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TRUNCATED MICROSUPPORT AND HOLOMORPHIC SOLUTIONS OF D-MODULES

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ABSTRACT. – We study the truncated microsupport SS_k of sheaves on a real manifold. Applying our results to the case of $F = R\mathcal{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$, the complex of holomorphic solutions of a coherent $\mathscr{D}$-module $\mathscr{M}$, we show that $\mathrm{SS}_k(F)$ is completely determined by the characteristic variety of $\mathscr{M}$. As an application, we obtain an extension theorem for the sections of $H^j(F)$, $j < d$, defined on an open subset whose boundary is non-characteristic outside of a complex analytic subvariety of codimension d. We also give a characterization of the perversity for $\mathbb{C}$-constructible sheaves in terms of their truncated microsupports.

RÉSUMÉ. – Nous étudions le micro-support tronqué SS_k des faisceaux sur une variété réelle. Appliquant nos résultats au cas du complexe $F = R\mathcal{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$ des solutions holomorphes d'un $\mathscr{D}$-module cohérent $\mathscr{M}$, nous montrons que $\mathrm{SS}_k(F)$ est complètement determiné par la variété caractéristique de $\mathscr{M}$.
Comme application, nous obtenons un théorème d'extension pour les sections de $H^j(F)$, $j < d$, definies sur un ouvert dont la frontière est non caractéristique en dehors d'un ensemble analytique complexe de codimension d. Nous donnons aussi une caractérisation de la perversité des faisceaux $\mathbb{C}$-constructibles en terme de leurs micro-supports tronqués.

1. Introduction

The notion of microsupport of sheaves was introduced in the course of the study of the theory of linear partial differential equations (LPDE), and it is now applied in various domains of mathematics. References are made to [8].

For an object F of the derived category of abelian sheaves on a real manifold X, its microsupport $\mathrm{SS}(F)$ is a closed conic subset of the cotangent bundle $\pi : T^*X \to X$ which describes the direction of "non-propagation" of F. In particular, for a smooth closed submanifold Y of X, the support $\mathrm{Supp}(\mu_Y(F))$ of the Sato microlocalization $\mu_Y(F)$ of F along Y is contained in $\mathrm{SS}(F) \cap T^*_Y X$, where $T^*_Y X$ denotes the conormal bundle to Y in X.

If X is a complex manifold and $\mathscr{M}$ is a system of LPDE, that is, a coherent module over the sheaf $\mathscr{D}_X$ of holomorphic differential operators, the complex F of holomorphic solutions of this system is given by $R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$, and the microsupport of F is then the characteristic variety $\mathrm{Ch}(\mathscr{M})$ of $\mathscr{M}$.

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However some phenomena of propagation may happen in specific degrees, related to the principle of unique continuation for holomorphic functions, and this leads to a variant of the notion of microsupport, that of “truncated microsupport”, a notion introduced by the authors of [8] but never published.

Following a suggestion of these authors, Tonin [12] was able to regain in the language of the truncated microsupport a result of Ebenfelt, Khavinson and Shapiro [2,3]. In their papers, these authors obtained the extension of holomorphic solutions of a differential operator when the solutions are defined on an open subset with smooth boundary non-characteristic outside a smooth complex hypersurface. Note that the problem of extending holomorphic solutions across non-characteristic real hypersurfaces plays a crucial role in the theory of LPDE, and was initiated by Leray [9], followed by [13,1,5] (see also [4] and [10] for an exposition of these results).

The truncated microsupport is defined as follows. Let $\mathbf{k}$ be a field, let X be a real manifold and let $F \in D^b(\mathbf{k}_X)$ be an object of the derived category of sheaves of $\mathbf{k}$-vector spaces on X. For an integer $k \in \mathbb{Z}$, a point $p \in T^*X$ does not belong to the truncated microsupport $\mathrm{SS}_k(F)$ if and only if F is microlocally at p isomorphic to an object of $D^{>k}(\mathbf{k}_X)$. Hence $\{\mathrm{SS}_k(F)\}_{k\in\mathbb{Z}}$ is an increasing sequence, while $\mathrm{SS}_k(F)$ is an empty set for $k \ll 0$ and $\mathrm{SS}_k(F)$ coincides with $\mathrm{SS}(F)$ for $k \gg 0$.

In this paper, we give equivalent definitions of the truncated microsupport and we study its behavior under exterior tensor product, smooth inverse image and proper direct image. We introduce then the closed subset $\mathrm{SS}_k^Y(F)$ of $\pi^{-1}(Y)$ which describes the support of the microlocalization of F along submanifolds of Y and prove (see Theorem 5.3):

$$\mathrm{SS}_k(F) = \overline{\mathrm{SS}_k(F) \setminus \pi^{-1}(Y)} \cup \mathrm{SS}_k^Y(F).$$

We then apply this result to the complex $F = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ of holomorphic solutions of a coherent $\mathscr{D}_X$-module $\mathscr{M}$ on a complex manifold X.

Let S be a closed complex analytic subset of codimension greater than or equal to d and let S' be a closed complex analytic subset of S of codimension greater than d such that $S_0 := S \setminus S'$ is a smooth submanifold of codimension d. We first prove the estimate below (see Proposition 6.2):

$$\begin{aligned}\mathrm{SS}_{d-1}(F) &= \overline{\mathrm{SS}_{d-1}(F) \setminus \pi^{-1}(S)},\\ \mathrm{SS}_d(F) &= \overline{\mathrm{SS}_d(F) \setminus \pi^{-1}(S)} \cup \overline{\mathrm{SS}_d(F) \cap T^*_{S_0}X}.\end{aligned}$$

In particular, if $j : \Omega \hookrightarrow X$ is the embedding of a pseudo-convex open subset with smooth boundary $\partial\Omega$, and if $\partial\Omega$ is transversal to S (i.e. $T^*_{\partial\Omega}X \cap \overline{T^*_{S_0}X} \subset T^*_X X$) and non-characteristic for $\mathscr{M}$ outside of S, one has

$$\mathcal{E}xt^j_{\mathscr{D}_X}\big(\mathscr{M}, \mathscr{O}_X^+/\mathscr{O}_X\big) = 0 \quad \text{for any } j < d,$$

where $\mathscr{O}_X^+/\mathscr{O}_X = j_* j^{-1}\mathscr{O}_X/\mathscr{O}_X$.

Next we calculate $\mathrm{SS}_k(F)$ in terms of the characteristic variety of $\mathscr{M}$ (see Theorem 6.7). Letting $\mathrm{Ch}(\mathscr{M}) = \bigcup_{\alpha\in A} V_\alpha$ be the decomposition of $\mathrm{Ch}(\mathscr{M})$ into irreducible components, one has

$$\mathrm{SS}_k(F) = \Bigg(\bigcup_{\operatorname{codim}\pi(V_\alpha)<k} V_\alpha\Bigg) \cup \Bigg(\bigcup_{\operatorname{codim}\pi(V_\alpha)=k} T^*_{\pi(V_\alpha)}X\Bigg). \tag{1.1}$$

In particular, if F is a perverse sheaf (i.e., $\mathscr{M}$ is holonomic), then letting $\mathrm{SS}(F)=\bigcup_{\alpha\in A}\Lambda_\alpha$ be the decomposition into irreducible components, one has

$$\mathrm{SS}_k(F)=\bigcup_{\operatorname{codim}\pi(\Lambda_\alpha)\leqslant k}\Lambda_\alpha.$$

Conversely if $F\in D^b(\mathbb{C}_X)$ is $\mathbb{C}$-constructible and if it satisfies

$$\mathrm{SS}_k(F)\cup\mathrm{SS}_k\big(R\mathcal{H}om(F,\mathbb{C}_X)\big)\subset\bigcup_{\operatorname{codim}\pi(\Lambda_\alpha)\leqslant k}\Lambda_\alpha\quad\text{for every } k,$$

then F is a perverse sheaf.

2. Notations and review

We will mainly follow the notations in [8].

Let X be a real analytic manifold. We denote by $\tau: TX\to X$ the tangent bundle to X and by $\pi: T^*X\to X$ the cotangent bundle. We identify X with the zero section of T^*X and set $\dot{T}^*X=T^*X\setminus X$. We denote by $\dot\pi:\dot{T}^*X\to X$ the restriction of π to $\dot{T}^*X$. For a smooth submanifold Y of X, T_YX denotes the normal bundle to Y and T^*_YX the conormal bundle. In particular, T^*_XX is identified with X.

For a submanifold Y of X and a subset S of X, we denote by $C_Y(S)$ the normal cone to S along Y, a closed conic subset of T_YX.

For a morphism $f: X\to Y$ of real manifolds, we denote by

$$f_\pi: X\times_Y T^*Y\to T^*Y\quad\text{and}\quad f_d: X\times_Y T^*Y\to T^*X$$

the associated morphisms.

