Ann. Scient. Éc. Norm. Sup.,
4e série, t. 36, 2003, p. 431 à 462.

ANDRÉ–QUILLEN HOMOLOGY OF ALGEBRA RETRACTS

BY LUCHEZAR L. AVRAMOV [1] AND SRIKANTH IYENGAR [2]

ABSTRACT. – Given a homomorphism of commutative noetherian rings $\varphi : R \to S$, Daniel Quillen conjectured in 1970 that if the André–Quillen homology functors $\mathrm{D}_n(S \mid R; -)$ vanish for all $n \gg 0$, then they vanish for all $n \geqslant 3$. We prove the conjecture under the additional hypothesis that there exists a homomorphism of rings $\psi : S \to R$ such that $\varphi \circ \psi = \mathrm{id}_S$. More precisely, in this case we show that ψ is a complete intersection at $\varphi^{-1}(\mathfrak{n})$ for every prime ideal $\mathfrak{n}$ of S. Using these results, we describe all algebra retracts $S \to R \to S$ for which the algebra $\mathrm{Tor}^R_\bullet(S, S)$ is finitely generated over $\mathrm{Tor}^R_0(S, S) = S$.

RÉSUMÉ. – Étant donné un homomorphisme $\varphi : R \to S$ d'anneaux commutatifs noethériens, Daniel Quillen a conjecturé en 1970 que si les foncteurs $\mathrm{D}_n(S \mid R; -)$ d'homologie d'André–Quillen sont nuls pour tout $n \gg 0$, alors ils sont nuls pour tout $n \geqslant 3$. Nous démontrons cette conjecture sous l'hypothèse supplémentaire qu'il existe un homomorphisme d'anneaux $\psi : S \to R$ tel que $\varphi \circ \psi = \mathrm{id}_S$. Plus précisément, nous montrons que dans ce cas ψ est d'intersection complète en $\varphi^{-1}(\mathfrak{n})$ pour tout idéal premier $\mathfrak{n}$ de S. En utilisant ces résultats, nous décrivons toutes les algèbres scindées $S \to R \to S$ pour lesquelles l'algèbre $\mathrm{Tor}^R_\bullet(S, S)$ est finiment engendrée sur $\mathrm{Tor}^R_0(S, S) = S$.

Introduction

Let $\varphi : R \to S$ be a homomorphism of commutative noetherian rings.

For each $n \geqslant 0$, let $\mathrm{D}_n(S \mid R; -)$ denote the nth cotangent homology functor on the category of S-modules, defined by André [1] and Quillen [25]. To study how the vanishing of these André–Quillen homology functors relates to the structure of φ, we define the *André–Quillen dimension* of S over R to be the number

$$\text{AQ-dim}_R S = \sup\{n \in \mathbb{N} \mid \mathrm{D}_n(S \mid R; -) \neq 0\};$$

in particular, $\text{AQ-dim}_R S = -\infty$ if and only if $\mathrm{D}_n(S \mid R; -) = 0$ for all $n \in \mathbb{Z}$.

The vanishing of André–Quillen homology in low dimensions characterizes important classes of homomorphisms of noetherian rings. Recall that φ is *regular* if it is flat with geometrically regular fibers. It is *étale* if, in addition, it is of finite type and unramified. A general *locally complete intersection*, or *l.c.i.*, property is defined in 7.2; when φ is of finite type, it means that

[1] Partly supported by a grant from the NSF.
[2] Supported by a grant from the EPSRC.

in some (equivalently, every) factorization of φ as an inclusion into a polynomial ring followed by a surjection, the kernel of the second map is locally generated by a regular sequence. The following results were proved in [1,25] for maps φ of finite type, and in [4,10] in general:

(A) $\text{AQ-dim}_R\, S = -\infty$ and φ is of finite type if and only if φ is étale.

(B) $\text{AQ-dim}_R\, S \leqslant 0$ if and only if $\mathrm{D}_1(S \mid R; -) = 0$, if and only if φ is regular.

(C) $\text{AQ-dim}_R\, S \leqslant 1$ if and only if $\mathrm{D}_2(S \mid R; -) = 0$, if and only φ is l.c.i.

Further research on homomorphisms of finite André–Quillen dimension has been driven by two conjectures, stated by Quillen in 1970. One of them, [25, (5.7)], is for maps *locally of finite flat dimension*: For each prime ideal $\mathfrak{n}$ of S the R-module $S_{\mathfrak{n}}$ has a finite resolution by flat R-modules. That conjecture was proved in [10]:

(D) $\text{AQ-dim}_R\, S < \infty$ and φ is locally of finite flat dimension if and only if φ is l.c.i.

As a consequence, if φ is locally of finite flat dimension, then $\text{AQ-dim}_R\, S < \infty$ implies $\text{AQ-dim}_R\, S \leqslant 1$. The remaining conjecture, [25, (5.6)], predicts the behavior of André–Quillen dimension when no flatness hypothesis is available.

QUILLEN'S CONJECTURE. – *If* $\text{AQ-dim}_R\, S < \infty$, *then* $\text{AQ-dim}_R\, S \leqslant 2$.

No structure theorem is known for R-algebras S with $\text{AQ-dim}_R\, S \leqslant 2$, so the conjecture presents a significant challenge beyond the generic difficulty of computing the modules $\mathrm{D}_n(S \mid R; M)$, defined in terms of simplicial resolutions. This partly explains why so few cases have been settled. In [10] the conjecture is proved when one of the rings R or S is a locally complete intersection. Indirect evidence is obtained in [21]: If φ is a large homomorphism of local rings in the sense of [23], R has characteristic 0, and $\text{AQ-dim}_R\, S$ is an odd integer, then $\text{AQ-dim}_R\, S = 1$.

Our main result establishes Quillen's Conjecture when S is an *algebra retract* of R, meaning that there exists a homomorphism of rings $\psi : S \to R$ such that $\varphi \circ \psi = \mathrm{id}_S$; any homomorphism ψ with this property is called a *section* of φ. Algebra retracts frequently arise from geometric considerations. For instance, to study a morphism of schemes $X \to Y$ one often uses the induced diagonal embedding $X \to X \times_Y X$. The underlying algebraic construction is the homomorphism of rings $\varphi : S \otimes_A S \to S$ defined by $\varphi(s' \otimes s'') = s's''$; the ring S is an algebra retract of $R = S \otimes_A S$, with section $\psi(s) = s \otimes 1$. A different type of retracts arises in constructions of projective schemes. They typically involve a graded S-algebra $R = \bigoplus_{i=0}^{\infty} R_i$ with $R_0 = S$; the relevant homomorphisms φ and ψ are, respectively, the canonical surjection $R \to (R/R_{\geqslant 1}) = S$ and the inclusion $S = R_0 \subseteq R$.

An important aspect of our result is that it connects the homological conditions in the conjecture through the structure of retracts of finite André–Quillen dimension. Let, as always, $\operatorname{Spec} S$ denote the set of prime ideals of S. If φ has a section ψ, then for every $\mathfrak{n} \in \operatorname{Spec} S$ one can find a set $\boldsymbol{x}$ of formal indeterminates over $S_{\mathfrak{n}}$ and an ideal $\mathfrak{b}$ contained in $\mathfrak{n}(\boldsymbol{x}) + (\boldsymbol{x})^2$ that fit into a commutative diagram

(E$_n$)
$$\begin{array}{ccccc} & & (R_{\mathfrak{m}})^* & & \\ & \overset{(\psi_{\mathfrak{m}})^*}{\nearrow} & \cong\uparrow & \overset{(\varphi_{\mathfrak{n}})^*}{\searrow} & \\ S_{\mathfrak{n}} & \xrightarrow{\psi'} & S_{\mathfrak{n}}[[\boldsymbol{x}]]/(\mathfrak{b}) & \xrightarrow{\varphi'} & S_{\mathfrak{n}} \end{array}$$

of homomorphisms of rings, where $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$, asterisks * denote $(\operatorname{Ker}(\varphi))$-adic completion, ψ' is the natural injection and φ' the surjection with kernel $(\boldsymbol{x})$.

For every real number c set $\lfloor c \rfloor = \sup\{i \in \mathbb{Z} \mid i \leqslant c\}$.

THEOREM I. – *Let $\varphi: R \to S$ be a homomorphism of rings and set $\mathfrak{a} = \operatorname{Ker}(\varphi)$. If φ admits a section and R is noetherian, then the following conditions are equivalent.*

(i) $\text{AQ-dim}_R S < \infty$.
(ii) $\text{AQ-dim}_R S \leqslant 2$.
(iii) $\mathrm{D}_3(S \mid R; -) = 0$.
(iv) $\mathrm{D}_n(S \mid R; -) = 0$ *for some* $n \geqslant 3$ *such that* $\lfloor \frac{n-1}{2} \rfloor!$ *is invertible in* S.
(v) *For each* $\mathfrak{n} \in \operatorname{Spec} S$*, the ideal* $\mathfrak{b}$ *in some* (*respectively, every*) *commutative diagram* (E_n) *is generated by a regular sequence.*

We apply the results discussed above in concrete cases, illustrating the known fact that all dimensions allowed under Quillen's Conjecture do occur.

