_{m \uparrow +\infty} \mathbf{M}_{\mathbf{x}}^\omega \left[\frac{U_m}{m} \right] \leq \frac{1}{\delta}, \quad \text{for all } \mathbf{x} \in \mathbb{Z}^d, \mathbb{P}\text{-a.s.} \tag{3.24}$$

Proof. – We set $\mathbf{x} = \mathbf{0}$. Since $\mathcal{M}_n, n \geq 1$, is an (\mathcal{F}_n) -martingale with respect to P_0^ω for any $N \geq 1$ we obtain

$$0 = \mathbf{M}_0^\omega[\mathbf{1} \cdot \mathcal{M}_{N \wedge U_m}] \leq m + c_* - \delta \mathbf{M}_0^\omega(N \wedge U_m) \tag{3.25}$$

thus (3.24) follows. \square

LEMMA 5. –

$$P_{\mathbf{x}}^\omega[R_k < +\infty] \leq (1 - \gamma)^k, \quad \text{for all } k \geq 1, \mathbf{x} \in \mathbb{Z}^d, \mathbb{P}\text{-a.s.} \tag{3.26}$$

Proof. – It suffices to show (3.26) for $\mathbf{x} = \mathbf{0}$. We have

$$\begin{aligned} P_0^\omega[R_k < +\infty] &= P_0^\omega[S_k + D \circ \theta_{S_k} < +\infty] \\ &\stackrel{\text{strong Markov prop.}}{=} \sum_{\mathbf{x} \in \mathbb{Z}^d} P_0^\omega[S_k < +\infty, X_{S_k} = \mathbf{x}] P_{\mathbf{x}}^\omega[D < +\infty] \\ &\stackrel{\text{Lemma 1}}{\leq} (1 - \gamma) P_0^\omega[S_k < +\infty] \leq (1 - \gamma) P_0^\omega[R_{k-1} < +\infty] \end{aligned}$$

and the conclusion of the lemma follows by induction. \square

Let $K := \inf[k \geq 1, R_k = +\infty]$, or $K = +\infty$ if the set of which we take the infimum is empty. From the previous lemma and Lemma 3 we immediately conclude the following.

COROLLARY 1. – (1) $P_{\mathbf{x}}^\omega[K < +\infty] = 1$, for all $\mathbf{x} \in \mathbb{Z}^d, \mathbb{P}\text{-a.s.}$ and
 (2) $P_{\mathbf{x}}^\omega[S_K < +\infty] = 1$.

Proof. – Part (1) is an immediate application of Lemma 5 and the Borel–Cantelli lemma. To show part (2) note that

$$\begin{aligned} P_{\mathbf{x}}^\omega[S_K < +\infty] &= \sum_{k=1}^{+\infty} P_{\mathbf{x}}^\omega[R_{k-1} < +\infty, U_{M_{k-1}+r+1} \circ \theta_{R_{k-1}} < +\infty, K \circ \theta_{R_{k-1}} = 1] \\ &= \sum_{k,l=1}^{+\infty} \int_{\mathbb{R}} \mathbf{M}_{\mathbf{x}}^\omega[R_{k-1} = l, M_{k-1} \in [m, m + dm), \\ &\quad P_{X_l}^\omega[U_{m+r+1} < +\infty, K = 1]]. \end{aligned} \tag{3.27}$$

However, using (3.25), we can easily conclude that $P_{\mathbf{y}}^\omega[U_m < +\infty] = 1$ for all $\mathbf{y} \in \mathbb{Z}^d, m \in \mathbb{R}, \mathbb{P}$ a.s., hence the utmost right hand side of (3.27) equals

$$\begin{aligned} &\sum_{k,l=1}^{+\infty} \int_{\mathbb{R}} \mathbf{M}_{\mathbf{x}}^\omega[R_{k-1} = l, M_{k-1} \in [m, m + dm), P_{X_l}^\omega[K = 1]] \\ &= P_{\mathbf{x}}^\omega[K < +\infty] = 1. \quad \square \end{aligned}$$

Remark 1. – After [18] we define the first non-retraction time

$$\tau_1 := S_K < +\infty, \quad \mathbb{P}_0\text{-a.s.}$$

We note that the random variable τ_1 is not an (\mathcal{F}_n) -stopping time. The subsequent times of non-retraction $\tau_n, n \geq 2$, are defined by induction using the relation

$$\tau_{n+1} = \tau_n + \tau_1 \circ \theta_{\tau_n} \quad \text{for } n \geq 1. \tag{3.28}$$

In contrast to [18] these times are not renewal times, due to the possible dependence of sites on the lattice.