For a subset A of T^*X, we denote by A^a the image of A by the antipodal map

$$a:(x;\xi)\mapsto(x;-\xi).$$

The closure of A is denoted by $\overline{A}$. For a cone $\gamma\subset TX$, the polar cone γ° to γ is the convex cone in T^*X defined by

$$\gamma^\circ=\big\{(x;\xi)\in T^*X;\ x\in\pi(\gamma)\text{ and }\langle v,\xi\rangle\geqslant 0\text{ for any }(x;v)\in\gamma\big\}.$$

Let $\mathbf{k}$ be a field. We denote by $D(\mathbf{k}_X)$ the derived category of complexes of sheaves of $\mathbf{k}$-vector spaces on X, and by $D^b(\mathbf{k}_X)$ the full subcategory of $D(\mathbf{k}_X)$ consisting of complexes with bounded cohomologies.

For $k\in\mathbb{Z}$, we denote as usual by $D^{\geqslant k}(\mathbf{k}_X)$ (resp. $D^{\leqslant k}(\mathbf{k}_X)$) the full additive subcategory of $D^b(\mathbf{k}_X)$ consisting of objects F satisfying $H^j(F)=0$ for any $j<k$ (resp. $H^j(F)=0$ for any $j>k$). The category $D^{\geqslant k+1}(\mathbf{k}_X)$ is sometimes denoted by $D^{>k}(\mathbf{k}_X)$.

We denote by $\tau^{\leqslant k}: D(\mathbf{k}_X)\to D^{\leqslant k}(\mathbf{k}_X)$ the truncation functor. Recall that for $F\in D(\mathbf{k}_X)$ the morphism $\tau^{\leqslant k}F\to F$ induces isomorphisms $H^j(\tau^{\leqslant k}F)\xrightarrow{\sim}H^j(F)$ for $j\leqslant k$ and $H^j(\tau^{\leqslant k}F)=0$ for $j>k$.

If F is an object of $D^b(\mathbf{k}_X)$, $\mathrm{SS}(F)$ denotes its microsupport, a closed $\mathbb{R}^+$-conic involutive subset of T^*X. For $p\in T^*X$, $D^b(\mathbf{k}_X;p)$ denotes the localization of $D^b(\mathbf{k}_X)$ by the full triangulated subcategory consisting of objects F such that $p\notin\mathrm{SS}(F)$.

If Y is a submanifold, $\mu_Y(F)$ denotes the Sato microlocalization of F along Y. Recall that $\mu_Y(F)\in D^b(\mathbf{k}_{T^*_Y X})$ and

$$H^j\big(\mu_Y(F)\big)_p \simeq \varinjlim_{Z} H^j_Z(F)_{\pi(p)} \quad \text{for } p\in T^*_Y X \text{ and } j\in\mathbb{Z}, \tag{2.1}$$

where Z runs through the family of closed subsets of X such that

$$C_Y(Z)_{\pi(p)}\setminus\{0\}\subset\big\{v\in(T_Y X)_{\pi(p)};\ \langle v,p\rangle>0\big\}. \tag{2.2}$$

On a complex manifold X, we consider the sheaf $\mathscr{O}_X$ of holomorphic functions and the sheaf $\mathscr{D}_X$ of linear holomorphic differential operators of finite order. Concerning the theory of $\mathscr{D}$-modules, references are made to [6].

3. Truncated microsupport

We shall give here several equivalent definitions of the truncated microsupport.

For a closed cone $\gamma\subset\mathbb{R}^n$, one sets

$$Z_\gamma := \big\{(x;y)\in\mathbb{R}^n\times\mathbb{R}^n;\ x-y\in\gamma\big\}.$$

Let $q_1,\, q_2:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be the first and the second projections.

One defines the integral transform with kernel $\mathbf{k}_{Z_\gamma}$,

$$\mathbf{k}_{Z_\gamma}\circ : D^b(\mathbf{k}_{\mathbb{R}^n})\to D^b(\mathbf{k}_{\mathbb{R}^n}),\qquad \mathbf{k}_{Z_\gamma}\circ G = Rq_{1!}\big(\mathbf{k}_{Z_\gamma}\otimes q_2^{-1}G\big).$$

If G has compact support, one has the following formula for the stalk of $\mathbf{k}_{Z_\gamma}\circ G$ at $x\in\mathbb{R}^n$:

$$(\mathbf{k}_{Z_\gamma}\circ G)_x \simeq R\Gamma\big(\mathbb{R}^n;\ \mathbf{k}_{x+\gamma^a}\otimes G\big).$$

Recall that a closed convex cone γ is called *proper* if $0\in\gamma$ and $\mathrm{Int}(\gamma^\circ)\neq\emptyset$.

For $(x_0;\xi_0)\in\mathbb{R}^n\times(\mathbb{R}^n)^*$ and $\varepsilon\in\mathbb{R}$ we set:

$$H_\varepsilon(x_0,\xi_0)=\big\{x\in\mathbb{R}^n;\ \langle x-x_0,\xi_0\rangle>-\varepsilon\big\}.$$

If there is no risk of confusion, we write H_ε instead of $H_\varepsilon(x_0,\xi_0)$ for short. The following result is proved in [8].

LEMMA 3.1. – *Let X be an open subset of $\mathbb{R}^n$ and let $G\in D^b(\mathbf{k}_X)$. Let $p=(x_0;\xi_0)\in T^*X$. Then $p\notin \mathrm{SS}(G)$ if and only if there exist an open neighborhood W of x_0, a proper closed convex cone γ and $\varepsilon>0$ such that $\xi_0\in\mathrm{Int}(\gamma^\circ)$, $(W+\gamma^a)\cap\overline{H_\varepsilon}\subset X$ and*

$$H^j(X;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon}\otimes G)=0 \quad \textit{for any } j\in\mathbb{Z} \textit{ and } x\in W.$$

We shall give a similar version of the above lemma for the truncated microsupport.

PROPOSITION 3.2. – *Let X be a real analytic manifold and let $p\in T^*X$. Let $F\in D^b(\mathbf{k}_X)$ and $k\in\mathbb{Z}$, $\alpha\in\mathbb{Z}_{\geqslant 1}\cup\{\infty,\omega\}$. Then the following conditions are equivalent*:

(i)$_k$ *There exist $F'\in D^{>k}(\mathbf{k}_X)$ and an isomorphism $F\simeq F'$ in $D^b(\mathbf{k}_X;p)$.*

(ii)$_k$ *There exist* $F' \in D^{>k}(\mathbf{k}_X)$ *and a morphism* $F' \to F$ *in* $D^b(\mathbf{k}_X)$ *which is an isomorphism in* $D^b(\mathbf{k}_X; p)$.

(iii)$_{k,\alpha}$ *There exists an open conic neighborhood* U *of* p *such that for any* $x \in \pi(U)$ *and for any* $\mathbb{R}$*-valued* C^α*-function* φ *defined on a neighborhood of* x *such that* $\varphi(x) = 0$, $d\varphi(x) \in U$, *one has*

$$H^j_{\{\varphi \geqslant 0\}}(F)_x = 0 \quad \textit{for any } j \leqslant k. \tag{3.1}$$

When X *is an open subset of* $\mathbb{R}^n$ *and* $p = (x_0; \xi_0)$, *the above conditions are also equivalent to*

(iv)$_k$ *There exist a proper closed convex cone* $\gamma \subset \mathbb{R}^n$, $\varepsilon > 0$ *and an open neighborhood* W *of* x_0 *with* $\xi_0 \in \operatorname{Int}(\gamma^\circ)$ *such that* $(W + \gamma^a) \cap \overline{H_\varepsilon} \subset X$ *and*

$$H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes F) = 0 \quad \textit{for any } j \leqslant k \textit{ and } x \in W. \tag{3.2}$$

Proof. – We may assume $X = \mathbb{R}^n$.

(ii)$_k \Rightarrow$ (i)$_k$ is obvious.

(i)$_k \Rightarrow$ (iv)$_k$ By the hypothesis, there exist distinguished triangles

$$G \to F \to K \xrightarrow{+1} \quad \text{and} \quad G \to F' \to K' \xrightarrow{+1}$$

in $D^b(\mathbf{k}_X)$ such that $p \notin \operatorname{SS}(K)$ and $p \notin \operatorname{SS}(K')$. By Lemma 3.1, there exist an open neighborhood W of x_0, a proper closed convex cone γ such that $\xi_0 \in \operatorname{Int}(\gamma^\circ)$, and $\varepsilon > 0$ such that

$$H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes K) = H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes K') = 0$$

for any $j \in \mathbb{Z}$ and $x \in W$. Hence one has

$$H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes F) \simeq H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes F').$$

Since $\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes F'$ belongs to $D^{>k}(\mathbf{k}_X)$, we get (3.2).

(i)$_k \Rightarrow$ (iii)$_{k,1}$ Same proof as (i)$_k \Rightarrow$ (iv)$_k$, replacing $H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes G)$ with $H^j_{\{\varphi \geqslant 0\}}(G)_x$ where $G = F', F, K, K'$.