Examples. – Let x, y be indeterminates over S. The natural homomorphisms

$$\begin{array}{ccccccccc} S & \longrightarrow & S[x,y] & \longrightarrow & S[x,y]/(x^2,xy,y^2) & \longrightarrow & S[x]/(x^2) & \longrightarrow & S \\ & & \| & & \| & & \| & & \\ & & P & & R & & T & & \end{array}$$

provide the following list of André–Quillen dimensions:

$$\begin{aligned} &\text{AQ-dim}_S S = \text{AQ-dim}_P P = \text{AQ-dim}_R R = \text{AQ-dim}_T T = -\infty, \\ &\text{AQ-dim}_S P = 0, \\ &\text{AQ-dim}_S T = \text{AQ-dim}_P T = \text{AQ-dim}_P S = 1, \\ &\text{AQ-dim}_T S = 2, \\ &\text{AQ-dim}_S R = \text{AQ-dim}_P R = \text{AQ-dim}_R T = \text{AQ-dim}_R S = \infty. \end{aligned}$$

Indeed, (A), (B), and (C) yield the equalities in the first three lines; (C) also implies $\text{AQ-dim}_T S \geqslant 2$. Because S is a retract of T, Theorem I provides the converse inequality; since T and S are retracts of R, the theorem also computes the last two dimensions on the last line. The two remaining dimensions on that line are given by (D), because R has finite flat dimension over S and over P.

We use Theorem I together with our results in [13] in a situation that does not *a priori* involve André–Quillen homology – the classical homology of an algebra retract $S \to R \to S$. In that case $\operatorname{Tor}_0^R(S,S) = S$ and $\operatorname{Tor}_\bullet^R(S,S)$ is a graded-commutative algebra with divided powers, but precise information on its structure is available in two instances only: when S is a field, cf. [22,19], or when $R \to S$ is locally complete intersection. Our second main result contains a description of all noetherian algebra retracts with finitely generated homology algebra.

Let $\operatorname{Max} S$ denote the set of maximal ideals of S.

THEOREM II. – *Let $S \xrightarrow{\psi} R \xrightarrow{\varphi} S$ be an algebra retract with noetherian ring R, and set $\operatorname{Max}' S = \{\mathfrak{n} \in \operatorname{Max} S \mid \operatorname{char}(S/\mathfrak{n}) > 0\}$. The following conditions are equivalent.*

(i) *The S-algebra $\operatorname{Tor}_\bullet^R(S,S)$ is finitely generated.*
(ii) *For every S-algebra T there exists an isomorphism of graded T-algebras*

$$\operatorname{Tor}_\bullet^R(S,T) \cong \Big(\bigwedge_S D_1 \otimes_S \operatorname{Sym}_S D_2\Big) \otimes_S T$$

where D_1 and D_2 are projective S-modules concentrated in degrees 1 and 2, respectively, and $(D_2)_{\mathfrak{n}'} = 0$ for all $\mathfrak{n}' \in \operatorname{Max}' S$.

(iii) *The* S*-modules* $\mathrm{D}_1(S \mid R; S)$ *and* $\mathrm{D}_2(S \mid R; S)$ *are projective,* $\mathrm{D}_3(S \mid R; S) = 0$*, and* $\mathrm{D}_2(S \mid R; S)_{\mathfrak{n}} = 0$ *for all* $\mathfrak{n} \in \operatorname{Max}' S$*.*
(iv) *For each* $\mathfrak{n} \in \operatorname{Spec} S$*, the ideal* $\mathfrak{b}$ *in some* (*respectively, every*) *commutative diagram* (E_n) *is generated by a regular sequence contained in* $(\boldsymbol{x})^2$*, and* $\mathfrak{b} = 0$ *if* $\mathfrak{n}$ *is contained in some* $\mathfrak{n}' \in \operatorname{Max}' S$*.*

If S is a flat algebra over some ring A, then $\operatorname{Tor}^{S \otimes_A S}_{\bullet}(S, S)$ is isomorphic to the *Hochschild homology algebra* $\mathrm{HH}_{\bullet}(S|A)$ of S over A. Our main result in [13] shows that if the ring $R = S \otimes_A S$ is noetherian, and $\mathrm{HH}_{\bullet}(S|A)$ is finitely generated as an algebra over S, then S is regular over A. On the other hand, by the Hochschild–Kostant–Rosenberg Theorem [20], as generalized by André [3], if S is regular over A then $\mathrm{HH}_{\bullet}(S|A) \cong \bigwedge_S D_1$. Thus, in the context of Hochschild homology the module D_2 in Theorem II is trivial. It is also trivial for algebra retracts where all the residue fields of S have positive characteristic. However, $\mathbb{Q} \to \mathbb{Q}[x]/(x^2) \to \mathbb{Q}$ has finitely generated Tor algebra with $D_2 \neq 0$.

We proceed with an overview of the contents of the article. Although its main topic is the simplicially defined André–Quillen homology theory, many arguments are carried out in the context of DG (= differential graded) homological algebra.

Section 1 contains basic definitions and results on DG algebras.

In Section 2 we recall the construction and first properties of non-negative integers $\varepsilon_n(\varphi)$, attached in [10] to every local homomorphism φ. These *deviations*, whose vanishing characterizes regularity and c.i. properties of φ, are linked to certain André–Quillen homology modules, but are easier to compute. Section 3 contains a general theorem on morphisms of minimal models of local rings. Its proof is long and difficult. Its applications go beyond the present discussion.

The next two sections are at the heart of our investigation.

In Section 4 we define a class of local homomorphisms, that we call *almost small*. It contains the small homomorphisms introduced in [8], and its larger size offers technical advantages that are essential to our study. We provide various characterizations of almost small homomorphisms and give examples. The key result established in this section is a structure theorem for surjective almost small homomorphisms of complete rings in terms of morphisms of DG algebras.

The proof of Theorem I depends on another new concept – that of *weak category* of a local homomorphism. It is defined in Section 5, where arguments from [10] are adapted in order to obtain information on the positivity and growth of deviations of homomorphisms with finite weak category. To apply these results to almost small homomorphisms we prove that they have finite weak category; the proof involves most of the material developed up to that point.

In Section 6 we return to André–Quillen homology, focusing on local homomorphisms of local rings. We show that vanishing of homology with coefficients in the residue field characterizes complete intersection homomorphisms among the homomorphisms having finite weak category. This leads to local versions of Theorems I and II above. The theorems themselves are proved in the final Section 7.

The main results of this paper were announced in [14], cf. also Remark 7.6. That article provides historical background, a more leisurely discussion of applications of André–Quillen homology to the structure of commutative algebras, and new proofs of some earlier results on the subject. Recently, J. Turner [28] has started a study of nilpotency in the homotopy of simplicial commutative algebras over a field of characteristic 2, with a view towards applications to Quillen's Conjecture.

1. Differential graded algebras

We use the theory of Eilenberg–Moore derived functors as described in [12, §1,§2]. We recall a minimum of material, referring for details to *loc. cit*.

1.1. Every graded object is concentrated in non-negative degrees, the differential of every complex has degree -1, and each DG algebra C is *graded commutative*:

$$c'c'' = (-1)^{|c'||c''|}c''c' \quad \text{for all } c', c'' \in C \quad \text{and} \quad c^2 = 0 \quad \text{for all } c \in C \text{ with } |c| \text{ odd}$$

where $|c|$ denotes the degree of c. The graded algebra underlying C is denoted $C^\natural$.

We set $C^{[2]} = C_0 + \partial(C_1)C_{\geqslant 1} + (C_{\geqslant 1})^2$ and $\operatorname{ind}(C) = C/C^{[2]}$. This is a complex of $\mathrm{H}_0(C)$-modules and every morphism of DG algebras $\gamma : C \to D$ induces a morphism of complexes of $\mathrm{H}_0(D)$-modules $\operatorname{ind}(\gamma) : \operatorname{ind}(C) \otimes_{\mathrm{H}_0(C)} \mathrm{H}_0(D) \to \operatorname{ind}(D)$.

1.2. A morphism $\gamma : C \to C'$ of DG algebras is a *quasiisomorphism* if it induces an isomorphism in homology; this is often signaled by the appearance of the symbol $\simeq$ next to its arrow. Let $C \to E$ be a morphism of DG algebras, such that the $C^\natural$-module $E^\natural$ is flat. If γ is a quasiisomorphism, then so is $\gamma \otimes_C E : E \to C' \otimes_C E$. If $\varepsilon : E \to E'$ is a quasiisomorphism and the graded $C^\natural$-module $E'^\natural$ is flat as well, then $C' \otimes_C \varepsilon : C' \otimes_C E \to C' \otimes_C E'$ is a quasiisomorphism.

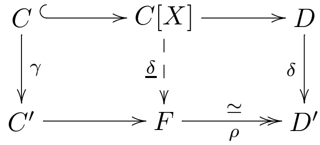
1.3. A *semifree extension* of C is a DG algebra $C[X]$ such that $C[X]^\natural$ is isomorphic to the tensor product over $\mathbb{Z}$ of $C^\natural$ with the *symmetric algebra* of a free $\mathbb{Z}$-module with basis $\bigsqcup_{i \geqslant 0} X_{2i}$ and the *exterior algebra* of a free $\mathbb{Z}$-module with basis $\bigsqcup_{i \geqslant 0} X_{2i+1}$; the differential of $C[X]$ extends that of C.

A *semifree* Γ-*extension* of C is a DG algebra $C\langle X' \rangle$ such that $C\langle X' \rangle^\natural$ is isomorphic to the tensor product over $\mathbb{Z}$ of $C^\natural$ with the *symmetric algebra* of a free $\mathbb{Z}$-module with basis X'_0, the *exterior algebra* of a free $\mathbb{Z}$-module with basis $\bigsqcup_{i \geqslant 0} X'_{2i+1}$ and the *divided powers algebra* of a free $\mathbb{Z}$-module with basis $\bigsqcup_{i \geqslant 1} X'_{2i}$; the differential of $C\langle X' \rangle$ extends that of C, and for every $x' \in X'_{2i}$ with $i \geqslant 1$ the jth divided power $x'^{(j)}$ satisfies $\partial(x'^{(j)}) = \partial(x')x'^{(j-1)}$ for all $j \geqslant 1$.