4. A transport operator

From condition (H3) we conclude that there exists a linear mapping $\mathcal{Z} : L^1(\Omega, \mathcal{G}, \mathbb{P}) \rightarrow L^1(\Omega \times \Omega, \mathcal{G}_0 \otimes \mathcal{R}, \mathbb{P} \otimes \mathbb{P})$ that satisfies

(Z1) $\iint \mathcal{Z}F(\omega, \omega') \mathbb{P}(d\omega) \mathbb{P}(d\omega') = \int F(\omega) \mathbb{P}(d\omega)$ for all $F \in L^1(\Omega, \mathcal{G}, \mathbb{P})$,

(Z2) $\mathcal{Z}F \geq 0$, when $F \geq 0$ and $\mathcal{Z}\mathbf{1} = \mathbf{1}$,

(Z3) $\mathcal{Z}(FG) = \mathcal{Z}(F)\mathcal{Z}(G)$ for any $F, G \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$,

(Z4) $\mathcal{Z}(F)(\omega, \omega') = F(\omega)$, $\mathcal{Z}(G)(\omega, \omega') = G(\omega')$ for any $F \in L^1(\Omega, \mathcal{G}_0, \mathbb{P})$, $G \in L^1(\Omega, \mathcal{R}, \mathbb{P})$ and $\mathcal{Z}(F)$ is $\mathcal{G}_0 \otimes \mathcal{R}^t$ -measurable provided that F is \mathcal{G}_t -measurable, for any $t \geq 0$,

(Z5) $\mathcal{Z}F(T_{\mathbf{x}}(\omega), T_{\mathbf{x}}(\omega')) = \mathcal{Z}U^{\mathbf{x}}F(\omega, \omega')$ for all $F \in L^1(\Omega, \mathcal{G}, \mathbb{P})$, $\mathbf{x} \in \mathbb{R}^d$. Here $U^{\mathbf{x}}F(\omega) := F(T_{\mathbf{x}}(\omega))$.

Let $A_k \in \mathcal{F}$ denote the event that $\min[X_0 \cdot \mathbf{1}, \dots, X_{S_k} \cdot \mathbf{1}] = X_0 \cdot \mathbf{1}$. For an arbitrary bounded F that is \mathcal{G}_0 -measurable we define

$$QF(\omega') := \sum_{\substack{k \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \int \mathcal{Z}_k(\mathbf{x}, \omega, T_{-\mathbf{x}}(\omega')) F(\omega) \mathbb{P}(d\omega), \tag{4.1}$$

where

$$\mathcal{Z}_k(\mathbf{x}, \omega, \omega') := \mathcal{Z}P_0^k[B_k(\mathbf{x})](\omega, \omega'), \tag{4.2}$$

with

$$B_k(\mathbf{x}) := [X_{S_k} = \mathbf{x}, S_k < +\infty, A_k]. \tag{4.3}$$

Notice that by virtue of (Z4), (3.4) and (G2) QF is \mathcal{G}_0 -measurable. Let

$$\mathbb{P}_D(d\omega) := P_0^\omega[D = +\infty] \mathbb{P}(d\omega) / \mathbb{P}_0[D = +\infty].$$

The following proposition holds.

PROPOSITION 2. –

$$\int QF(\omega') \mathbb{P}_D(d\omega') = \int F(\omega') \mathbb{P}_D(d\omega') \tag{4.4}$$

for any F bounded and \mathcal{G}_0 -measurable.

Proof. – Notice that the left hand side of (4.4) equals

$$\begin{aligned}
 & \frac{1}{\mathbb{P}_0[D = +\infty]} \sum_{\substack{k \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \iint \mathcal{Z}_k(\mathbf{x}, \omega, T_{-\mathbf{x}}(\omega')) F(\omega) P_0^{\omega'}[D = +\infty] \mathbb{P}(d\omega) \mathbb{P}(d\omega') \\
 &= \frac{1}{\mathbb{P}_0[D = +\infty]} \sum_{\substack{k \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \iint \mathcal{Z}_k(\mathbf{x}, \omega, \omega') F(\omega) P_0^{T_{\mathbf{x}}(\omega')} [D = +\infty] \mathbb{P}(d\omega) \mathbb{P}(d\omega') \\
 &= \frac{1}{\mathbb{P}_0[D = +\infty]} \sum_{\substack{k \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \int P_0^\omega [B_k(\mathbf{x})] F(\omega) P_{\mathbf{x}}^\omega [D = +\infty] \mathbb{P}(d\omega). \tag{4.5}
 \end{aligned}$$

The last equality in (4.5) follows from (Z1), (Z3) and (Z4) and the translation invariance of \mathbb{P} . The utmost right hand side of (4.5) equals

$$\frac{1}{\mathbb{P}_0[D = +\infty]} \int P_0^\omega [\tau_1 < +\infty, D = +\infty] F(\omega) \mathbb{P}(d\omega).$$

The equality (4.4) now follows from the fact that $\tau_1 < +\infty$ is a sure event in light of Corollary 1. \square

As a consequence of the above proposition we can extend Q to a density preserving linear operator $Q : L^1(\Omega, \mathcal{G}_0, \mathbb{P}_D) \rightarrow L^1(\Omega, \mathcal{G}_0, \mathbb{P}_D)$. We shall call it a *transport operator*.

H_* is an invariant density for Q if $H_* \geq 0$, $\int H_* d\mathbb{P}_D = 1$ and $QH_* = H_*$. The following proposition will be of crucial importance for us in the sequel.