(iii)$_{k,1} \Rightarrow$ (iii)$_{k,\omega}$ is obvious.

(iv)$_k \Rightarrow$ (ii)$_k$ To start with, note that (3.2) entails

$$(\mathbf{k}_{Z_\gamma} \circ F_{H_\varepsilon})_W \in D^{>k}(\mathbf{k}_X). \tag{3.3}$$

Let Δ denote the diagonal of $X \times X$. Then the morphism $\mathbf{k}_{Z_\gamma} \to \mathbf{k}_\Delta$ induces the morphism in $D^b(\mathbf{k}_X)$

$$\mathbf{k}_{Z_\gamma} \circ F_{H_\varepsilon} \to F_{H_\varepsilon}, \tag{3.4}$$

which is an isomorphism in $D^b(\mathbf{k}_X; p)$ by [8, Theorem 7.1.2]. Therefore, the composition

$$(\mathbf{k}_{Z_\gamma} \circ F_{H_\varepsilon})_W \to (F_{H_\varepsilon})_W \to F$$

is an isomorphism in $D^b(\mathbf{k}_X; p)$ and $(\mathbf{k}_{Z_\gamma} \circ F_{H_\varepsilon})_W$ belongs to $D^{>k}(\mathbf{k}_X; p)$.

(iii)$_{k,\omega} \Rightarrow$ (iv)$_k$ We already know that (iv)$_k$ is equivalent to (i)$_k$ for every k. Hence arguing by induction on k, we may assume that (i)$_{k-1}$ holds. Therefore we may assume $F \in D^{\geqslant k}(\mathbf{k}_X)$. Then we have

$$H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes F) = 0 \quad \text{for any } j \leqslant k-1$$

and

$$H^k(X;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon}\otimes F)\simeq H^0\big(X;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon}\otimes H^k(F)\big),$$
$$H^k_{\{\varphi\geqslant 0\}}(F)\simeq \Gamma_{\{\varphi\geqslant 0\}}\big(H^k(F)\big).$$

We may assume that $\mathrm{Int}(\gamma)\neq\emptyset$, $W\times(\gamma^\circ\setminus\{0\})\subset U$ and $(W+\gamma^a)\cap\overline{H_\varepsilon}\subset W$. Let $s\in\Gamma(X;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon}\otimes H^k(F))$. Then there exists $y\in\mathbb{R}^n$ such that

$$x+\gamma^a\subset y+\mathrm{Int}(\gamma^a)\subset W\cup(X\setminus H_\varepsilon)$$

and s extends to a section

$$\tilde{s}\in\Gamma\big(y+\mathrm{Int}(\gamma^a);\mathbf{k}_{H_\varepsilon}\otimes H^k(F)\big)\subset\Gamma\big(y+\mathrm{Int}(\gamma^a);H^k(F)\big).$$

Set $S=\mathrm{supp}(\tilde{s})\subset H_\varepsilon\cap(y+\mathrm{Int}(\gamma^a))$. Then the following lemma asserts $S=\emptyset$, and hence $H^k(X;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon}\otimes F)=0$. □

LEMMA 3.3. – *Let γ be a proper closed convex cone in $\mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^n$ such that $\Omega+\gamma^a=\Omega$, and let S be a closed subset of Ω such that $S\Subset\mathbb{R}^n$. Assume the following condition: for any $x\in\mathbb{R}^n$ and any real analytic function φ defined on $\mathbb{R}^n$, the three conditions $S\cap\varphi^{-1}(\mathbb{R}_{<0})=\emptyset$, $\varphi(x)=0$ and $d\varphi(x)\in\mathrm{Int}(\gamma^\circ)$ imply $x\notin S$.*

Then S is an empty set.

Proof. – If $\gamma=\{0\}$, then by taking $\varphi=0$, the lemma is trivially true. Hence we may assume that $\{0\}\subsetneqq\gamma$. Let us take ξ such that $\gamma\setminus\{0\}\subset\{x;\ \langle x,\xi\rangle>0\}$. Then there is a real number a such that $S\subset\{x;\ \langle x,\xi\rangle>a\}$. Set $H_-=\{x;\ \langle x,\xi\rangle<a\}$. By replacing Ω with $\Omega\cup H_-$, we may assume from the beginning $H_-\subset\Omega$.

For a proper closed convex cone γ' such that $\gamma\setminus\{0\}\subset\mathrm{Int}(\gamma')$ and $\gamma'\setminus\{0\}\subset\{x;\ \langle x,\xi\rangle>0\}$, set $\Omega_{\gamma'}=\{x\in\Omega;\ x+\gamma'^a\in\Omega\}$. Since $H_-\subset\Omega$, $\Omega_{\gamma'}$ is an open subset and $\Omega=\bigcup_{\gamma'}\Omega_{\gamma'}$. Set $S_{\gamma'}=S\cap\Omega_{\gamma'}$. Since $S=\bigcup_{\gamma'}S_{\gamma'}$, it is enough to show the assertion for $S_{\gamma'}$. Since $\gamma'^\circ\setminus\{0\}\subset\mathrm{Int}(\gamma^\circ)$, by replacing Ω, S, γ with $\Omega_{\gamma'}$, $S_{\gamma'}$ and γ', we may assume from the beginning

(3.5) $\qquad \mathrm{Int}(\gamma)\neq\emptyset,$

(3.6) $\qquad S\cap\varphi^{-1}(\mathbb{R}_{<0})=\emptyset,\quad \varphi(x)=0\quad\text{and}\quad d\varphi(x)\in\gamma^\circ\setminus\{0\}\quad\Rightarrow\quad x\notin S.$

Let us set $\psi(x)=\mathrm{dist}(x,\gamma^a):=\inf\{\|y-x\|;\ y\in\gamma^a\}$. It is well known that ψ is a continuous function on $\mathbb{R}^n$, and C^1 on $\mathbb{R}^n\setminus\gamma^a$. More precisely for any $x\in\mathbb{R}^n\setminus\gamma^a$, there exists a unique $y\in\gamma^a$ such that $\psi(x)=\|x-y\|$. Moreover $d\psi(x)=\|x-y\|^{-1}(x-y)\in\gamma^\circ\setminus\{0\}$. Furthermore $B_{\psi(x)}(y):=\{z\in\mathbb{R}^n;\ \|z-y\|<\psi(x)\}$ is contained in $\{z\in\mathbb{R}^n;\ \psi(z)<\psi(x)\}$.

For $\varepsilon>0$, we set $\gamma^a_\varepsilon=\{x\in\mathbb{R}^n;\ \psi(x)<\varepsilon\}$. Then γ^a_ε is an open convex set. Moreover $\gamma^a_\varepsilon+\gamma^a=\gamma^a_\varepsilon$. Set $\Omega_\varepsilon=\{x;x+\gamma^a_\varepsilon\subset\Omega\}$. Then $\Omega=\bigcup_{\varepsilon>0}\Omega_\varepsilon$. Set $S_\varepsilon=S\cap\Omega_\varepsilon$. It is enough to show that $S_\varepsilon=\emptyset$.

Assuming $S_\varepsilon\neq\emptyset$, we shall derive a contradiction. Let us take $x_0\in S_\varepsilon$ and $v\in\mathrm{Int}(\gamma)$. Set $V_t=x_0+\gamma^a_{\varepsilon/2}+tv$ for $t\in\mathbb{R}$. Then one has

(3.7) $$V_t=\bigcup_{t'<t}V_{t'}\quad\text{and}\quad\overline{V_t}=\bigcap_{t'>t}V_{t'}=\bigcap_{t'>t}\overline{V_{t'}},$$

(3.8) $\qquad x_0\in V_t\cap S\quad\text{for } t\geqslant 0,\quad\text{and}\quad V_t\subset H_-\ \text{for } t\ll 0,$

(3.9) $\qquad \overline{V_t}\subset\Omega\quad\text{for any } t\leqslant 0.$

Hence, for any compact set K and $t \in \mathbb{R}$ such $K \cap \overline{V_t} = \emptyset$, there exists $t' > t$ such that $K \cap V_{t'} = \emptyset$.

Let us set

$$c = \sup\{t; V_t \cap S = \emptyset\}.$$

Then $c \leqslant 0$ and $V_c \cap S = \emptyset$. By (3.9), one has $\overline{V_c} \cap \overline{S} \subset \overline{V_c} \cap S$. Since $\overline{S}$ is a compact set, there exists $x_1 \in S \cap \partial V_c$. Here $\partial V_c := \overline{V_c} \setminus V_c$ is the boundary of V_c. As seen before, there exists a ball $B_{\varepsilon/2}(y) := \{x; \|x-y\| < \varepsilon/2\}$ such that $B_{\varepsilon/2}(y) \subset V_c$, $\|x_1 - y\| = \varepsilon/2$ and $x_1 - y \in \gamma^\circ$. This is a contradiction by taking $\varphi(x) = \|x-y\|^2 - (\varepsilon/2)^2$. □

DEFINITION 3.4. – (i) Let $F \in D^b(\mathbf{k}_X)$. The closed conic subset $\mathrm{SS}_k(F)$ of T^*X is defined by: $p \notin \mathrm{SS}_k(F)$ if and only if F satisfies the equivalent conditions in Proposition 3.2.