1.4. Any morphism of DG algebras $C \to D$ factors as the canonical injection

$$C \hookrightarrow C[X]$$

followed by a surjective quasiisomorphism $C[X] \twoheadrightarrow D$. If $\rho : F \to D'$ is a surjective quasiisomorphism, then for each commutative diagram



of morphisms of DG algebras displayed by solid arrows there exists a unique up to C-linear homotopy morphism $\underline{\delta}$ preserving commutativity.

1.5. The diagrams $D \leftarrow C \to E$ of DG algebras are the objects of a category, whose morphisms are commutative diagrams of DG algebras

$$\begin{array}{ccccc} D & \longleftarrow & C & \longrightarrow & E \\ {\scriptstyle\delta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\varepsilon}\big\downarrow \\ D' & \longleftarrow & C' & \longrightarrow & E' \end{array}$$

In view of 1.4, $\operatorname{Tor}^C_\bullet(D,E) = \mathrm{H}(C[X] \otimes_C E)$ and $\operatorname{Tor}^\gamma_\bullet(\delta,\varepsilon) = \mathrm{H}(\underline{\delta} \otimes_\gamma \varepsilon)$ define a functor from this category to that of graded algebras. A fundamental property of this functor is: If γ, δ, ε above are quasiisomorphisms, then $\operatorname{Tor}^\gamma_\bullet(\delta,\varepsilon)$ is bijective. By 1.2, each factorization $C \to F \xrightarrow{\simeq} D$ with $F^\natural$ flat over $C^\natural$ yields a unique isomorphism $\operatorname{Tor}^C_\bullet(D,E) \to \mathrm{H}(F \otimes_C E)$ of graded algebras.

1.6. A *DG* Γ*-algebra* is a DG algebra K in which a sequence $\{x^{(j)} \in K_{jn}\}_{j\geqslant 0}$ of *divided powers* is defined for each $x \in K_n$ with n even positive, and satisfies a list of standard identities; it can be found in full, say, in [19, (1.7.1), (1.8.1)]. A *morphism of DG* Γ*-algebras* $\varkappa : K \to L$ is a morphism of DG algebras such that $\varkappa(x^{(j)}) = (\varkappa(x))^{(j)}$ for all $x \in K$ with $|x|$ even positive and all $j \in \mathbb{N}$.

Let $K^{(2)}$ denote the K_0-submodule of K generated by $K^{[2]}$ and all $x^{(j)}$, where $|x|$ is even positive and $j \geqslant 2$. Set $\Gamma\text{-ind}(K) = K/K^{(2)}$. This is a complex of $\mathrm{H}_0(K)$-modules. Every morphism of Γ-algebras $\varkappa : K \to L$ induces a morphism

$$\Gamma\text{-ind}(\varkappa) : \Gamma\text{-ind}(K) \otimes_{\mathrm{H}_0(K)} \mathrm{H}_0(L) \to \Gamma\text{-ind}(L)$$

of complexes of $\mathrm{H}_0(L)$-modules.

1.7. If K is a DG Γ-algebra, then $K\langle X'\rangle$ has a unique structure of DG Γ-algebra extending that of K and preserving the divided powers of the variables $x' \in X'_{2i}$ with $i > 0$. Every morphism of DG Γ-algebras $\varkappa : K \to L$ can be factored as $K \hookrightarrow K\langle X'\rangle \twoheadrightarrow L$ with second map a surjective quasiisomorphism of DG Γ-algebras. If $\zeta : M \to L'$ is a surjective quasiisomorphism, then for each commutative diagram

$$\begin{array}{ccccc} K & \hookrightarrow & K\langle X'\rangle & \longrightarrow & L \\ {\scriptstyle\varkappa}\big\downarrow & & {\scriptstyle\underline{\lambda}}\,\vdots & & {\scriptstyle\lambda}\big\downarrow \\ K' & \longrightarrow & M & \underset{\zeta}{\overset{\simeq}{\twoheadrightarrow}} & L' \end{array}$$

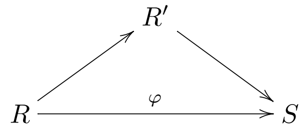
of morphisms of DG algebras displayed by solid arrows there exists a unique up to K-linear homotopy morphism of DG Γ-algebras $\underline{\lambda}$ making both squares commute.

1.8. Divided powers of a cycle are cycles, but divided powers of a boundary need not be boundaries. If they are, then the DG Γ-algebra K is called *admissible*, and $\mathrm{H}(K)$ inherits from K a structure of Γ-algebra. This notion of admissibility is less restrictive than the one adopted in [12], and lacks some of the desirable properties the latter posesses, but it suffices for the needs of this paper.

Let $K \twoheadrightarrow k$ be a surjective morphism of DG Γ-algebras, where k is a field concentrated in degree 0, and let $k \hookrightarrow l$ be a field extension. If $K \hookrightarrow K\langle X'\rangle \twoheadrightarrow k$ is a factorization as in 1.7, then the unique DG Γ-algebra structure on $l\langle X'\rangle = K\langle X'\rangle \otimes_K l$ is admissible, cf. [12, (2.6)] or [14, (3.4)]. Thus, Tor defines a functor from the category of diagrams $K \twoheadrightarrow k \hookrightarrow l$, with the obvious morphisms, to the category of Γ-algebras and their morphisms.

2. Factorizations of local homomorphisms

Let $\varphi:(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a homomorphism of local rings, which is *local* in the sense that $\varphi(\mathfrak{m})\subseteq\mathfrak{n}$. A *regular factorization* of φ is a commutative diagram

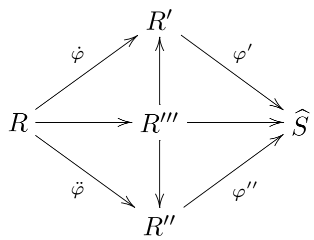


of local homomorphisms such that the R-module R' is flat, the ring $R'/\mathfrak{m}R'$ is regular, and the map $R'\to S$ is surjective.

Regular factorizations are often easily found, for instance, when φ is essentially of finite type (in particular, surjective), or when φ is the canonical embedding of R in its completion with respect to the maximal ideal. In this paper they are mostly used through the following construction of Avramov, Foxby, and Herzog [11].

2.1. If $\grave{\varphi}:R\to\widehat{S}$ is the composition of φ with the canonical inclusion $S\to\widehat{S}$, then by [11, (1.1)], $\grave{\varphi}$ has a regular factorization $R\xrightarrow{\dot{\varphi}}R'\xrightarrow{\varphi'}\widehat{S}$ with a complete local ring R'; it is called a *Cohen factorization* of $\grave{\varphi}$. By [11, (1.5)], it can be chosen to satisfy the additional condition $\operatorname{edim}R'/\mathfrak{m}R'=\operatorname{edim}S/\mathfrak{m}S$; we say that such a Cohen factorization is *reduced* (it is called minimal in [11]). Clearly, any regular factorization $\varphi=\pi\circ\iota$ gives rise to a Cohen factorization $\grave{\varphi}=\widehat{\pi}\circ\grave{\iota}$.

Cohen factorizations need not be isomorphic. However, if $R\xrightarrow{\ddot{\varphi}}R''\xrightarrow{\varphi''}\widehat{S}$ also is a Cohen factorization of $\grave{\varphi}$, then by [11, (1.2)] there exists a commutative diagram



of local homomorphisms, where the horizontal row is a Cohen factorization, and the vertical maps are surjections with kernels generated by regular sequences whose images in $R'''/\mathfrak{m}R'''$ can be completed to regular systems of parameters.

2.2. Let $(A,\mathfrak{p},k)$ be a local ring. We say that a semifree extension $A[X]$ has *decomposable differential* if $X=X_{\geqslant 1}$ and

$$\partial(X)\subseteq\mathfrak{p}A[X]+(X)^2A[X].$$

When this condition holds, for each $n\geqslant 1$ there are equalities

$$\mathrm{H}_n\big(A[X]/(\mathfrak{p},X_{<n})\big)=\mathrm{Z}_n\big(A[X]/(\mathfrak{p},X_{<n})\big)=kX_n.$$

2.3. Let $\varphi:(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism.

A *minimal model* of φ is a diagram $R\xrightarrow{\widetilde{\iota}}R'[U]\xrightarrow{\widetilde{\varphi}}S$ where the differential of $R'[U]$ is decomposable, $\widetilde{\varphi}$ is a quasiisomorphism, and $\varphi=\widetilde{\varphi}_0\circ\widetilde{\iota}_0$ is a regular factorization. If φ has a regular factorization (in particular, if $S = \widehat{S}$), then φ has a minimal model: The DG algebra $R'[U]$ is obtained by successively adjoining to R' sets of variables U_n of degree $n \geqslant 1$, so that $\partial(U_1)$ minimally generates $\mathrm{Ker}(\pi)$ and $\partial(U_n)$ is a minimal set of generators for $\mathrm{H}_{n-1}(R[U_{<n}])$, cf. [9, (2.1.10)].

The next proposition elaborates on [10, (3.1)].