PROPOSITION 3. – *There exists an $H_* > 0$, \mathbb{P}_D -a.s., that is a unique invariant density for Q .*

Proof. – For any $q, \rho > 0$ we denote by $\mathcal{C}(\rho, q)$ the set of all paths $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^d$ such that $\sigma(0) = \mathbf{0}$,

$$[\sigma(n + 1) - \sigma(n)] \cdot \mathbf{1} > \rho, \quad \text{for all } n = 0, \dots, U_{q+1}.$$

There exists a deterministic number s depending only on r and δ such that $U_{r+1}(\sigma) \leq s$ for all $\sigma \in \mathcal{C}(\delta/2, r)$. We introduce also

$$E := [\mathbf{y} \in \mathbb{Z}^d : \exists \sigma \in \mathcal{C}(\delta/2, r), \text{ such that } \mathbf{y} = \sigma(U_{r+1})].$$

Note that the set is of finite cardinality.

For any $\mathbf{x}_0 \in E$ we have

$$\begin{aligned}
 & P_0^\omega [X_{S_1} = \mathbf{x}_0, S_1 < +\infty, A_1] \\
 & \geq P_0^\omega [\sigma \in \mathcal{C}(\delta/2, r), \sigma(U_{r+1}) = \mathbf{x}_0] \\
 & = \sum_{\sigma \in \mathcal{C}(\delta/2, \mathbf{x}_0, \mathbf{1})} \prod_{i=0}^{U_{\mathbf{x}_0, \mathbf{1}}-1} p^\omega(\sigma(i), \sigma(i + 1) - \sigma(i)). \tag{4.6}
 \end{aligned}$$

Applying properties (Z2), (Z3) and (Z5) we get

$$\begin{aligned} & \sum_{\mathbf{x}_0 \in E} \mathcal{Z}_1(\mathbf{x}_0, \omega, T_{-\mathbf{x}_0}(\omega')) \\ & \geq \sum_{\mathbf{x}_0 \in E} \sum_{\sigma \in \mathcal{C}(\delta/2, \mathbf{x}_0, 1)} \prod_{i=0}^{U_{\mathbf{x}_0, 1}-1} \mathcal{Z} p^*(\sigma(i), \sigma(i+1) - \sigma(i))(\omega, T_{-\mathbf{x}_0}(\omega')). \end{aligned} \tag{4.7}$$

From the non-nestling condition (H2)

$$\sum_{\substack{e: 1 \geq \delta/2 \\ e \in \mathfrak{Y}_d}} p^\omega(\mathbf{y}, e) \geq \frac{\delta}{2c_*}, \quad \text{for all } \mathbf{y} \in \mathbb{Z}^d.$$

Thus, according to (Z2), we also have

$$\sum_{\substack{e: 1 \geq \delta/2 \\ e \in \mathfrak{Y}_d}} \mathcal{Z} p^*(\mathbf{y}, e)(\omega, \omega') \geq \frac{\delta}{2c_*}, \quad \text{for all } \mathbf{y} \in \mathbb{Z}^d, \mathbb{P} \otimes \mathbb{P}\text{-a.s.}$$

This and the fact that $U_{\mathbf{x}_0, 1} \leq s$ for some deterministic s imply that the expression on the right hand side of (4.7) is bounded from below by a deterministic constant $(\frac{\delta}{2c_*})^s$.

Hence, from (4.1) we conclude that for a nonnegative valued $F \in L^1(\Omega, \mathcal{G}_0, \mathbb{P}_D)$

$$QF(\omega') \geq \left(\frac{\delta}{2c_*}\right)^s \int F d\mathbb{P}.$$

The conclusion of the proposition holds upon an application of Theorem 5.6.2 of [7]. Moreover, by virtue of the aforementioned theorem

$$\lim_{n \uparrow +\infty} \int \left| Q^n F(\omega) - H_*(\omega) \int F d\mathbb{P}_D \right| \mathbb{P}_D(d\omega) = 0. \quad \square \tag{4.8}$$

We set $h_*(\omega, \sigma) := H_*(\omega) \mathbf{1}_{[D=+\infty]}(\sigma)$, $(\omega, \sigma) \in \Omega \times (\mathbb{Z}^d)^\mathbb{N}$. On the measurable space $(\Omega \times (\mathbb{Z}^d)^\mathbb{N}, \overline{\mathcal{B}(\Omega)} \otimes \mathcal{F})$ we define the following two probability measures $\tilde{\mathbb{P}}, \tilde{\mathbb{P}}_{h_*}$ by

$$\tilde{\mathbb{P}}(d\omega, d\sigma) := \mathbf{1}_{[D=+\infty]}(\sigma) P_0^\omega(d\sigma) \mathbb{P}(d\omega) / \mathbb{P}[D = +\infty]$$

and

$$\tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) := h_*(\omega, \sigma) P_0^\omega(d\sigma) \mathbb{P}(d\omega) / \mathbb{P}[D = +\infty].$$

The following theorem is the main result of this section.