(ii) Let $p \in T^*X$ and $k \in \mathbb{Z}$. Then $D^{>k}(\mathbf{k}_X;p)$ denotes the full additive subcategory of $D^b(\mathbf{k}_X;p)$ consisting of F satisfying $p \notin \mathrm{SS}_k(F)$. We write sometimes $D^{\geqslant k}(\mathbf{k}_X;p)$ for $D^{>k-1}(\mathbf{k}_X;p)$.

Note that $\mathrm{SS}_k(F) \cap T^*_X X = \pi(\mathrm{SS}_k(F)) = \mathrm{Supp}(\tau^{\leqslant k} F)$.

Remark 3.5. – The truncated microsupport has the following properties, similarly to those of the microsupport.

(i) For any $F \in D^b(\mathbf{k}_X)$, one has $\mathrm{SS}_k(F[n]) = \mathrm{SS}_{k+n}(F)$.

(ii) If $F' \to F \to F'' \xrightarrow{+1}$ is a distinguished triangle, then one has

$$\begin{aligned} &\mathrm{SS}_k(F) \subset \mathrm{SS}_k(F') \cup \mathrm{SS}_k(F''), \\ &\big(\mathrm{SS}_k(F') \setminus \mathrm{SS}_{k-1}(F'')\big) \cup \big(\mathrm{SS}_k(F'') \setminus \mathrm{SS}_{k+1}(F')\big) \subset \mathrm{SS}_k(F). \end{aligned} \tag{3.10}$$

(iii) For any $F \in D^b(\mathbf{k}_X)$, one has

$$\mathrm{SS}_k(F) = \mathrm{SS}_k(\tau^{\leqslant k} F). \tag{3.11}$$

Indeed, one has a distinguished triangle $\tau^{\leqslant k} F \to F \to \tau^{>k} F \xrightarrow{+1}$. Then (ii) implies

$$\begin{aligned} &\mathrm{SS}_k(F) \subset \mathrm{SS}_k(\tau^{\leqslant k} F) \cup \mathrm{SS}_k(\tau^{>k} F) \quad \text{and} \\ &\mathrm{SS}_k(\tau^{\leqslant k} F) \setminus \mathrm{SS}_{k-1}(\tau^{>k} F) \subset \mathrm{SS}_k(F). \end{aligned}$$

Hence the assertion follows from $\mathrm{SS}_{k-1}(\tau^{>k} F) = \mathrm{SS}_k(\tau^{>k} F) = \emptyset$.

Remark 3.6. – (i) If $F \in D^{>k}(\mathbf{k}_X)$, then $\mathrm{SS}_k(F) = \emptyset$.

(ii) If $F \in D^{\leqslant k}(\mathbf{k}_X)$, then $\mathrm{SS}_{k+d_X}(F) = \mathrm{SS}(F)$. Here d_X is the dimension of X.

The last statement follows from the characterization $(\mathrm{iv})_k$ in Proposition 3.2 and the fact that $H^j(X;F)$ vanishes for any $F \in D^{\leqslant k}(\mathbf{k}_X)$ and $j > k + d_X$.

Remark 3.7. – It is not true that $F \in D^{\geqslant k}(\mathbf{k}_X;p)$ implies the existence of a morphism $F \to F'$ in $D^b(\mathbf{k}_X)$ which is an isomorphism in $D^b(\mathbf{k}_X;p)$ and $F' \in D^{\geqslant k}(\mathbf{k}_X)$. For example take $X = \mathbb{R}$, $p = (0;1)$, $Z = \{x \in X;\ x < 0\}$, $F_1 = \mathbf{k}_{\{0\}}$ and $F = \mathbf{k}_Z[1]$. Then there is a morphism $F_1 \to F$ which is an isomorphism in $D(\mathbf{k}_X;p)$. Hence one has $F \in D^{\geqslant 0}(\mathbf{k}_X;p)$. Assume that there is a morphism $u\colon F \to F'$ in $D^b(\mathbf{k}_X)$ which is an isomorphism in $D^b(\mathbf{k}_X;p)$ and $F' \in D^{\geqslant 0}(\mathbf{k}_X)$. Since $H^0(F) = 0$, the morphism $H^0(F_1) \to H^0(F')$ vanishes, and hence the composition $F_1 \to F \xrightarrow{u} F'$ vanishes. This is a contradiction.

Example 3.8. – (i) One has

$$\mathrm{SS}_k(\mathbf{k}_X) = \begin{cases} \emptyset & \text{for } k < 0, \\ T^*_X X & \text{for } k \geqslant 0. \end{cases}$$

(ii) Let $X = \mathbb{R}$ and $Z_1 = \{x \in X;\ x \geqslant 0\}$, $Z_2 = \{x \in X;\ x > 0\}$. Then one has

$$\mathrm{SS}_k(\mathbf{k}_{Z_1}) = \begin{cases} \emptyset & \text{for } k < 0, \\ \{(x;\xi);\ \xi = 0,\ x \geqslant 0\} \cup \{(x;\xi);\ x = 0,\ \xi \geqslant 0\} & \text{for } k \geqslant 0, \end{cases}$$

$$\mathrm{SS}_k(\mathbf{k}_{Z_2}) = \begin{cases} \emptyset & \text{for } k < 0, \\ \{(x;\xi);\ \xi = 0,\ x \geqslant 0\} & \text{for } k = 0, \\ \{(x;\xi);\ \xi = 0,\ x \geqslant 0\} \cup \{(x;\xi);\ x = 0,\ \xi \leqslant 0\} & \text{for } k \geqslant 1. \end{cases}$$

(iii) Let X be a complex manifold. Then

$$\mathrm{SS}_k(\mathscr{O}_X) = \begin{cases} \emptyset & \text{for } k < 0, \\ T^*_X X & \text{for } k = 0, \\ T^* X & \text{for } k \geqslant 1. \end{cases}$$

(iv) Let M be a real analytic manifold, X a complexification of M, $\mathscr{M}$ a coherent $\mathscr{D}_X$-module, and let $\mathscr{B}_M$ denote the sheaf of Sato's hyperfunctions on M. Regarding T^*M as a subset of T^*X, one has

$$\mathrm{SS}_0\big(R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)\big) \subset \mathrm{Ch}(\mathscr{M}) \cap T^*M.$$

This follows immediately from the Holmgren theorem.

4. Functorial properties

In this section, we study in Propositions 4.1–4.4 below, the behavior of SS_k under external tensor products, proper direct image and smooth inverse image. These properties are proved similarly to the corresponding properties of the microsupport (cf. Chapter V of [8]), using Proposition 3.2. However the property of the microsupport corresponding to Proposition 4.1 was only known in a weaker form.

PROPOSITION 4.1. – *Let X and Y be real analytic manifolds. Then for $F \in D^b(\mathbf{k}_X)$, $G \in D^b(\mathbf{k}_Y)$ and $k \in \mathbb{Z}$, one has*

$$\mathrm{SS}_k(F \boxtimes G) = \bigcup_{i+j=k} \mathrm{SS}_i(F) \times \mathrm{SS}_j(G). \tag{4.1}$$

Proof. – Let us first show that $(p,p') \notin \bigcup_{i+j=k} \mathrm{SS}_i(F) \times \mathrm{SS}_j(G)$ implies

$$(p,p') \notin \mathrm{SS}_k(F \boxtimes G).$$

Since $\mathrm{SS}_k(F \boxtimes G) \subset \mathrm{SS}(F \boxtimes G) \subset \mathrm{SS}(F) \times \mathrm{SS}(G)$, we may assume that

$$(p,p') \in \mathrm{SS}(F) \times \mathrm{SS}(G).$$

Since $\mathrm{SS}_i(F) = \mathrm{SS}(F)$ for $i \gg 0$ and $\mathrm{SS}_i(F) = \emptyset$ for $i \ll 0$, there exists i such that $p \in \mathrm{SS}_i(F)$ and $p \notin \mathrm{SS}_{i-1}(F)$. Set $j = k - i$. Then $p' \notin \mathrm{SS}_j(G)$. Hence there exist a morphism $F' \to F$ which is an isomorphism in $D^b(\mathbf{k}_X; p)$ with $F' \in D^{>i-1}(\mathbf{k}_X)$, and $G' \to G$ which is an isomorphism in $D^b(\mathbf{k}_Y; p')$ with $G' \in D^{>j}(\mathbf{k}_X)$. Hence $F' \boxtimes G' \to F \boxtimes G$ is an isomorphism in $D^b(\mathbf{k}_{X\times Y}; (p,p'))$ and $F' \boxtimes G' \in D^{>i+j}(\mathbf{k}_{X\times Y})$.