2.4. PROPOSITION. – *Let $\varphi:(R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a local homomorphism and let $R \to R'[U'] \to \widehat{S}$ and $R \to R''[U''] \to \widehat{S}$ be minimal models of $\grave{\varphi}$.*

For each integer $n \geqslant 2$ there are equalities

$$\mathrm{card}(U'_1) - \mathrm{edim}(R'/\mathfrak{m}R') = \mathrm{card}(U''_1) - \mathrm{edim}(R''/\mathfrak{m}R''),$$
$$\mathrm{card}(U'_n) = \mathrm{card}(U''_n),$$

and there exist isomorphisms of DG algebras over the field l

$$l[U'_{\geqslant n}] = R'[U']/(\mathfrak{m}', U'_{<n}) \cong R''[U'']/(\mathfrak{m}'', U''_{<n}) = l[U''_{\geqslant n}].$$

Proof. – By 2.1 we may assume there is a surjection $R'' \to R'$ with kernel generated by a regular sequence $\boldsymbol{x}$ that extends to a minimal generating set of the maximal ideal $\mathfrak{m}''$ of R''. Changing U''_1 if need be, we may assume that $U'' = V \sqcup U$ with $\partial(V) = \boldsymbol{x}$. The canonical map $R''[V] \to R'$ is a quasiisomorphism, $R''[V]$ is a DG subalgebra of $R''[U'']$ and the $R''[V]^\natural$-module $R''[U'']^\natural$ is free, so the induced map

$$R''[U''] \to R''[U'']/\big(V, \partial(V)\big) = R'[U]$$

is a quasiisomorphism, cf. 1.2. Thus, $\mathrm{H}(R'[U]) \cong \widehat{S}$, and the differential of $R'[U]$ is decomposable because it is induced by that of $R''[U'']$. By [9, (7.2.3)] there exists an isomorphism $R'[U'] \cong R'[U]$ of DG algebras over R', so we get

$$R'[U']/(\mathfrak{m}', U'_{<n}) \cong R'[U]/(\mathfrak{m}', U_{<n})$$

for all $n \geqslant 1$. The algebra on the right is equal to $R''[U'']/(\mathfrak{m}'', U''_{<n})$ for $n \geqslant 2$, so we have proved the last assertion. In view of 2.2, it implies

$$lU'_n = \mathrm{H}_n\big(R'[U']/(\mathfrak{m}', U'_{<n})\big) \cong \mathrm{H}_n\big(R'[U]/(\mathfrak{m}', U_{<n})\big) = lU_n.$$

Thus, we obtain numerical equalities

$$\begin{aligned} \mathrm{card}(U'_n) &= \mathrm{card}(U_n) = \mathrm{card}(U''_n) \quad \text{for } n \geqslant 2; \\ \mathrm{card}(U'_1) &= \mathrm{card}(U_1) = \mathrm{card}(U''_1) - \mathrm{card}(V) \\ &= \mathrm{card}(U''_1) - \big(\mathrm{edim}(R''/\mathfrak{m}R'') - \mathrm{edim}(R'/\mathfrak{m}R')\big). \end{aligned}$$

All the assertions of the proposition have now been established. □

2.5. DEFINITION. – Let $\varphi:(R,\mathfrak{m},k) \to S$ be a local homomorphism, and let $R \to R'[U] \to \widehat{S}$ be a minimal model of $\grave{\varphi}$. The nth *deviation* of φ is the number

$$\varepsilon_n(\varphi) = \begin{cases} \mathrm{card}(U_1) - \mathrm{edim}(R'/\mathfrak{m}R') + \mathrm{edim}(S/\mathfrak{m}S) & \text{for } n = 2; \\ \mathrm{card}(U_{n-1}) & \text{for } n \geqslant 3. \end{cases}$$

By Proposition 2.4, these are invariants of φ. Deviations were defined in [10, §3] with a typo in the expression for $\varepsilon_2(\varphi)$, which is corrected above.

Note that $\varepsilon_n(\varphi) \geqslant 0$ for all n: this is clear for $n \geqslant 3$; for $n = 2$, use the equalities

$$\operatorname{card}(U_1) = \operatorname{rank}_l\left(\frac{\operatorname{Ker}(\varphi')}{\mathfrak{m}' \operatorname{Ker}(\varphi')}\right);$$

$$\operatorname{edim}(R'/\mathfrak{m}R') - \operatorname{edim}(S/\mathfrak{m}S) = \operatorname{rank}_l\left(\frac{\operatorname{Ker}(\varphi')}{\operatorname{Ker}(\varphi') \cap (\mathfrak{m}'^2 + \mathfrak{m}R')}\right).$$

The vanishing of deviations is linked to the structure of φ. We reproduce [10, (3.2)]:

2.6. PROPOSITION. – *If $\varphi : (R, \mathfrak{m}, k) \to S$ is a local homomorphism, then the following conditions are equivalent.*

(i) *φ is flat and $S/\mathfrak{m}S$ is regular.*
(ii) *$\varepsilon_n(\varphi) = 0$ for all $n \geqslant 2$.*
(iii) *$\varepsilon_2(\varphi) = 0$.*

Proof. – (i) $\Rightarrow$ (ii) The diagram $R \to \widehat{S} = \widehat{S}$ is a Cohen factorization of $\grave{\varphi}$, so $\grave{\varphi}$ has a minimal model with $U = \emptyset$.

(iii) $\Rightarrow$ (i) Choose a reduced Cohen factorization. By definition, $\varepsilon_2(\varphi) = 0$ entails $U_1 = \emptyset$, so $\widehat{S} = \mathrm{H}_0(R'[U]) = R'$, hence $\widehat{S}$ is flat over R and $\widehat{S}/\mathfrak{m}\widehat{S}$ is regular; these properties descend to S and $S/\mathfrak{m}S$. $\square$

The following notion is basic for the rest of the paper.

2.7. DEFINITION. – A local homomorphism $\varphi : R \to (S, \mathfrak{n}, l)$ is a *complete intersection* (or *c.i.*) at $\mathfrak{n}$, if in some Cohen factorization $R \to R' \xrightarrow{\varphi'} \widehat{S}$ of $\grave{\varphi}$ the ideal $\operatorname{Ker}(\varphi')$ is generated by an R'-regular sequence.

Other definitions of c.i. homomorphisms require additional hypotheses on φ; when they hold, the general concept specializes properly, cf. [10, (5.2), (5.3)]. The next proposition amplifies [10, (3.3)]; it shows, in particular, that the c.i. property is detected by every Cohen factorization.

2.8. PROPOSITION. – *If $\varphi : R \to (S, \mathfrak{n}, l)$ is a local homomorphism, then the following conditions are equivalent.*

(i) *φ is a complete intersection at $\mathfrak{n}$.*
(ii) *$\varepsilon_n(\varphi) = 0$ for all $n \geqslant 3$.*
(iii) *$\varepsilon_3(\varphi) = 0$.*

Proof. – In any minimal model $R \to R'[U] \to \widehat{S}$ of $\grave{\varphi}$ the DG algebra $R'[U_1]$ is the Koszul complex on a minimal set of generators of $\operatorname{Ker}(\varphi')$. If (i) holds, then $U = U_1$, so (i) implies (ii). If (iii) holds, then $\mathrm{H}_1(R'[U_1]) = 0$, so the ideal $\operatorname{Ker}(\varphi')$ is generated by a regular sequence. $\square$

3. Indecomposables

In this section we analyze the divided powers in Tor.

3.1. If $(R, \mathfrak{m}, k)$ is a local ring, then $\operatorname{Tor}^R_\bullet(k, k)$ is a Γ-algebra, cf. 1.8.

Using the functor $\Gamma\text{-ind}(-)$ of Γ-indecomposables defined in 1.6, we set

$$\pi_\bullet(R) = \Gamma\text{-ind}\big(\operatorname{Tor}^R_\bullet(k, k)\big).$$

If $\overline{\varphi} : k \rightarrowtail l$ is a field extension, then the canonical isomorphism

$$\mathrm{Tor}^R_\bullet(k,k)\otimes_k l \cong \mathrm{Tor}^R_\bullet(k,l)$$

is one of Γ-algebras, and so induces an isomorphism of graded l-vector spaces

$$\pi_\bullet(R)\otimes_k l \cong \Gamma\text{-ind}\big(\mathrm{Tor}^R_\bullet(k,l)\big)$$

that we use as identification. Thus, every local homomorphism $\varphi\colon R\to(S,\mathfrak{n},l)$ defines an l-linear homomorphism of graded vector spaces

$$\pi_\bullet(\varphi)\colon \pi_\bullet(R)\otimes_k l \xrightarrow{\Gamma\text{-ind}(\mathrm{Tor}^\varphi_\bullet(\overline{\varphi},l))} \pi_\bullet(S).$$

3.2. *Example*. – Let $(R,\mathfrak{m},k)$ be a local ring. An *acyclic closure* of k is a factorization $R\to R\langle X'\rangle\to k$ of the epimorphism $R\to k$, as in 1.7, constructed so that $\partial(X'_1)$ minimally generates $\mathfrak{m}$ and $\partial(X'_n)$ minimally generates $\mathrm{H}_{n-1}(R\langle X'_{<n}\rangle)$ for each $n\geqslant 2$, cf. [9, (6.3)]. By an important theorem of Gulliksen [18] and Schoeller [26], in this case $\partial(R\langle X'\rangle)\subseteq \mathfrak{m}R\langle X'\rangle$, cf. also [9, (6.3.4)]. This yields isomorphisms

$$\pi_n(R)\cong kX'_n \quad \text{for all } n\in\mathbb{Z}.$$

The nth *deviation* of R is the number $\varepsilon_n(R)=\operatorname{card}X'_n$. They measure the singularity of R: $\varepsilon_n(R)=0$ for all $n\geqslant 2$ if and only if $\varepsilon_2(R)=0$, if and only if R is regular; $\varepsilon_n(R)=0$ for all $n\geqslant 3$ if and only if $\varepsilon_3(R)=0$, if and only if R is c.i., cf. [19, Ch. III], [9, §7]. These results can be derived from Propositions 2.6 and 2.8, since by [9, (7.2.5)] deviations of rings and of homomorphisms are linked as follows:

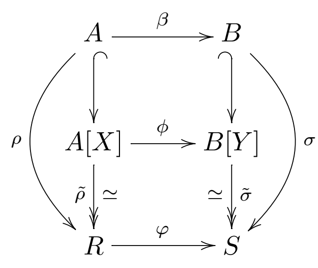
3.3. If $\varphi\colon A\to R$ is a surjective local homomorphism with A regular, then

$$\varepsilon_n(\varphi)=\varepsilon_n(R) \quad \text{for all } n\geqslant 2.$$

The next result is a functorial enhancement of the numerical equality above.