THEOREM 2. – *The sequence $(\tau_{n+1} - \tau_n, X_{\tau_{n+1}} - X_{\tau_n})$, $n \geq 1$, is stationary and ergodic under $\tilde{\mathbb{P}}_{h_*}$, i.e. for any $n \geq 1$, bounded functions $F_1, \dots, F_n : \mathbb{Z}^d \times \mathbb{Z} \rightarrow \mathbb{R}$ and $k \geq 0$ we have*

$$\iint \prod_{p=1}^n F_p(X_{\tau_{p+1+k}} - X_{\tau_{p+k}}, \tau_{p+1+k} - \tau_{p+k}) \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma)$$

$$= \iint \prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p) \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) \tag{4.9}$$

and any bounded function $\mathcal{B}((\mathbb{Z}^d \times \mathbb{Z})^{\mathbb{N}})$ -measurable function F that satisfies

$$\iint |F((X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n)_{n \geq 2}) - F((X_{\tau_n} - X_{\tau_{n-1}}, \tau_n - \tau_{n-1})_{n \geq 2})| \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) = 0 \tag{4.10}$$

must be equal to a constant $\tilde{\mathbb{P}}_{h_*}$ -a.s., cf. [5, pp. 291–293].

Proof. – Stationarity. We only prove the stationarity for $k = 1$, the proof for a general k poses no additional difficulty. The left hand side of (4.9) equals

$$\begin{aligned} & \frac{1}{\mathbb{P}[D = +\infty]} \iint \prod_{p=1}^n F_p(X_{\tau_{p+2}} - X_{\tau_{p+1}}, \tau_{p+2} - \tau_{p+1}) h_*(\omega, \sigma) P_0^\omega(d\sigma) \mathbb{P}(d\omega) \\ &= \frac{1}{\mathbb{P}[D = +\infty]} \sum_{\substack{k \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \int \mathbf{M}_0^\omega \left[\prod_{p=1}^n F_p((X_{\tau_{p+1} \circ \theta_{S_k}} - X_{\tau_p \circ \theta_{S_k}}) \circ \theta_{S_k}, (\tau_{p+1} - \tau_p) \circ \theta_{S_k}), \right. \\ & \quad \left. B_k(\mathbf{x}), D \circ \theta_{S_k} = +\infty \right] H_*(\omega) \mathbb{P}(d\omega), \end{aligned} \tag{4.11}$$

with $B_k(\mathbf{x})$ as in (4.3). An application of the strong Markov property, yields the following expression for the right hand side of (4.11)

$$\begin{aligned} & \frac{1}{\mathbb{P}[D = +\infty]} \sum_{\substack{k \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \int P_0^\omega [B_k(\mathbf{x})] \\ & \quad \times \mathbf{M}_x^\omega \left[\prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p), D = +\infty \right] H_*(\omega) \mathbb{P}(d\omega). \end{aligned} \tag{4.12}$$

Using operator \mathcal{Z} we can rewrite (4.12) as being equal to

$$\begin{aligned} & \frac{1}{\mathbb{P}[D = +\infty]} \sum_{\substack{k \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \iint \mathcal{Z}_k(\mathbf{x}, \omega, \omega') \\ & \quad \times \mathbf{M}_x^{\omega'} \left[\prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p), D = +\infty \right] H_*(\omega) \mathbb{P}(d\omega) \mathbb{P}(d\omega') \\ &= \frac{1}{\mathbb{P}[D = +\infty]} \int \mathbf{M}_0^{\omega'} \left[\prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p), D = +\infty \right] \\ & \quad \times QH_*(\omega') \mathbb{P}(d\omega'), \end{aligned} \tag{4.13}$$

where $\mathcal{Z}_k(\mathbf{x}, \omega, \omega')$ is given by (4.2). From Proposition 3 we have $QH_* = H_*$ and in conclusion the right hand side of (4.13) equals

$$\begin{aligned} & \frac{1}{\mathbb{P}[D = +\infty]} \int \mathbf{M}_0^{\omega'} \left[\prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p), D = +\infty \right] H_*(\omega') \mathbb{P}(d\omega') \\ &= \iint \prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p) \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma). \end{aligned}$$

Ergodicity. We start with the following lemma.

LEMMA 6. – Suppose that $n \geq 1$, $F_1, \dots, F_n: \mathbb{Z}^d \times \mathbb{Z} \rightarrow \mathbb{R}$ are bounded, G is bounded and $\mathcal{B}((\mathbb{Z}^d \times \mathbb{Z})^{\mathbb{N}})$ -measurable and $R \in L^1(\Omega, \mathcal{G}_0, \mathbb{P}_D)$. Then, there exists $Y \in L^1(\Omega, \mathcal{G}_0, \mathbb{P}_D)$ such that

$$\int Y \mathbb{P}_D(d\omega) = \iint R(\omega) \prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p) \tilde{\mathbb{P}}(d\omega, d\sigma)$$

and for any $k \geq 1$

$$\begin{aligned} & \iint R(\omega) \prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p) \\ & \quad \times G((X_{\tau_{m+2+k}} - X_{\tau_{m+1+k}}, \tau_{m+2+k} - \tau_{m+1+k})_{m \geq n}) \tilde{\mathbb{P}}(d\omega, d\sigma) \\ &= \iint Q^{k-1} Y G((X_{\tau_{m+2}} - X_{\tau_{m+1}}, \tau_{m+2} - \tau_{m+1})_{m \geq 0}) \tilde{\mathbb{P}}(d\omega, d\sigma). \end{aligned} \tag{4.14}$$