Next let us show $\mathrm{SS}_i(F) \times \mathrm{SS}_j(G) \subset \mathrm{SS}_{i+j}(F \boxtimes G)$. If $i \ll 0$ or $j \ll 0$, then the assertion is trivial. By the induction on i and j, we may assume that $\mathrm{SS}_{i-1}(F) \times \mathrm{SS}_j(G) \subset \mathrm{SS}_{i+j-1}(F \boxtimes G)$ and $\mathrm{SS}_i(F) \times \mathrm{SS}_{j-1}(G) \subset \mathrm{SS}_{i+j-1}(F \boxtimes G)$.

Let $(p,p') \in \mathrm{SS}_i(F) \times \mathrm{SS}_j(G)$. We shall prove $(p,p') \in \mathrm{SS}_{i+j}(F \boxtimes G)$. If $p \in \mathrm{SS}_{i-1}(F)$, then the assertion is trivial. Hence we may assume $p \notin \mathrm{SS}_{i-1}(F)$. Thus, by replacing F with a sheaf microlocally isomorphic at p, we may assume $F \in D^{\geqslant i}(\mathbf{k}_X)$ from the beginning. Similarly we may assume $G \in D^{\geqslant j}(\mathbf{k}_Y)$. For any open neighborhood U of p, we can find $x_1 \in X$ and a C^1-function φ such that $(x_1; d\varphi(x_1)) \in U$ and

$$H^i_{\{\varphi \geqslant 0\}}(F)_{x_1} = \Gamma_{\{\varphi \geqslant 0\}}(H^i(F))_{x_1} \neq 0.$$

Similarly, for any open neighborhood U' of p', we can find $y_1 \in Y$ and a C^1-function ψ such that $(y_1; d\psi(y_1)) \in U'$ and $H^j_{\{\psi \geqslant 0\}}(G)_{y_1} = \Gamma_{\{\psi \geqslant 0\}}(H^j(G))_{y_1} \neq 0$. Set $z = (x_1, y_1) \in X \times Y$ and $\eta(x,y) = \varphi(x) + \psi(y)$. Then $(z; d\eta(z)) \in U \times U'$ and

$$H^{i+j}_{\{\eta \geqslant 0\}}(F \boxtimes G)_z = \Gamma_{\{\eta \geqslant 0\}}\bigl(H^i(F) \boxtimes H^j(G)\bigr)_z,$$

as a subspace of $H^i(F)_{x_1} \otimes H^j(G)_{y_1}$, contains $(\Gamma_{\{\varphi \geqslant 0\}}(H^i(F))_{x_1}) \otimes (\Gamma_{\{\psi \geqslant 0\}}(H^j(G))_{y_1})$. Hence $H^{i+j}_{\{\eta \geqslant 0\}}(F \boxtimes G)_z \neq 0$. Since $U \times U'$ forms a neighborhood system of (p,p'), we can conclude $(p,p') \in \mathrm{SS}_{i+j}(F \boxtimes G)$. $\square$

COROLLARY 4.2. – *Let X and Y be real analytic manifolds. Then for $F \in D^b(\mathbf{k}_X)$ and $G \in D^b(\mathbf{k}_Y)$, one has*

$$\mathrm{SS}(F \boxtimes G) = \mathrm{SS}(F) \times \mathrm{SS}(G). \tag{4.2}$$

PROPOSITION 4.3. – *Let $f : X \to Y$ be a morphism of real analytic manifolds and let $F \in D^b(\mathbf{k}_X)$ such that f is proper on the support of F. Then for any $k \in \mathbb{Z}$,*

$$\mathrm{SS}_k\bigl(Rf_*(F)\bigr) \subset f_\pi {f_d}^{-1}\bigl(\mathrm{SS}_k(F)\bigr). \tag{4.3}$$

The equality holds in case f is a closed embedding.

Proof. – We shall follow the method of proof of Proposition 5.4.4 of [8].

Let $y \in Y$ and let φ be a real C^1-function on Y such that $\varphi(y) = 0$ and $d(\varphi \circ f)(x) \notin \mathrm{SS}_k(F)$ for every $x \in f^{-1}(y)$. Therefore

$$H^j R\Gamma_{\{\varphi\circ f \geqslant 0\}}(F)|_{f^{-1}(y)} = 0 \quad \text{for any } j \leqslant k.$$

We have

$$\begin{aligned} H^j R\Gamma_{\{\varphi \geqslant 0\}}\bigl(Rf_*(F)\bigr)_y &\simeq H^j Rf_*\bigl(R\Gamma_{\{\varphi\circ f \geqslant 0\}}(F)\bigr)_y \\ &\simeq H^j\bigl(f^{-1}(y); R\Gamma_{\{\varphi\circ f \geqslant 0\}}(F)\bigr) = 0 \end{aligned}$$

for every $j \leqslant k$. This proves (4.3).

Let us now assume that f is a closed embedding. Let $p \notin \mathrm{SS}_k(Rf_*F)$. We may assume that Y is a real vector space and X is a linear subspace of Y. Let $\gamma \subset Y$, $W \subset Y$, ε be chosen as in Proposition 3.2(iv)$_k$ with respect to p and Rf_*F, that is,

$$H^j(Y;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes Rf_*F) = 0 \quad \text{for any } j \leqslant k \text{ and } x \in W.$$

Since f is a closed embedding, one has

$$H^j(Y;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes Rf_*F) \simeq H^j(X;\mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon \cap X} \otimes F).$$

Hence one has

$$H^j(X;\mathbf{k}_{(x+\gamma^a\cap X)\cap(H_\varepsilon\cap X)} \otimes F) = 0 \quad \text{for any } j \leqslant k,$$

and the interior of the polar set of $\gamma \cap X$ contains $f_d f_\pi^{-1}(p)$, and therefore

$$\mathrm{SS}_k(F) \cap f_d f_\pi^{-1}(p) = \emptyset. \qquad \square$$

PROPOSITION 4.4. – *Let X and Y be real analytic manifolds and let $f: X \to Y$ be a smooth morphism. Let $G \in D^b(\mathbf{k}_Y)$. Then, for any $k \in \mathbb{Z}$,*

$$\mathrm{SS}_k\big(f^{-1}G\big) = f_d f_\pi^{-1}\big(\mathrm{SS}_k(G)\big). \tag{4.4}$$

Proof. – The problem being local on X, we may assume that $X = Y \times Z$, Y and Z are vector spaces and f is the projection. Then we have to show

$$\mathrm{SS}_k(G \boxtimes \mathbf{k}_Z) = \mathrm{SS}_k(G) \times T_Z^*Z. \tag{4.5}$$

This follows from Proposition 4.1. $\square$

5. Estimates for the truncated microsupport

Let Y be a smooth submanifold of X. In this section we will give an estimate for $\mathrm{SS}_k(F) \cap \pi^{-1}(Y)$. Recall that $\mu_Y(F)$ denotes the microlocalization of F along Y. Note that for $F \in D^{\geqslant k}(\mathbf{k}_X)$, $H^k(\mu_Y(F)) \simeq H^0(\mu_Y(H^k(F)))$ is a subsheaf of $\pi^{-1}H^k(F)|_{T_Y^*X}$ and

$$\begin{aligned} H^k\big(\mu_Y(F)\big)_p \simeq \big\{ s \in H^k(F)_{\pi(p)};\ C_Y\big(\mathrm{supp}(s)\big)_{\pi(p)} \setminus \{0\} \\ \subset \big\{ v \in (T_Y X)_{\pi(p)};\ \langle v, p\rangle > 0 \big\}\big\} \end{aligned} \tag{5.1}$$

for $p \in T_Y^*X$.

The following result is a generalization of [7, Theorem 5.7.1] to SS_k.

THEOREM 5.1. – *Let X be a real analytic manifold and Y a smooth submanifold. Let $F \in D^b(\mathbf{k}_X)$. Then*

$$\mathrm{SS}_k(F) \cap T_Y^*X = \big(T_Y^*X \cap \overline{\mathrm{SS}_k(F) \setminus \pi^{-1}(Y)}\big) \cup \mathrm{Supp}\big(\tau^{\leqslant k}\mu_Y(F)\big). \tag{5.2}$$

Proof. – It is evident that the right hand side of (5.2) is contained in the left hand side. Let us show the converse inclusion. Assuming that $p \in T_Y^*X$ satisfies

$$p \notin \overline{\mathrm{SS}_k(F) \setminus \pi^{-1}(Y)} \cup \mathrm{Supp}\big(\tau^{\leqslant k}\big(\mu_Y(F)\big)\big),$$

we shall prove $p \notin \mathrm{SS}_k(F)$. Arguing by induction on k, one has $p \notin \mathrm{SS}_{k-1}(F)$ and by Proposition 3.2(ii)$_{k-1}$ we may assume that $F \in D^{\geqslant k}(\mathbf{k}_X)$.