3.4. THEOREM. – *Consider a commutative diagram of morphisms of DG algebras*

(1)



where $(R,\mathfrak{m},k)$ and $(S,\mathfrak{n},l)$ are local rings, $(A,\mathfrak{p},k)$ and $(B,\mathfrak{q},l)$ are regular local rings, the homomorphisms φ and β are local, the homomorphisms ρ and σ are surjective, $\mathrm{Ker}(\rho)\subseteq\mathfrak{p}^2$ *and* $\mathrm{Ker}(\sigma)\subseteq\mathfrak{q}^2$*, and the triangles are minimal models.*

For each $n\geqslant 2$ there exists a commutative diagram of homomorphisms

$$\begin{array}{ccc} \pi_n(R)\otimes_k l & \xrightarrow{\ \pi_n(\varphi)\ } & \pi_n(S) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ \operatorname{ind}_{n-1}(l[X]) & \xrightarrow{\ \operatorname{ind}_{n-1}(\phi\otimes_\beta l)\ } & \operatorname{ind}_{n-1}(l[Y]) \end{array}$$

of l-vector spaces, where the vertical arrows are isomorphisms.

The theorem shows that $\pi_n(\varphi)$ and $\operatorname{ind}_{n-1}(\phi\otimes_\beta l)$ determine each other. These are very different maps: the first is induced by a morphism of DG Γ-algebras, while divided powers have no role in the construction of the second. This accounts for the intricacies of the proof. In it, and later in the paper, it is convenient to suppress the effect of $\operatorname{Tor}_1^R(k,k)$ on $\operatorname{Tor}_\bullet^R(k,k)$. We do that in a systematic way.

3.5. The *reduced torsion algebra* of a local ring $(R,\mathfrak{m},k)$ is the k-algebra

$$\operatorname{tor}_\bullet^R(k,k) = \frac{\operatorname{Tor}_\bullet^R(k,k)}{\operatorname{Tor}_\bullet^R(k,k)\cdot\operatorname{Tor}_1^R(k,k)}.$$

Since $\operatorname{Tor}_\bullet^R(k,k)$ is a Γ-algebra and the ideal $J=\operatorname{Tor}_\bullet^R(k,k)\cdot\operatorname{Tor}_1^R(k,k)$ is generated by elements of degree 1, basic properties of divided powers imply that each element of even degree $a\in J$ satisfies $a^{(i)}\in J$ for all $i\geqslant 1$. It follows that $\operatorname{tor}_\bullet^R(k,k)$ admits a unique Γ-structure for which the canonical surjection $\operatorname{Tor}_\bullet^R(k,k)\to\operatorname{tor}_\bullet^R(k,k)$ becomes a morphism of Γ-algebras, hence

$$\pi_{\geqslant 2}(R)=\Gamma\text{-ind}\big(\operatorname{tor}_\bullet^R(k,k)\big).$$

If $\varphi:R\to(S,\mathfrak{n},l)$ is a local homomorphism, then $\operatorname{Tor}_\bullet^\varphi(\overline{\varphi},l)$ induces a morphism

$$\operatorname{tor}_\bullet^\varphi(\overline{\varphi},l):\operatorname{tor}_\bullet^R(k,l)\to\operatorname{tor}_\bullet^S(l,l)$$

Γ-algebras, so for $n\geqslant 2$ we get commutative diagrams of l-linear homomorphisms

$$\begin{array}{ccc} \pi_n(R)\otimes_k l & \xrightarrow{\ \pi_n(\varphi)\ } & \pi_n(S) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ \Gamma\text{-ind}_n(\operatorname{tor}_\bullet^R(k,l)) & \xrightarrow{\ \Gamma\text{-ind}_n(\operatorname{tor}_\bullet^\varphi(\overline{\varphi},l))\ } & \Gamma\text{-ind}_n(\operatorname{tor}_\bullet^S(l,l)) \end{array}$$

The *proof* of Theorem 3.4 takes up the rest of the section. Only its statement is used later, so the reader may skip to the next section without loss of continuity.

We start by forming a diagram of morphisms of DG Γ-algebras

$$\begin{array}{ccc} A & \xrightarrow{\ \beta\ } & B \\ \big\downarrow & & \big\downarrow \\ A\langle X_1'\rangle & \xrightarrow{\ \varkappa\ } & B\langle Y_1'\rangle \\ {\scriptstyle\varepsilon}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\eta} \\ k & \xrightarrow{\ \overline{\varphi}\ } & l \end{array} \tag{2}$$

in the following order. First we form the vertical sides by choosing them to be acyclic closures of the respective residue fields. Next we note that since both A and B are regular local rings, the DG algebras $A\langle X_1'\rangle$ and $B\langle Y_1'\rangle$ are Koszul complexes on minimal sets of generators of $\mathfrak{p}$ and $\mathfrak{q}$, respectively. Finally, we use 1.7 to choose a morphism $\varkappa$ that preserves the commutativity of the rectangle.

Base change from Diagram (2) yields the central rectangles in the diagram

(3)
$$\begin{array}{ccccccc}
 & & & \overset{\underline{\varphi}}{\frown} & & & \\
R\langle X'\rangle & \hookleftarrow & R & \xrightarrow{\varphi} & S & \hookrightarrow & S\langle Y'\rangle \\
\| & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\theta} & & \| \\
R\langle X'\rangle & \hookleftarrow & R\langle X_1'\rangle & \xrightarrow{\varphi\otimes_\beta\varkappa} & S\langle Y_1'\rangle & \hookrightarrow & S\langle Y'\rangle \\
 & \searrow{\scriptstyle\simeq} & \downarrow & & \downarrow & \swarrow{\scriptstyle\simeq} & \\
 & & k & \xrightarrow{\overline{\varphi}} & l & &
\end{array}$$

of morphisms of DG Γ-algebras. The rest is constructed as follows. In view of the hypotheses on ρ and σ, minimal sets of generators of $\mathfrak{p}$ and $\mathfrak{q}$ map to minimal sets of generators of $\mathfrak{m}$ and $\mathfrak{n}$, respectively. By Example 3.2 the DG algebras $R\langle X_1'\rangle = R\otimes_A A\langle X_1'\rangle$ and $S\langle Y_1'\rangle = S\otimes_B B\langle Y_1'\rangle$ can be extended to acyclic closures $R\hookrightarrow R\langle X'\rangle \twoheadrightarrow k$ and $S\hookrightarrow S\langle Y'\rangle \twoheadrightarrow l$. Finally, the morphism $\underline{\varphi}$ is chosen so as to preserve the commutativity of the diagram: this is possible by 1.7.

3.6. LEMMA. – *Diagram* (3) *induces a commutative diagram*

$$\begin{array}{ccccccc}
l\langle X'_{\geqslant 2}\rangle & \xrightarrow{\cong} & \operatorname{tor}^R_\bullet(k,l) & \xrightarrow{\operatorname{tor}^{\varphi}_\bullet(\overline{\varphi},l)} & \operatorname{tor}^S_\bullet(l,l) & \xleftarrow{\cong} & l\langle Y'_{\geqslant 2}\rangle \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\
 & & \operatorname{Tor}^{R\langle X_1'\rangle}_\bullet(k,l) & \xrightarrow{\operatorname{Tor}^{\varphi\otimes_\beta\varkappa}_\bullet(\overline{\varphi},l)} & \operatorname{Tor}^{S\langle Y_1'\rangle}_\bullet(l,l) & &
\end{array}$$

of homomorphisms of Γ*-algebras.*

Proof. – By construction, $R\langle X'\rangle$ and $S\langle Y'\rangle$ are acyclic closures. In view of 3.2, this means that there are inclusions $\partial(R\langle X'\rangle)\subseteq \mathfrak{m}R\langle X'\rangle$ and $\partial(S\langle Y'\rangle)\subseteq \mathfrak{n}S\langle Y'\rangle$. These inclusions provide the equalities in the commutative diagram

$$\begin{array}{ccccccc}
 & & & \overset{\underline{\varphi}\otimes_\varphi l}{\frown} & & & \\
l\langle X'\rangle & = & \operatorname{Tor}^R_\bullet(k,l) & \xrightarrow{\operatorname{Tor}^{\varphi}_\bullet(\overline{\varphi},l)} & \operatorname{Tor}^S_\bullet(l,l) & = & l\langle Y'\rangle \\
\downarrow & & \downarrow{\scriptstyle\operatorname{Tor}^{\tau}_\bullet(k,l)} & & \downarrow{\scriptstyle\operatorname{Tor}^{\theta}_\bullet(l,l)} & & \downarrow \\
l\langle X'_{\geqslant 2}\rangle & = & \operatorname{Tor}^{R\langle X_1'\rangle}_\bullet(k,l) & \xrightarrow{\operatorname{Tor}^{\varphi\otimes_\beta\varkappa}_\bullet(\overline{\varphi},l)} & \operatorname{Tor}^{S\langle Y_1'\rangle}_\bullet(l,l) & = & l\langle Y'_{\geqslant 2}\rangle
\end{array}$$