Proof. – Calculation done in (4.11)–(4.13) shows that the left hand side of (4.14) equals

$$\begin{aligned} & \iint QR(\omega) \prod_{p=1}^n F_p(X_{\tau_p} - X_{\tau_{p-1}}, \tau_p - \tau_{p-1}) \\ & \quad \times G((X_{\tau_{m+1+k}} - X_{\tau_{m+k}}, \tau_{m+1+k} - \tau_{m+k})_{m \geq n}) \tilde{\mathbb{P}}(d\omega, d\sigma) \end{aligned} \tag{4.15}$$

with the convention $\tau_0 := 0$. The expression (4.15) can be further rewritten as

$$\begin{aligned} & \frac{1}{\mathbb{P}[D = +\infty]} \sum_{\substack{l \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \int QR(\omega) \mathbf{M}_0^\omega \left[F_1(\mathbf{x}, S_l) \prod_{p=2}^n F_p((X_{\tau_{p-1} \circ \theta_{S_l}} - X_{\tau_{p-2} \circ \theta_{S_l}}) \circ \theta_{S_l}, \right. \\ & \quad (\tau_{p-1} - \tau_{p-2}) \circ \theta_{S_l}) G(((X_{\tau_{m+k} \circ \theta_{S_l}} - X_{\tau_{m+k-1} \circ \theta_{S_l}}) \circ \theta_{S_l}, \\ & \quad \left. \tau_{m+k} \circ \theta_{S_l} - \tau_{m+k-1} \circ \theta_{S_l})_{m \geq n}), B_l(\mathbf{x}), D \circ \theta_{S_l} = +\infty \right] \mathbb{P}(d\omega). \end{aligned} \tag{4.16}$$

Repeating the strong Markov property argument from the proof of stationarity we conclude that the right hand side of (4.15) equals

$$\begin{aligned} & \iint R_1(\omega) \prod_{p=2}^n F_p(X_{\tau_{p-1}} - X_{\tau_{p-2}}, \tau_{p-1} - \tau_{p-2}) \\ & \quad \times G((X_{\tau_{m+k}} - X_{\tau_{m+k-1}}, \tau_{m+k} - \tau_{m+k-1})_{m \geq n}) \tilde{\mathbb{P}}(d\omega, d\sigma), \end{aligned} \tag{4.17}$$

with

$$R_1(\omega') := \sum_{\substack{l \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \int Q R(\omega) \mathcal{Z} \{ M_0 [F_1(\mathbf{x}, S_l), B_l(\mathbf{x})] \} (\omega, T_{-\mathbf{x}} \omega') \mathbb{P}(d\omega)$$

\mathcal{G}_0 -measurable and

$$\int R_1 d\mathbb{P}_D = \iint R(\omega) F_1(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1) \tilde{\mathbb{P}}(d\omega, d\sigma).$$

Proceeding $(n - 1)$ -times analogously as in (4.15)–(4.17) we eventually obtain R_n that is \mathcal{G}_0 -measurable and satisfies

$$\int R_n \mathbb{P}_D(d\omega) = \iint R(\omega) \prod_{p=1}^n F_p(X_{\tau_{p+1}} - X_{\tau_p}, \tau_{p+1} - \tau_p) \tilde{\mathbb{P}}(d\omega, d\sigma).$$

The right hand side of (4.14) equals

$$\iint R_n(\omega) G((X_{\tau_{m+1+k}} - X_{\tau_{m+k}}, \tau_{m+1+k} - \tau_{m+k})_{m \geq 1}) \tilde{\mathbb{P}}(d\omega, d\sigma). \tag{4.18}$$

Thus, repeating once more the calculation done in (4.11)–(4.13), we deduce (4.14) with $Y := R_n$. \square

We proceed now with the proof of ergodicity. Suppose that F is such a function for which (4.10) holds. Let

$$F_n : \underbrace{\mathbb{Z}^d \times \mathbb{Z} \times \dots \times \mathbb{Z}^d \times \mathbb{Z}}_{n\text{-times}} \rightarrow \mathbb{R}$$

be bounded and such that

$$\lim_{n \uparrow +\infty} \iint |F_n(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1, \dots, X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n) - Z| \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) = 0. \tag{4.19}$$

Here $Z := F((X_{\tau_n} - X_{\tau_{n-1}}, \tau_n - \tau_{n-1})_{n \geq 2})$. By virtue of Lemma 6 and formula (4.10) we conclude that there exists a bounded random variable Y_n that is \mathcal{G}_0 -measurable such that

$$\begin{aligned} & \iint F_n(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1, \dots, X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n) Z \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) \\ &= \iint H_*(\omega) \mathbf{1}_{[D=+\infty]}(\sigma) F_n(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1, \dots, X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n) \\ & \quad \times F((X_{\tau_{m+k+2}} - X_{\tau_{m+k+1}}, \tau_{m+k+2} - \tau_{m+k+1})_{m \geq n}) \mathbb{P}(d\omega) \mathbf{P}_0(d\sigma) \\ &= \iint Q^{k-1} Y_n Z \tilde{\mathbb{P}}(d\omega, d\sigma). \end{aligned} \tag{4.20}$$