There exists an open conic neighborhood U of p in T^*X such that $U \cap \mathrm{SS}_k(F) \subset \pi^{-1}(Y)$ and $H^j(\mu_Y(F))|_U = 0$ for any $j \leqslant k$. Furthermore, we may assume that $X = \mathbb{R}^n$, Y is a linear subspace of X, $p = (x_0; \xi_0)$. Let us take an open neighborhood W of x_0, a proper closed convex cone γ and $\varepsilon > 0$ such that $W \times \mathrm{Int}(\gamma^\circ) \subset U$, $\xi_0 \in \mathrm{Int}(\gamma^\circ)$ and $(W + \gamma^a) \cap \overline{H_\varepsilon} \subset W$. Hence one has

$$H^j(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes F) \simeq H^{j-k}\big(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes H^k(F)\big) \quad \text{for any } j \leqslant k.$$

Thus it is enough to check that

$$\Gamma\big(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes H^k(F)\big) = 0. \tag{5.3}$$

Let $s \in \Gamma(X; \mathbf{k}_{(x+\gamma^a)\cap H_\varepsilon} \otimes H^k(F))$. Then there exists an open set Ω_0 such that $\Omega_0 + \gamma^a = \Omega_0$, $x + \gamma^a \subset \Omega_0$ and s extends to a section $\tilde{s} \in \Gamma(\Omega_0; \mathbf{k}_{H_\varepsilon} \otimes H^k(F))$. Moreover we may assume that $(\Omega_0 \cap \overline{H_\varepsilon}) \times \gamma^\circ \subset U$. Set $S = \mathrm{supp}(\tilde{s})$. Then $S \setminus (Y + \gamma)$ satisfies the condition in Lemma 3.3 with $\Omega = \Omega_0 \setminus (Y + \gamma)$. Hence we have $S \setminus (Y + \gamma) = \emptyset$ and hence $S \subset Y + \gamma$. Since $H^k(\mu_Y(F))|_U = 0$ and

$$C_Y(Y + \gamma) \subset Y \times (Y + \gamma) \subset \big\{v \in T_Y X; \langle v, \xi_0\rangle > 0\big\} \cup \big(Y \times \{0\}\big),$$

the formula (5.1) implies $\tilde{s}|_Y = 0$. One has therefore $S \cap Y = \emptyset$. Then S satisfies the condition in Lemma 3.3, and we can conclude $S = \emptyset$, which implies $s = 0$. $\square$

We shall need the following definition:

DEFINITION 5.2. – Let Y be a closed submanifold of X, let $k \in \mathbb{Z}$ and let $F \in D^b(\mathbf{k}_X)$. The closed subset $\mathrm{SS}_k^Y(F)$ of $\pi^{-1}(Y)$ is defined by: $p \notin \mathrm{SS}_k^Y(F)$ if and only if there exists an open conic neighborhood U of p in $\pi^{-1}(Y)$ satisfying the following two conditions:
(i) $\tau^{\leqslant k}\mu_Y(F)|_{U \cap T_Y^* X} = 0$,
(ii) for any smooth real analytic hypersurface Z of Y,

$$\tau^{\leqslant k}\mu_Z(F)|_{U \cap T_Z^* X \setminus T_Y^* X} = 0.$$

We remark that $\mathrm{SS}_k^Y(F)$ is a conic closed set obviously contained in $\mathrm{SS}_k(F) \cap \pi^{-1}(Y)$.

THEOREM 5.3. – *Let X be a real analytic manifold and Y a closed submanifold. Let $F \in D^b(\mathbf{k}_X)$. Then*

$$\mathrm{SS}_k(F) = \overline{\mathrm{SS}_k(F) \setminus \pi^{-1}(Y)} \cup \mathrm{SS}_k^Y(F). \tag{5.4}$$

Proof. – The left hand side obviously contains the right hand side. Let us prove the converse inclusion. Assuming that $p \in \pi^{-1}(Y)$ satisfies $p \notin \overline{\mathrm{SS}_k(F) \setminus \pi^{-1}(Y)} \cup \mathrm{SS}_k^Y(F)$, let us show $p \notin \mathrm{SS}_k(F)$.

If $p \in T_Y^* X$, Theorem 5.1 implies the assertion. Hence we may assume $p \notin T_Y^* X$. Let U be an open conic neighborhood of p in T^*X such that $\mathrm{SS}_k(F) \cap U \subset \pi^{-1}(Y)$ and $U \cap \mathrm{SS}_k^Y(F) = \emptyset$.

We may assume that $X = \{x = (u, v, t);\ u \in \mathbb{R}^n,\ v \in \mathbb{R}^m,\ t \in \mathbb{R}\}$, $Y = \{(u, v, t) \in X;\ u = 0\}$ and $p = ((0,0,0); (0,0,1))$. We may assume $W \times \gamma^\circ \subset U$ with $\gamma = \{t \geqslant \sqrt{\|u\|^2 + \|v\|^2}\}$ and an open neighborhood W of the origin. Set $H_\varepsilon = \{(u, v, t);\ t > -\varepsilon\}$, and choose W and a sufficiently small ε such that $(x + \gamma^a) \cap \overline{H_\varepsilon} \subset W$ for any $x \in W$.

By the induction on k, we may assume $F \in D^{\geqslant k}(\mathbf{k}_X)$. It is enough to show for any $x_0 \in W$

$$\Gamma\big(x_0 + \mathrm{Int}(\gamma^a); \mathbf{k}_{H_\varepsilon} \otimes H^k(F)\big) = 0.$$

Let $s \in \Gamma(x_0 + \mathrm{Int}(\gamma^a); \mathbf{k}_{H_\varepsilon} \otimes H^k(F))$. Set $S = \mathrm{supp}(s) \subset (x_0 + \mathrm{Int}(\gamma^a)) \cap H_\varepsilon$. Assuming $S \neq \emptyset$, we shall derive a contradiction. Set $x_0 = (u_0, v_0, t_0)$ and set

$$\varphi(x) = \|u - u_0\|^2 + \|v - v_0\|^2 - (t - t_0)^2.$$

Then $\Omega := x_0 + \mathrm{Int}(\gamma^a) = \{x; t < t_0, \varphi(x) < 0\}$, and $\varphi(\Omega \cap H_\varepsilon)$ is bounded from below. Moreover one has $d\varphi(x) \in \mathrm{Int}(\gamma^\circ)$ for any $x \in \Omega$, and $d\varphi(x) \notin T^*_Y X$ for any $x \in \Omega \cap Y$. Set $c = \inf\{\varphi(x);\ x \in S\} < 0$. Since $\varphi|_S : S \to \mathbb{R}_{<0}$ is a proper map, one has $c \in \varphi(S)$. Let Z_c be the closed subset $\{x = (0, v, t) \in Y;\ t < t_0,\ \varphi(x) = c\}$ of Y and set $\Omega' = \Omega \setminus (Z_c + \gamma)$. Since one has

$$Z_c + \gamma = \Big\{(t, u, v); t - t_0 \geqslant \|u\| - \sqrt{\|v - v_0\|^2 + \|u_0\|^2 - c}\,\Big\},$$

φ takes values smaller than c on $Y \setminus (Z_c + \gamma)$, and hence $S' := S \cap \Omega'$ does not intersect Y. Therefore S' satisfies the condition in Lemma 3.3. Hence $S' = \emptyset$, which means $S \subset Z_c + \gamma$. Since $C_{Z_c}(Z_c + \gamma)_x \setminus \{0\} \subset \{d\varphi(x) > 0\}$ for any $x \in Z_c$, $H^0(\mu_{Z_c} H^k(F))|_{U \setminus T^*_Y X} = 0$ implies $s|_{Z_c} = 0$. Therefore $S \cap Z_c = \emptyset$. Since $\{x \in (Z_c + \gamma) \cap \Omega; \varphi(x) = c\} \subset Z_c$, one has $c \notin \varphi(S)$, which is a contradiction. $\square$

6. Applications to $\mathscr{D}$-modules

In this section, X denotes a complex manifold.

Before stating our main result, let us recall a classical lemma on the vanishing of the microlocalization of $\mathscr{O}_X$ along submanifolds.

LEMMA 6.1. – *Let Y be a closed complex submanifold of codimension d of X and let S be a smooth real analytic hypersurface of Y. Then*

$$H^k\big(\mu_Y(\mathscr{O}_X)\big) = 0 \quad \textit{for any } k \neq d, \tag{6.1}$$

$$H^k\big(\mu_S(\mathscr{O}_X)\big)\big|_{T^*_S X \setminus T^*_Y X} = 0 \quad \textit{for any } k \leqslant d. \tag{6.2}$$

Proof. – The vanishing property (6.1) is proved in [11] and (by a different method) in [7, Proposition 11.3.4].