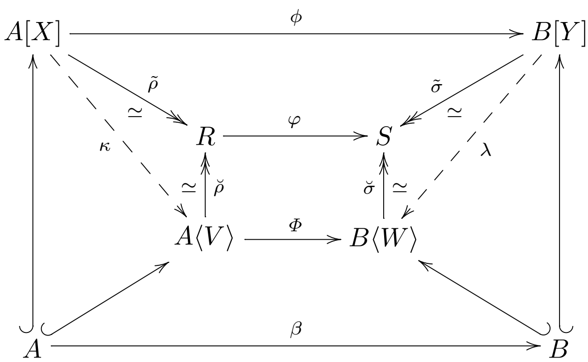
induced by Diagram (3). By 1.8, all the maps are morphisms of Γ-algebras.

The inclusions noted above also show that the external vertical maps are the canonical surjections of graded algebras, whose kernels are the ideals generated by X_1' and Y_1' respectively.

It follows that $\mathrm{Ker}(\mathrm{Tor}^{\tau}_{\bullet}(k,l))$ is generated by $\mathrm{Tor}^{R}_{1}(k,l)$, and $\mathrm{Ker}(\mathrm{Tor}^{\theta}_{\bullet}(l,l))$ is generated by $\mathrm{Tor}^{S}_{1}(l,l)$. In view of the definition of the reduced Tor functor in 3.5, the diagram above induces the desired diagram. □

We refine Diagram (1) to a commutative diagram of morphisms of DG algebras

(4)



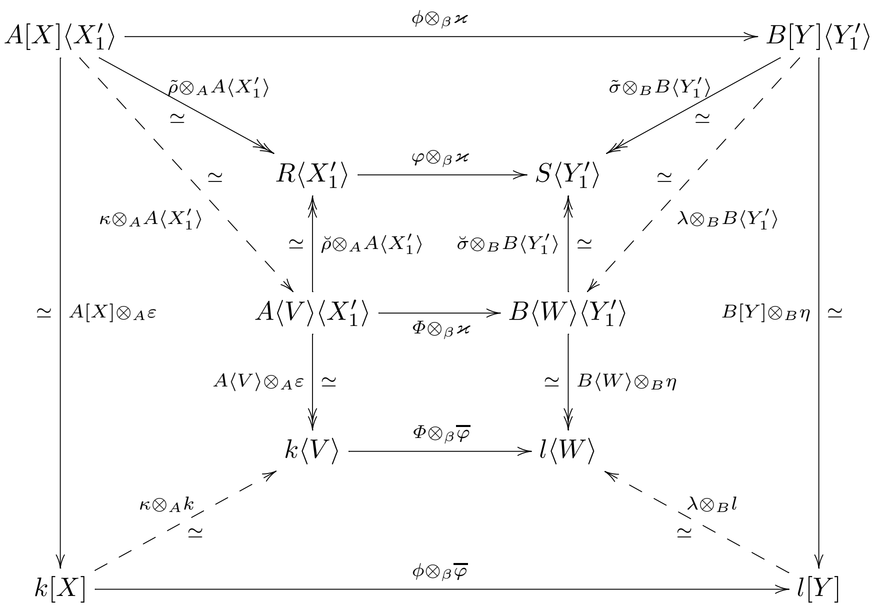
by performing the following steps. First we invoke 1.7 to construct factorizations

$$A \hookrightarrow A\langle V\rangle \xrightarrow{\breve\rho} R \quad \text{of } \rho \quad \text{and} \quad B \hookrightarrow B\langle W\rangle \xrightarrow{\breve\sigma} S \quad \text{of } \sigma.$$

Next we choose by 1.7 a morphism of DG Γ-algebras Φ so as to preserve the commutativity of the already constructed part of the diagram. Finally, we use 1.4 to obtain morphisms of DG algebras κ and λ which preserve the commutativity of the lateral trapezoids.

It should be noted at this point that, in general, $\Phi\kappa \neq \lambda\phi$. Using Diagrams (2) and (4) we produce a diagram of morphisms of DG algebras

(5)



where the central rectangles are formed by morphisms of DG Γ-algebras, all non-horizontal arrows are quasiisomorphisms due to 1.2, and almost all paths commute – the possible exception being the paths around the two trapezoids with horizontal bases and hyphenated sides.

From 1.5 and 1.8 we deduce the following result.

3.7. LEMMA. – *The maps in Diagram* (5) *induce a commutative diagram*

$$\begin{array}{ccc} \operatorname{Tor}_\bullet^{R\langle X_1'\rangle}(k,l) & \xrightarrow{\operatorname{Tor}_\bullet^{\varphi\otimes_\beta\varkappa}(\overline{\varphi},l)} & \operatorname{Tor}_\bullet^{S\langle Y_1'\rangle}(l,l) \\ \cong\downarrow & & \downarrow\cong \\ \operatorname{Tor}_\bullet^{k\langle V\rangle}(k,l) & \xrightarrow{\operatorname{Tor}_\bullet^{\Phi\otimes_\beta\overline{\varphi}}(\overline{\varphi},l)} & \operatorname{Tor}_\bullet^{l\langle W\rangle}(l,l) \end{array}$$

of homomorphisms of Γ*-algebras, where the vertical maps are isomorphisms.*

We pause to recall some classical material on bar-constructions.

3.8. Let C be a *connected* DG algebra over the field k, which means that $C_0 = k$ and $\partial(C_1) = 0$. The *bar construction* $(\overline{\mathrm{B}}_k(C), \bar{\partial})$ is a connected DG Γ-algebra over k, with multiplication (called *shuffle product*) and divided powers constructed in [15, Exp. 7, §1)]; cf. also [24, Chapter X, §12]. It has a basis consisting of symbols $[c_1|c_2|\cdots|c_p]$ of degree $p + |c_1| + \cdots + |c_p|$, where the c_i range independently over a basis of $C_{\geqslant 1}$ and $p = 0, 1, 2 \ldots.$ The element $[c_1|c_2|\cdots|c_p]$ has *weight* p; the weight of $x \cdot y$ is the sum of those of x and y; if $|x|$ is even positive, then the weight of $x^{(i)}$ is i times that of x. In general, the DG Γ-algebra $\overline{\mathrm{B}}_k(C)$ is not admissible.

There exists a DG algebra $(\mathrm{B}_k(C), \partial)$ such that $\mathrm{B}_k(C)^\natural = C^\natural \otimes_k \overline{\mathrm{B}}_k(C)^\natural$ as graded algebras, ∂ extends the differential of C, the isomorphism $\mathrm{B}_k(C) \otimes_C k \cong \overline{\mathrm{B}}_k(C)$ is one of DG algebras, and the augmentation $\mathrm{B}_k(C) \to k$ is a quasiisomorphism of DG algebras. If C is a DG Γ-algebra, then by [15, Exp. 7, §5)] so is $\mathrm{B}_k(C)$, the map $\mathrm{B}_k(C) \to \overline{\mathrm{B}}_k(C)$ is a morphism of DG Γ-algebras, and $\overline{\mathrm{B}}_k(C)$ is admissible.

The bar construction is natural for morphisms $\gamma : C \to C'$ of connected DG k-algebras; a morphism of DG Γ-algebras $\overline{\mathrm{B}}_k(\gamma) : \overline{\mathrm{B}}_k(C) \to \overline{\mathrm{B}}_k(C')$ is given by

$$\overline{\mathrm{B}}_k(\gamma)\bigl([c_1|\cdots|c_p]\bigr) = \bigl[\gamma(c_1)|\cdots|\gamma(c_p)\bigr]. \tag{6}$$

There is a canonical isomorphism $\overline{\mathrm{B}}_k(C) \otimes_k l \cong \overline{\mathrm{B}}_l(C \otimes_k l)$ of DG Γ-algebras over l. In conjunction with 1.5, it induces isomorphisms of graded algebras

$$\operatorname{Tor}_\bullet^C(k,l) \cong \mathrm{H}\bigl(\mathrm{B}_k(C) \otimes_C l\bigr) = \mathrm{H}\bigl(\overline{\mathrm{B}}_k(C) \otimes_k l\bigr) \cong \mathrm{H}\bigl(\overline{\mathrm{B}}_l(C \otimes_k l)\bigr)$$