According to (4.8) the utmost right hand side of (4.20) tends, upon the subsequent application of limits when $k \uparrow +\infty$ and $n \uparrow +\infty$, to

$$\left[\iint Z \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) \right]^2.$$

In conclusion we obtain that

$$\iint Z^2 \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) = \left[\iint Z \tilde{\mathbb{P}}_{h_*}(d\omega, d\sigma) \right]^2,$$

which shows that $Z \equiv \text{const}$, $\tilde{\mathbb{P}}_{h_*}$ -a.s. \square

5. The proof of Theorem 1

The proof is a modification of the argument used to prove Theorem 2.3 of [18]. We start with the following.

PROPOSITION 4. – *We have*

$$\iint \tau_1 d\tilde{\mathbb{P}}_{h_*} < +\infty \tag{5.1}$$

and

$$\iint |X_{\tau_1}| d\tilde{\mathbb{P}}_{h_*} < +\infty. \tag{5.2}$$

Proof. – Since $|X_{\tau_1}| \leq \tau_1$ it suffices to show (5.1). First we show the following.

LEMMA 7. –

$$w_* := \iint \mathbf{1} \cdot X_{\tau_1} d\tilde{\mathbb{P}}_{h_*} < +\infty. \tag{5.3}$$

Proof. – Repeating word by word the argument of the proof of Proposition 1.6 cf. formula (1.33) of [18], we conclude that

$$\mathbf{1} \cdot X_{\tau_1} \leq c_* + \sum_{1 \leq k' < K} (c_* + M_{k'} - \mathbf{1} \cdot X_{S_{k'}})$$

with K as defined in Corollary 1 and c_* given by (3.15). Hence

$$\begin{aligned} & \iint \mathbf{1} \cdot X_{\tau_1} d\tilde{\mathbb{P}}_{h_*} \\ & \leq c_* + \sum_{1 \leq k' < k} \iint (c_* + M_{k'} - \mathbf{1} \cdot X_{S_{k'}}) \mathbf{1}_{[S_k < +\infty, D \circ \theta_{S_k} = +\infty]} d\tilde{\mathbb{P}}_{h_*} \\ & \leq c_* + \sum_{1 \leq k' < k} \iint (c_* + M_{k'} - \mathbf{1} \cdot X_{S_{k'}}) \mathbf{1}_{[R_{k-1} < +\infty, D \circ \theta_{S_k} = +\infty]} d\tilde{\mathbb{P}}_{h_*}. \end{aligned} \tag{5.4}$$

Since $R_{k-1} = D \circ \theta_{S_{k-1}} + S_{k-1}$ we obtain, upon a multiple application of strong Markov property for random walks and (3.5) that the right hand side of (5.4) is less than or equal to

$$\begin{aligned}
 & c_* + \sum_{1 \leq k' < k} (1 - \gamma)^{k-1-k'} \iint (c_* + M_{k'} - \mathbf{1} \cdot X_{S_{k'}}) \mathbf{1}_{[R_{k'} < +\infty]} d\tilde{\mathbb{P}}_{h_*} \\
 & \leq c_* + \frac{1}{\mathbb{P}_0[D = +\infty]} \sum_{\substack{1 \leq k' < k \\ \mathbf{x} \in \mathbb{Z}^d}} (1 - \gamma)^{k-1-k'} \\
 & \quad \times \int H_*(\omega) \mathbf{M}_0^\omega [X_{S_{k'}} = \mathbf{x}, S_{k'} < +\infty, \mathbf{M}_\mathbf{x}^\omega [c_* + M_*, D < +\infty]] \mathbb{P}(d\omega). \tag{5.5}
 \end{aligned}$$

By virtue of Lemma 3 we conclude that the right hand side of (5.5) is less than or equal to

$$\begin{aligned}
 & c_* + \frac{c_* + \gamma_3}{\mathbb{P}_0[D = +\infty]} \sum_{1 \leq k' < k} (1 - \gamma)^{k-1-k'} \int H_*(\omega) P_0^\omega [S_{k'} < +\infty] \mathbb{P}(d\omega) \\
 & \stackrel{\text{Lemma 5}}{\leq} c_* + C \sum_{1 \leq k} k(1 - \gamma)^{k-1} < +\infty,
 \end{aligned}$$

for some constant $C > 0$, and (5.3) follows. \square

Let $(k_m)_{m \geq 1}$ be a random sequence of integers defined by

$$\tau_{k_m} \leq U_m < \tau_{k_m+1}. \tag{5.6}$$

Recall the convention that $\tau_0 := 0$. Then $\tilde{\mathbb{P}}_{h_*}$ -a.s. we have

$$\mathbf{1} \cdot X_{\tau_{k_m}} \leq \mathbf{1} \cdot X_{U_m} < \mathbf{1} \cdot X_{\tau_{k_m+1}}$$

and $|\mathbf{1} \cdot X_{U_m} - m| \leq c_*$, with c_* given by (3.15). By virtue of the individual ergodic theorem we conclude that

$$\lim_{k \uparrow +\infty} \frac{\mathbf{1} \cdot X_{\tau_k}}{k} = w_*, \quad \tilde{\mathbb{P}}_{h_*}\text{-a.s.}$$