The vanishing property (6.2) follows from [7, Proposition 11.3.1]. Let us recall this statement. Let $p \in T^*_S X$. Set $E_p = T_p(T^*X)$, $\lambda_S = T_p(T^*_S X)$, $\lambda_0 = T_p(\pi^{-1}\pi(p))$, and denote by ν the complex line in E_p, the tangent space to the Euler vector field in T^*X at p. Let c be the real codimension of the real submanifold S and let δ denote the complex dimension of $\lambda_S \cap \sqrt{-1}\,\lambda_S \cap \lambda_0$.

The result of loc. cit. asserts that if the real dimension of $\lambda_S \cap \nu$ is 1, then

$$H^j\big(\mu_S(\mathscr{O}_X)\big)_p = 0 \quad \text{for } j < c - \delta.$$

(The result in loc. cit. is more precise, involving the signature of the Levi form.) If $p \in T^*_S X \setminus T^*_Y X$, the real dimension of $\lambda_S \cap \nu$ is 1. Since $c = 2d + 1$ and $\delta = d$, we get the desired result. $\square$

Now we are ready to prove the following proposition.

PROPOSITION 6.2. – *Let X be a complex manifold, let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and let S be a closed complex analytic subset of X with $\operatorname{codim}_X S \geqslant d$. Set $F = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Then*

(i) $\mathrm{SS}_{d-1}(F) = \overline{\mathrm{SS}_{d-1}(F) \setminus \pi^{-1}(S)}$.

(ii) *Let S' be a closed complex analytic subset of S such that $\operatorname{codim}_X S' > d$ and $S_0 := S \setminus S'$ is a non-singular subvariety of codimension d. Then*

$$\mathrm{SS}_d(F) = \overline{\mathrm{SS}_d(F) \setminus \pi^{-1}(S)} \cup \overline{\mathrm{Supp}(\tau^{\leqslant d} \mu_{S_0}(F|_{X \setminus S'}))}.$$

In particular one has

$$\mathrm{SS}_d(F) = \overline{\mathrm{SS}_d(F) \setminus \pi^{-1}(S)} \cup \overline{\mathrm{SS}_d(F) \cap T^*_{S_0} X}.$$

Proof. – (i) By the induction on the codimension of S, we may assume that S is non-singular. By Theorem 5.3 one has

$$\mathrm{SS}_{d-1}(F) = \overline{\mathrm{SS}_{d-1}(F) \setminus \pi^{-1}(S)} \cup \mathrm{SS}^S_{d-1}(F).$$

Hence it is enough to show that $\mathrm{SS}^S_d(F) = \emptyset$, or equivalently $H^j(\mu_S(\mathscr{O}_X)) = 0$ for $j < d$ and $H^j(\mu_Z(\mathscr{O}_X))|_{T^*_Z X \setminus T^*_S X} = 0$ for $j < d$ for any real analytic hypersurface Z of S. This is a consequence of Lemma 6.1.

(ii) By (i), we may assume that S is non-singular of codimension d. Hence it is enough to show $\mathrm{SS}^S_d(F) = \mathrm{Supp}(\tau^{\leqslant d} \mu_S(F))$. By the definition, we are reduced to proving

$$H^j\big(\mu_Z(\mathscr{O}_X)\big)\big|_{T^*_Z X \setminus T^*_S X} = 0$$

for $j \leqslant d$ and for any real analytic hypersurface Z of S. This is again a consequence of Lemma 6.1. □

Remark 6.3. – When S is a closed smooth hypersurface, the inclusion

$$\mathrm{SS}_1(F) \subset \overline{\mathrm{SS}(F) \setminus \pi^{-1}(S)} \cup \big(\mathrm{SS}(F) \cap T^*_S X\big)$$

was obtained in [12].

Let Ω be an open subset of X. We shall say for short that Ω has a smooth boundary $\partial\Omega$ if there exists a real C^1-function φ such that $d\varphi \neq 0$ on the set $\{\varphi = 0\}$ and $\Omega = \{x \in X;\ \varphi(x) < 0\}$.

COROLLARY 6.4. – *Let Ω be an open subset of X with smooth boundary, let $\mathscr{M}$, F, S and S_0 be as in Proposition* 6.2 *and let Λ be a closed conic subset of T^*X. Assume that*

$$\mathrm{Ch}(\mathscr{M}) \subset \Lambda \cup \pi^{-1}(S),$$
$$\overline{T^*_{S_0} X} \cap T^*_{\partial\Omega} X \subset T^*_X X,$$
$$\Lambda \cap T^*_{\partial\Omega} X \subset T^*_X X.$$

Then one has

$$\mathrm{SS}_d(F) \cap T^*_{\partial\Omega} X \subset T^*_X X.$$

In particular one has

$$H^j\big(R\Gamma_{X \setminus \Omega}(F)\big)\big|_{\partial\Omega} = 0 \quad \textit{for } j \leqslant d.$$

Example 6.5. – Under the situation of Corollary 6.4, assume further that Ω is pseudo-convex. Let us denote by $j:\Omega \hookrightarrow X$ the open embedding. Then $H^k(Rj_*j^{-1}\mathscr{O}_X)=0$ for $k\neq 0$, and $R\Gamma_{X\setminus\Omega}(\mathscr{O}_X)|_{\partial\Omega}[1]$ is concentrated in degree 0. Let us set for short:

$$\mathscr{O}_X^+/\mathscr{O}_X = \big(j_*j^{-1}\mathscr{O}_X/\mathscr{O}_X\big)\big|_{\partial\Omega} \simeq R\Gamma_{X\setminus\Omega}(\mathscr{O}_X)|_{\partial\Omega}[1].$$

Applying Corollary 6.4, we find that

$$\mathcal{E}xt^j_{\mathscr{D}_X}\big(\mathscr{M},\mathscr{O}_X^+/\mathscr{O}_X\big)=0 \quad \text{for } j<d.$$

Example 6.6. – Let P be a differential operator on X whose principal symbol $\sigma(P)$ has the form $a(x)q(x,\xi)$ with $a\in\mathscr{O}_X(X)$ and $q\in\mathscr{O}_{T^*X}(T^*X)$. Then taking $\mathscr{D}_X/\mathscr{D}_XP$ as $\mathscr{M}$, the solution complex F is $\mathscr{O}_X\xrightarrow{P}\mathscr{O}_X$, where $\mathscr{O}_X$ is at degree 0 and 1. Taking $a^{-1}(0)$ and $q^{-1}(0)$ as S and Λ, Corollary 6.4 implies

$$\mathrm{SS}_1(F)\subset q^{-1}(0)\cup\overline{\{(x;\xi);a(x)=0,\xi\in\mathbb{C}da(x)\}}.$$

By (3.10) and the distinguished triangle

$$\mathrm{Ker}(\mathscr{O}_X\xrightarrow{P}\mathscr{O}_X)\to F\to\mathrm{Coker}(\mathscr{O}_X\xrightarrow{P}\mathscr{O}_X)[-1]\xrightarrow{+1},$$

one has also

$$\mathrm{SS}_1\big(\mathrm{Ker}(\mathscr{O}_X\xrightarrow{P}\mathscr{O}_X)\big)\subset q^{-1}(0)\cup\overline{\{(x;\xi);a(x)=0,\xi\in\mathbb{C}da(x)\}}.$$

Finally one has the following theorem which calculates $\mathrm{SS}_k(F)$ in terms of $\mathrm{Ch}(\mathscr{M})$. Here, for a closed complex subset Z of X, T^*_ZX means $\overline{T^*_{Z_{\mathrm{reg}}}X}$ where Z_{reg} is the non-singular locus of Z.

THEOREM 6.7. – *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, and let F be the solution complex $R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)$. Let $\mathrm{Ch}(\mathscr{M})=\bigcup_{\alpha\in A}V_\alpha$ be the decomposition of $\mathrm{Ch}(\mathscr{M})$ into irreducible components. Let Y_α be the irreducible complex analytic subset $\pi(V_\alpha)$ of X. Then for any integer k one has*

$$\mathrm{SS}_k(F)=\Big(\bigcup_{\operatorname{codim}Y_\alpha<k}V_\alpha\Big)\cup\Big(\bigcup_{\operatorname{codim}Y_\alpha=k}T^*_{Y_\alpha}X\Big). \tag{6.3}$$

Proof. – The inclusion $\subset$ is a consequence of Proposition 6.2. Let us show the converse inclusion. Note that both sides are empty sets for $k<0$. Hence arguing by induction on k, we can assume that (6.3) holds for $k-1$. Hence it is enough to show:

(6.4(i)) $\quad$ if $\operatorname{codim}Y_\alpha=k-1$ and $V_\alpha\neq T^*_{Y_\alpha}X$, $\quad$ then $V_\alpha\subset\mathrm{SS}_k(F)$,

(6.4(ii)) $\quad$ if $\operatorname{codim}Y_\alpha=k$, $\quad$ then $T^*_{Y_\alpha}X\subset\mathrm{SS}_k(F)$.