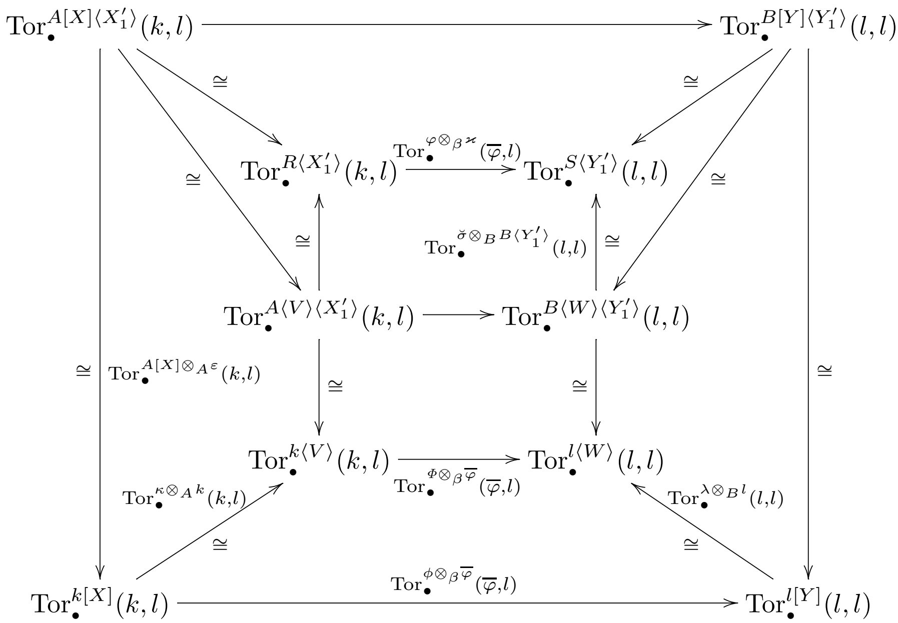
which are natural with respect to morphisms of connected DG k-algebras. When C is a DG Γ-algebra the isomorphisms above are of Γ-algebras, cf. 1.8.

3.9. LEMMA. – *The DG* Γ*-algebras* $\overline{\mathrm{B}}_l(l[X])$ *and* $\overline{\mathrm{B}}_l(l[Y])$ *are admissible, and the maps in Diagram* (5) *induce a commutative diagram*

$$\begin{array}{ccc} \operatorname{Tor}_\bullet^{k\langle V\rangle}(k,l) & \xrightarrow{\operatorname{Tor}_\bullet^{\Phi\otimes_\beta\overline{\varphi}}(\overline{\varphi},l)} & \operatorname{Tor}_\bullet^{l\langle W\rangle}(l,l) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}\overline{\mathrm{B}}_l(l[X]) & \xrightarrow{\mathrm{H}\overline{\mathrm{B}}_l(\phi\otimes_\beta l)} & \mathrm{H}\overline{\mathrm{B}}_l(l[Y]) \end{array}$$

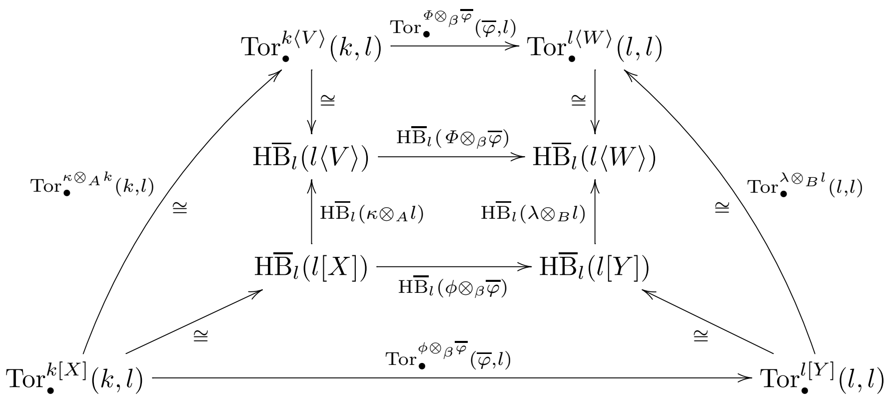
of morphisms of Γ*-algebras over* l*, where the vertical maps are isomorphisms.*

Proof. – Diagram (5) induces a diagram of homomorphisms of graded algebras



where the non-horizontal maps are bijective by 1.5. All paths commute, except possibly those around two trapezoids with horizontal bases – the one on the floor and the larger of the pair at the ceiling. Composing either path from $\mathrm{Tor}_\bullet^{A[X]\langle X_1'\rangle}(k,l)$ to $\mathrm{Tor}_\bullet^{B\langle W\rangle\langle Y_1'\rangle}(l,l)$ with the isomorphism $\mathrm{Tor}_\bullet^{\check{\sigma}\otimes_B B\langle Y_1'\rangle}(l,l)$ we get the same map, so the upper trapezoid commutes. Using this, we see that the isomorphism $\mathrm{Tor}_\bullet^{A[X]\otimes_A\varepsilon}(k,l)$, composed with either path from $\mathrm{Tor}_\bullet^{k[X]}(k,l)$ to $\mathrm{Tor}_\bullet^{l\langle W\rangle}(l,l)$, yields the same map, so the lower trapezoid commutes as well.

We inflate this trapezoid to a diagram of homomorphisms of graded algebras



From 3.8 we know that the maps pointing inward from the corners are bijective, and that the upper rectangle, both triangles, and the inner trapezoid commute. We conclude that the lower rectangle commutes and its vertical arrows are bijective.

Referring to 3.8 again, we note that all maps in the lower rectangle are induced by morphisms of DG Γ-algebras, and that $\overline{\mathrm{B}}_l(l\langle V\rangle)$ and $\overline{\mathrm{B}}_l(l\langle W\rangle)$ are admissible. As $\overline{\mathrm{B}}_l(\kappa\otimes_A l)$ and $\overline{\mathrm{B}}_l(\lambda\otimes_B l)$

are quasiisomorphisms, it follows that the DG Γ-algebras $\overline{\mathrm{B}}_l(l[X])$ and $\overline{\mathrm{B}}_l(l[Y])$ are admissible and the maps in the lower rectangle are isomorphisms of Γ-algebras. To finish the proof we remark, with a final reference to 3.8, that the upper rectangle is formed by homomorphisms of Γ-algebras. □

3.10. Let C be a connected DG algebra over l. The differential $\overline{\partial}$ of the bar construction $\overline{\mathrm{B}}_l(C)$ has the form $\partial' + \partial''$, where

$$\partial'\big([c_1|c_2|\cdots|c_p]\big) = \sum_{j=1}^{p-1}(-1)^{|c_1|+\cdots+|c_j|+j}[c_1|\cdots|c_jc_{j+1}|\cdots|c_p],$$

$$\partial''\big([c_1|c_2|\cdots|c_p]\big) = \sum_{j=1}^{p}(-1)^{|c_1|+\cdots+|c_{j-1}|+j}[c_1|\cdots|\partial(c_j)|\cdots|c_p].$$

Thus, the l-span $\mathrm{F}^q(C)$ of the elements $[c_1|\cdots|c_p]$ of degree at most $(p+q)$ for $p = 0,1,2\ldots,$ is a subcomplex of $\overline{\mathrm{B}}_l(C)$. The page ${}^0\mathrm{E}$ of the spectral sequence of the filtration $\{\mathrm{F}^q(C)\}$ is a complex of graded vector spaces with ${}^0d$ induced by ∂'. It can also be obtained by tensoring with l over $C^\natural$ the complex of graded $C^\natural$-modules

$$\cdots \to C^\natural \otimes_l \Sigma^p\big(C^{\natural}_{\geqslant 1}{}^{\otimes p}\big) \xrightarrow{\delta_p} C^\natural \otimes_l \Sigma^{p-1}\big(C^{\natural}_{\geqslant 1}{}^{\otimes (p-1)}\big) \to \cdots$$

(7)
$$\delta\big(\Sigma^p(c_0\otimes c_1\otimes\cdots\otimes c_p)\big) = \sum_{j=0}^{p-1}(-1)^j\Sigma^{p-1}(c_0\otimes\cdots\otimes c_jc_{j+1}\otimes\cdots\otimes c_p)$$

where for a graded vector space M we let $\Sigma^p M$ denote the graded space with $(\Sigma^p M)_n = M_{n-p}$ for all n, and $\Sigma^p : M \to \Sigma^p M$ be the degree p bijection defined by the maps $\mathrm{id}M_n$. The complex (7) is the standard resolution of l by free graded $C^\natural$-modules, so the spectral sequence of the filtration $\{\mathrm{F}^q(C)\}$ has

(8)
$${}^1\mathrm{E}_{p,q} = \mathrm{Tor}_p^{C^\natural}(l,l)_q \Rightarrow \mathrm{H}\overline{\mathrm{B}}_l(C).$$

In particular, the following equalities hold:

$${}^1\mathrm{E}_{p,q} = \begin{cases} 0 & \text{for } p \leqslant 0 \text{ and all } q \text{ except for } (p,q) = (0,0); \\ \mathrm{ind}_q(C) & \text{for } p = 1 \text{ and all } q. \end{cases}$$

The differentials of the spectral sequence act according to the pattern

$${}^r d_{p,q} : {}^r\mathrm{E}_{p,q} \to {}^r\mathrm{E}_{p+r-1,q-r} \quad \text{for each } r \geqslant 0$$

so for every $n \geqslant 1$ at the edge $p = 1$ the spectral sequence defines l-linear maps

$$\mathrm{H}\overline{\mathrm{B}}_l(C)_n \twoheadrightarrow {}^\infty\mathrm{E}_{1,n-1} \rightarrowtail \cdots \rightarrowtail {}^1\mathrm{E}_{1,n-1} = \mathrm{ind}_{n-1}(C),$$

where the kernel of the first one is the image of $\mathrm{H}_n(\mathrm{F}^{n-2}(C)) \to \mathrm{H}\overline{\mathrm{B}}_l(C)_n$. Shuffle products and divided powers in $\overline{\mathrm{B}}_k(C)$ are homogeneous with respect both to degree and to weight, cf. 3.8, so the subspace $\overline{\mathrm{B}}_l(C)_n^{(2)}$ of 1.6 is contained in $\mathrm{F}^{n-2}(C)_n$. Thus, if $\overline{\mathrm{B}}_l(C)$ is admissible, then for each $n \geqslant 1$ the maps above define a composition

(9)
$$\nu_n^C : \Gamma\text{-}\mathrm{ind}_n(\mathrm{H}\overline{\mathrm{B}}_l(C)) \twoheadrightarrow {}^\infty\mathrm{E}_{1,n-1} \rightarrowtail \cdots \rightarrowtail {}^1\mathrm{E}_{1,n-1} = \mathrm{ind}_{n-1}(C)$$

of l-linear homomorphisms. Formula (6) yields inclusions $\overline{\mathrm{B}}_l(\gamma)(\mathrm{F}^q(C)) \subseteq \mathrm{F}^q(C')$ for all q and every morphism $\gamma : C \to C'$ of connected DG algebras over l. It follows that the spectral sequence (8) above is natural with respect to such morphisms, and hence so are its edge homomorphisms ν_n^C.