But

$$\frac{\mathbf{1} \cdot X_{\tau_{k_m}}}{k_m} \leq \frac{\mathbf{1} \cdot X_{U_m}}{k_m} < \frac{\mathbf{1} \cdot X_{\tau_{k_m+1}}}{k_m}$$

therefore

$$\lim_{m \uparrow +\infty} \frac{k_m}{m} = \frac{1}{w_*}, \quad \tilde{\mathbb{P}}_{h_*}\text{-a.s.}$$

On the other hand

$$\frac{U_m}{m} \geq \frac{\tau_{k_m}}{k_m} \cdot \frac{k_m}{m}, \tag{5.7}$$

thus, in light of Lemma 4 and the ergodic theorem applied to the sequence

$$\left(\frac{\sum_{n=1}^N (\tau_{n+1} - \tau_n)}{N} \right)_{N \geq 1}$$

we obtain that the right hand side of (5.7) tends $\tilde{\mathbb{P}}_{h_*}$ -a.s. to

$$\frac{t}{w_*} > 0, \tag{5.8}$$

with

$$0 < t := \iint \tau_1 d\tilde{\mathbb{P}}_{h_*} \leq +\infty.$$

On the other hand however from (3.24)

$$\mathbf{M}_0^\omega \left[\liminf_{m \uparrow +\infty} \frac{U_m}{m} \right] \leq \frac{1}{\delta},$$

which, in turn, proves that $t < +\infty$ and (5.1) follows. \square

To finish the proof of Theorem 1 we consider a non-decreasing sequence $(l_n)_{n \geq 1}$, that tends to $+\infty$ $\tilde{\mathbb{P}}_{h_*}$ -a.s., defined by

$$\tau_{l_n} \leq n < \tau_{l_n+1}.$$

We have

$$\lim_{n \uparrow +\infty} \frac{n}{l_n} = t, \quad \tilde{\mathbb{P}}_{h_*}\text{-a.s.}$$

Writing

$$\frac{X_n}{n} = \frac{X_{\tau_{l_n}}}{l_n} \cdot \frac{l_n}{n} + \frac{X_n - X_{\tau_{l_n}}}{n}$$

we conclude, by virtue of the ergodic theorem, that

$$\lim_{n \uparrow +\infty} \frac{X_n}{n} = \frac{\iint X_{\tau_1} d\tilde{\mathbb{P}}_{h_*}}{t} =: \mathbf{v}, \quad \tilde{\mathbb{P}}_{h_*}\text{-a.s.}$$

Due to (5.8), we have also $\mathbf{v} \cdot \mathbf{1} > 0$.

We shall show now that the convergence holds in fact \mathbb{P}_0 -a.s. Let

$$A := \left[(\mathbf{x}_n, t_n)_{n \geq 1} \in (\mathbb{Z}^d \times \mathbb{Z})^{\mathbb{N}} : \frac{\sum_{m=1}^n \mathbf{x}_m}{\sum_{m=1}^n t_m} \not\rightarrow \mathbf{v}, \text{ or } \frac{\sum_{m=1}^n t_m}{n} \not\rightarrow t \text{ as } n \uparrow +\infty \right].$$

We know that

$$\int \mathbf{1}_A ((X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n)_{n \geq 1}) d\tilde{\mathbb{P}}_{h_*} = 0. \tag{5.9}$$

However, repeating the calculation made in (4.14) and (4.15) we obtain

$$\begin{aligned} & \int \mathbf{1}_A((X_{\tau_{n+2}} - X_{\tau_{n+1}}, \tau_{n+2} - \tau_{n+1})_{n \geq 1}) d\mathbb{P}_0 \\ &= \int H(\omega') \mathbf{M}_0^{\omega'} [\mathbf{1}_A((X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1} - \tau_n)_{n \geq 1}), D = +\infty] \mathbb{P}(d\omega') \end{aligned} \quad (5.10)$$

with

$$H(\omega') := \int \sum_{\substack{l \geq 1 \\ \mathbf{x} \in \mathbb{Z}^d}} \mathcal{Z} P_0[S_l < +\infty, X_{S_l} = \mathbf{x}](\omega, T_{-\mathbf{x}}(\omega')) d\mathbb{P}(\omega).$$

Taking into account (5.9) we conclude that the right hand side of (5.10) vanishes and therefore

$$\lim_{n \uparrow +\infty} \frac{X_{\tau_n}}{\tau_n} = \mathbf{v} \quad \text{and} \quad \lim_{n \uparrow +\infty} \frac{\tau_n}{n} = t, \quad \mathbb{P}_0\text{-a.s.} \quad \square$$