In both cases, we may assume that $Y:=Y_\alpha$ is a non-singular subvariety. Let $j:Y\to X$ be the inclusion map.

Proof of (6.4(i)). – It is enough to show that for any open subset U of T^*X with a non-empty intersection with V_α, $\mathrm{SS}_k(F)\cap U$ is non-empty. We may assume that $\mathrm{Ch}(\mathscr{M})\cap U=V_\alpha\cap U$ and $V_\alpha\cap U\to Y$ is a smooth morphism. Since $V_\alpha\subset\pi^{-1}(Y)$, we may assume, by shrinking U if necessary, that $\mathcal{E}_X^{\mathbb{R}} \otimes_{\mathscr{D}_X} \mathscr{M}|_U \simeq \mathcal{E}_{X\hookleftarrow Y}^{\mathbb{R}} \otimes_{\mathcal{E}_Y} \mathscr{N}|_U$ for a coherent $\mathcal{E}_Y$-module $\mathscr{N}$ [11]. For any smooth complex hypersurface Z of Y, one has by [11]

$$\begin{aligned}H^k\big(\mu_Z(F)\big) &\simeq \mathcal{H}om_{\mathscr{D}_X}\big(\mathscr{M}, H^k\big(\mu_Z(\mathscr{O}_X)\big)\big)\\ &\simeq \mathcal{H}om_{\mathcal{E}_X^{\mathbb{R}}}\big(\mathcal{E}_X^{\mathbb{R}} \otimes_{\mathscr{D}_X} \mathscr{M}, H^k\big(\mu_Z(\mathscr{O}_X)\big)\big)\\ &\simeq \mathcal{H}om_{\mathcal{E}_X^{\mathbb{R}}}\big(\mathcal{E}_{X\hookleftarrow Y}^{\mathbb{R}} \otimes_{\mathcal{E}_Y} \mathscr{N}, H^k\big(\mu_Z(\mathscr{O}_X)\big)\big)\\ &\simeq j_{d*}j_\pi^{-1}\mathcal{H}om_{\mathcal{E}_Y}\big(\mathscr{N}, H^1\big(\mu_Z(\mathscr{O}_Y)\big)\big)\end{aligned}$$

on $U \cap T_Z^*X$. Hence the result follows from the following lemma which is an easy consequence of the classification theorem for coherent $\mathcal{E}$-modules at generic points of their supports.

LEMMA 6.8. – *Let V be a non-empty* (*locally closed*) *smooth submanifold of $\dot{T}^*Y$ such that $V \to Y$ is smooth, and let $\mathscr{N}$ be a coherent $\mathcal{E}_Y$-module defined on a neighborhood of V such that* $\operatorname{Supp}(\mathscr{N}) = V$. *Then there is a smooth complex hypersurface Z of Y such that* $\mathcal{H}om_{\mathcal{E}_Y}(\mathscr{N}, H^1(\mu_Z(\mathscr{O}_Y)))|_{V\cap T_Z^*X} \neq 0$.

Proof. – By the generic classification theorem in [11], $\mathcal{E}_Y^\infty \otimes_{\mathcal{E}_Y} \mathscr{N}$ is a de Rham system by shrinking V if necessary. There exists a smooth complex hypersurface Z of Y such that $T_Z^*Y \subset V$. Then by a quantized contact transform, $\mathcal{E}_Y^\infty \otimes_{\mathcal{E}_Y} \mathscr{N}$ and $H^1(\mu_Z(\mathscr{O}_Y))$ are transformed to $(\mathcal{E}_{\mathbb{C}^n}^\infty/(\sum_{i=1}^s \mathcal{E}_{\mathbb{C}^n}^\infty \partial_i))^{\oplus m}$ and $H^1(\mu_{\{z_n=0\}}(\mathscr{O}_{\mathbb{C}^n}))$ with $s < n$ and $m > 0$. In this case, the assertion is obvious. □

Proof of (6.4(ii)). – The proof is similar to that of (6.4(i)). Note that V_α contains T_Y^*X. By shrinking Y if necessary, we may assume that $\operatorname{Ch}(\mathscr{M})$ is equal to V_α on a neighborhood of a point p in T_Y^*X. Then on a neighborhood of p, $\mathcal{E}_X^{\mathbb{R}} \otimes_{\mathscr{D}_X} \mathscr{M}$ is isomorphic to $\mathcal{E}_{X\hookleftarrow Y}^{\mathbb{R}} \otimes_{\mathscr{D}_Y} \mathscr{N}$ for a coherent $\mathscr{D}_Y$-module $\mathscr{N}$ with $\operatorname{Supp}(\mathscr{N}) = Y$ on a neighborhood of $\pi(p)$. One has then by [11]

$$\begin{aligned}H^k\big(\mu_Y(F)\big) &\simeq \mathcal{H}om_{\mathscr{D}_X}\big(\mathscr{M}, H^k\big(\mu_Y(\mathscr{O}_X)\big)\big)\\ &\simeq \mathcal{H}om_{\mathcal{E}_X^{\mathbb{R}}}\big(\mathcal{E}_X^{\mathbb{R}} \otimes_{\mathscr{D}_X} \mathscr{M}, H^k\big(\mu_Y(\mathscr{O}_X)\big)\big)\\ &\simeq \mathcal{H}om_{\mathcal{E}_X^{\mathbb{R}}}\big(\mathcal{E}_{X\hookleftarrow Y}^{\mathbb{R}} \otimes_{\mathscr{D}_Y} \mathscr{N}, H^k\big(\mu_Y(\mathscr{O}_X)\big)\big)\\ &\simeq \pi^{-1}\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{N}, \mathscr{O}_Y)|_{T_Y^*X}.\end{aligned}$$

Hence the assertion follows from the following well-known result. □

LEMMA 6.9. – *If a coherent $\mathscr{D}_X$-module $\mathscr{M}$ satisfies* $\operatorname{Supp}(\mathscr{M}) = X$, *then one has* $\operatorname{Supp}(\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) = X$.

By the Riemann–Hilbert correspondence of perverse sheaves and holonomic $\mathscr{D}$-modules, we have the following description of the truncated microsupport of perverse sheaves.

COROLLARY 6.10. – *Let $F \in D^b(\mathbb{C}_X)$ and let $\{X_\alpha\}_{\alpha\in A}$ be a family of complex submanifolds such that $\overline{X_\alpha}$ and $\overline{X_\alpha} \setminus X_\alpha$ are closed complex analytic subsets and*

$$\operatorname{SS}(F) = \bigcup_{\alpha\in A} \overline{T_{X_\alpha}^*X}.$$

If F is a perverse sheaf (*i.e. there is a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ such that*

$$F \simeq R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)),$$

then one has

$$\mathrm{SS}_k(F) = \bigcup_{\operatorname{codim} X_\alpha \leqslant k} \overline{T^*_{X_\alpha} X} \quad \textit{for any } k. \tag{6.6}$$

Conversely if $F \in D^b(\mathbb{C}_X)$ is $\mathbb{C}$-constructible and if it satisfies

$$\mathrm{SS}_k(F) \cup \mathrm{SS}_k\big(R\mathcal{H}om(F, \mathbb{C}_X)\big) \subset \bigcup_{\operatorname{codim} X_\alpha \leqslant k} \overline{T^*_{X_\alpha} X} \quad \textit{for any } k, \tag{6.7}$$

then F is a perverse sheaf.

Proof. – It is a direct consequence of Theorem 6.7 that the perversity of F implies (6.6). Conversely assume (6.7). In order to prove that F is a perverse sheaf, it is enough to show that F is microlocally isomorphic to $\mathbb{C}_{X_\alpha}[-\operatorname{codim} X_\alpha]^{\oplus m}$ for some m at a generic point of $T^*_{X_\alpha} X$ by [8, Theorem 10.3.12]. By [8], F is isomorphic to $\mathbb{C}_{X_\alpha}[-\operatorname{codim} X_\alpha] \otimes K$ at a generic point of $T^*_{X_\alpha} X$ for some $K \in D^b(\mathbb{C})$. Since $\mu_{X_\alpha}(F)$ must be in $D^{\geqslant \operatorname{codim} X_\alpha}(\mathbb{C}_{T^*_{X_\alpha} X})$ and

$$\mu_{X_\alpha}(F) \simeq \mathbb{C}_{T^*_{X_\alpha} X}[-\operatorname{codim} X_\alpha] \otimes K,$$

one has $K \in D^{\geqslant 0}(\mathbb{C})$. Similarly,

$$\mu_{X_\alpha}\big(R\mathcal{H}om(F, \mathbb{C}_X)\big) \simeq \mathbb{C}_{T^*_{X_\alpha} X}[-\operatorname{codim} X_\alpha] \otimes R\mathcal{H}om(K, \mathbb{C})$$

implies $K \in D^{\leqslant 0}(\mathbb{C})$. □

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