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Appendix A. The proof of Proposition 1

Let us assume that $\mathbb{P}[\cdot | \mathcal{G}_0]$ possesses atoms in \mathcal{G}_t on a set of a positive measure. For a certain $N \geq 1$ there exists then a set $L \in \mathcal{G}_0$ of positive measure such that the conditional distribuant $F_{\xi_t}(x | \xi_0(\omega)) := \Phi_{\xi_t}[\xi_t < x | \xi_0(\omega)]$, $x \in \mathbb{R}$, has a jump of size at least $1/N$ in the interval $[-N, N]$ for \mathbb{P} -a.s. $\omega \in L$. Let $f(\omega)$ be the minimum of such jump sites in $[-N, N]$ for a fixed ω . Then $A := [\omega \in L: \xi_t(\omega) = f(\omega)]$ is an atom with $\mathbb{P}[A | \mathcal{G}_0](\omega) \geq 1/N$ for \mathbb{P} -a.s. $\omega \in L$.

Let $\mathcal{G}_t^{(k)}$ be the sub σ -algebra of \mathcal{G}_t generated by the sites $|\mathbf{x}| \leq k$. Notice that, thanks to assumption (A), $\mathbb{P}[\cdot | \mathcal{G}_0^{(k)}]$ defined on $\mathcal{G}_t^{(m)}$ are atomless for any $k, m \geq 1$, \mathbb{P} -a.s. Let \mathcal{A}_k be a countable algebra of sets such that $\mathcal{G}_t^{(k)} = \mathcal{A}_k \vee \mathcal{N}$ and $\mathcal{A} := \bigvee_{k=1}^{+\infty} \mathcal{A}_k \vee \{A, A^c\}$, with $A^c := \Omega \setminus A$.

According to the martingale almost sure convergence theorem we have

$$\lim_{k \uparrow +\infty} \mathbb{P}[C | \mathcal{G}_0^{(k)}] = \mathbb{P}[C | \mathcal{G}_0] \quad \text{for all } C \in \mathcal{A} \text{ } \mathbb{P}\text{-a.s.}$$

Therefore for some \mathcal{G}_0 -measurable set $\tilde{L} \subset L$ of positive measure and sufficiently large k_0 , $\mathbb{P}[A | \mathcal{G}_0^{(k)}](\omega) \geq \frac{1}{2N}$ for \mathbb{P} -a.s. $\omega \in \tilde{L}$ and $k \geq k_0$.

For arbitrary $\varepsilon > 0$ we can find $\tilde{A} \in \mathcal{G}_t^{(m)}$ such that $\mathbb{P}[A \Delta \tilde{A}] \leq \varepsilon$ and we have

$$\mathbb{P}[A \Delta \tilde{A}] = \int \mathbb{P}[A \Delta \tilde{A} | \mathcal{G}_0^{(k)}] d\mathbb{P} \geq \int \liminf_{k \uparrow +\infty} \mathbb{P}[A \Delta \tilde{A} | \mathcal{G}_0^{(k)}] d\mathbb{P}.$$

With a suitable choice of ε (e.g. $\varepsilon = 1/(1000N)$)

$$\mathbb{P}[A \Delta \tilde{A} | \mathcal{G}_0^{(k_l)}](\omega) \leq \frac{1}{128N},$$

for ω from some \mathcal{G}_0 -measurable set $\widehat{L} \subset \widetilde{L}$ and some sequence of positive integers $(k_l)_{l \geq 1}$.

Since $\mathbb{P}[\cdot | \mathcal{G}_0^{(k_l)}](\omega)$ are atomless on $\mathcal{G}_t^{(m)}$, $m \geq 1$, one can find $B^{(1)}, B^{(2)} \in \mathcal{G}_t^{(m)}$ such that $B^{(1)} \cap B^{(2)} = \emptyset$, $B^{(1)}, B^{(2)} \subseteq \widetilde{A}$ and

$$\frac{1}{16N} < \mathbb{P}[B^{(i)} | \mathcal{G}_0^{(k_l)}](\omega) < \frac{1}{8N}, \quad i = 1, 2,$$

for some $k_l \geq k_0$. Then a.s. in Ω

$$\mathbb{P}[B^{(i)} | \mathcal{G}_0](\omega) = \mathbb{E}[\mathbb{P}[B^{(i)} | \mathcal{G}_0^{(k_l)}] | \mathcal{G}_0](\omega) \geq \mathbb{E}[\mathbb{P}[B^{(i)} | \mathcal{G}_0^{(k_l)}] \mathbf{1}_{\widehat{L}} | \mathcal{G}_0](\omega) \geq \frac{1}{16N} \mathbf{1}_{\widehat{L}}(\omega).$$

One can easily check that $A_i := B^{(i)} \cap A$, $i = 1, 2$, satisfy $\mathbb{P}[A_i | \mathcal{G}_0](\omega) \geq 1/(32N)$ and $A_1 \cap A_2 = \emptyset$, which contradicts the fact that A is an atom.

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