Proof of Theorem 3.4. – For each $n \geqslant 2$ we form a commutative diagram

$$
\begin{array}{ccccccc}
 & & \pi_n(R) \otimes_k l & \xrightarrow{\ \pi_n(\varphi)\ } & \pi_n(S) & & \\
 & & \| & & \| & & \\
lX'_n & \xrightarrow{\cong} & \Gamma\text{-ind}_n(\mathrm{tor}^R_\bullet(k,l)) & \xrightarrow{\ \Gamma\text{-ind}_n(\mathrm{tor}^\varphi_\bullet(\overline{\varphi},l))\ } & \Gamma\text{-ind}_n(\mathrm{tor}^S_\bullet(l,l)) & \xleftarrow{\cong} & lY'_n \\
 & & {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} & & \\
 & & \Gamma\text{-ind}_n(\mathrm{H}\overline{\mathrm{B}}_l(l[X])) & \xrightarrow{\ \Gamma\text{-ind}_n(\mathrm{H}\overline{\mathrm{B}}_l(\phi\otimes_\beta l))\ } & \Gamma\text{-ind}_n(\mathrm{H}\overline{\mathrm{B}}_l(l[Y])) & & \\
 & & {\scriptstyle\nu_n^{l[X]}}\downarrow & & \downarrow{\scriptstyle\nu_n^{l[Y]}} & & \\
 & & \mathrm{ind}_{n-1}(l[X])) & \xrightarrow{\ \mathrm{ind}_{n-1}(\phi\otimes_\beta l)\ } & \mathrm{ind}_{n-1}(l[Y]) & &
\end{array}
$$

of l-vector spaces as follows: The top rectangle comes from 3.5. The middle part is obtained by stacking the commutative diagrams of Lemmas 3.6, 3.7, and 3.9, then taking Γ-indecomposables, as in 1.6. The bottom rectangle reflects the naturality of the edge homomorphisms ν_n defined in (9).

To finish the proof we show that its vertical maps are bijective. It suffices to do this for $\nu_n^{l[X]}$. By the isomorphisms above and 3.3, its source and target have the same rank, so it is enough to prove that it is surjective. To this end we analyze the spectral sequence (8). A well known computation, cf. e.g. [9, (7.2.9)], gives

$$
{}^1\mathrm{E}_{p,q} = \mathrm{Tor}_p^{l[X]^\natural}(l,l)_q \cong l\langle X''\rangle_{p,q} \quad \text{where } \mathrm{card}\, X''_{m,n} = \begin{cases} \mathrm{card}\, X_n & \text{if } m=1; \\ 0 & \text{otherwise.} \end{cases}
$$

From 3.3 we know that $\mathrm{card}(X_n) = \mathrm{card}(X'_{n+1})$ for $n \geqslant 1$, so the graded vector space associated with the bigraded space ${}^1\mathrm{E}$ is isomorphic to $l\langle X'_{\geqslant 2}\rangle$. By Lemmas 3.6, 3.7, and 3.9 the latter space is isomorphic to $\mathrm{H}\overline{\mathrm{B}}_l(l[X])$. This is the abutment of the spectral sequence (8), so it is isomorphic to the graded vector space associated with the bigraded space ${}^\infty\mathrm{E}$. Putting these remarks together, for each n we get

$$
\sum_{p+q=n} \mathrm{rank}_l\, {}^1\mathrm{E}_{p,q} = \mathrm{rank}_l\big(l\langle X'_{\geqslant 2}\rangle_n\big) = \mathrm{rank}_l\, \mathrm{H}_n\big(\overline{\mathrm{B}}_l(l[X])\big) = \sum_{p+q=n} \mathrm{rank}_l\, {}^\infty\mathrm{E}_{p,q}.
$$

They imply that the spectral sequence (8) stops on the page ${}^1\mathrm{E}$, so in the decomposition (9) the injections ${}^{r+1}\mathrm{E}_{1,n-1} \rightarrowtail {}^r\mathrm{E}_{1,n-1}$ are bijective for all $n \geqslant 1$ and $1 \leqslant r \leqslant \infty$. As a consequence, the map $\nu_n^{l[X]}$ is surjective, as desired. □

4. Almost small local homomorphisms

We introduce a class of maps of major importance for this paper.

4.1. DEFINITION. – A local homomorphism $\varphi:(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ is said to be *almost small* if the kernel of the homomorphism $\operatorname{Tor}^{\varphi}_{\bullet}(\overline{\varphi},l):\operatorname{Tor}^{R}_{\bullet}(k,l)\to\operatorname{Tor}^{S}_{\bullet}(l,l)$ of graded algebras is generated by elements of degree 1.

The name reflects the relation of the new concept to that of *small homomorphism*, defined in [8] by the condition that the map $\operatorname{Tor}^{\varphi}_{\bullet}(\overline{\varphi},l)$ is injective.

4.2. *Example.* – It is proved in [8, (4.1)] that for every ring $(R,\mathfrak{m},k)$, and for each ideal $\mathfrak{a}$ contained in $\mathfrak{m}^s$ for some sufficiently large s, the canonical epimorphism $R\to R/\mathfrak{a}$ is small. An effective bound on s has been found recently by Liana Şega.

Namely, let G be the symmetric algebra of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$, let $\operatorname{gr}_{\mathfrak{m}}(R)$ be the associated graded ring of R, and extend the identity map of $\mathfrak{m}/\mathfrak{m}^2$ to a homomorphism of graded k-algebras $G\to\operatorname{gr}_{\mathfrak{m}}(R)$. Let $\operatorname{pol\,reg}R$ denote the Castelnuovo–Mumford regularity of the graded G-module $\operatorname{gr}_{\mathfrak{m}}(R)$, that is

$$\operatorname{pol\,reg}R=\sup_{i\in\mathbb{N}}\bigl\{j\in\mathbb{Z}\bigm|\operatorname{Tor}^{G}_{i}\bigl(\operatorname{gr}_{\mathfrak{m}}(R),k\bigr)_{i+j}\neq 0\bigr\}.$$

By [27, (6.2)] the epimorphism $R\to R/\mathfrak{m}^s$ is Golod for all $s\geqslant 2+\operatorname{pol\,reg}R$. Golod homomorphisms are small by [8, (3.5)], so the factorization $R\to R/\mathfrak{a}\to R/\mathfrak{m}^s$ and functoriality imply that $R\to R/\mathfrak{a}$ is small for every ideal $\mathfrak{a}$ contained in $\mathfrak{m}^s$.

By [8, (3.1)], φ is small if and only if $\pi_{\bullet}(\varphi)$ is injective. We characterize almost smallness in similar terms, and by means of reduced Tor-algebras, cf. 3.5.

4.3. PROPOSITION. – *Let $\varphi:(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism. The following conditions are equivalent.*

(i) *φ is almost small.*
(ii) *$\pi_{\geqslant 2}(\varphi)$ is injective.*
(iii) *$\operatorname{tor}^{\varphi}_{\bullet}(\overline{\varphi},l)$ is injective.*

Proof. – Using the fact that $\operatorname{Tor}^{\varphi}_{\bullet}(\overline{\varphi},l)$ is a homomorphism of *Hopf* Γ*-algebras*, it is proved in [8, (1.3)] that there exists a subset $G\subset\operatorname{Tor}^{R}_{\bullet}(k,l)$ such that

$$\operatorname{Ker}\bigl(\operatorname{Tor}^{\varphi}_{\bullet}(\overline{\varphi},l)\bigr)=\bigl(l\langle G\rangle\bigr)_{\geqslant 1}\operatorname{Tor}^{R}_{\bullet}(k,l)$$

and the following hold:

(1) The image of G in $\pi_{\bullet}(R)\otimes_k l$ is a basis of $\operatorname{Ker}(\pi_{\bullet}(\varphi))$.
(2) The graded $l\langle G\rangle$-module $\operatorname{Tor}^{R}_{\bullet}(k,l)$ is free.

Clearly, (i) ⇔ (ii) follows from (1). The freeness of $\operatorname{Tor}^{R}_{\bullet}(k,l)$ over $l\langle\operatorname{Tor}^{R}_{1}(k,l)\rangle$, that of $\operatorname{Tor}^{S}_{\bullet}(l,l)$ over $l\langle\operatorname{Tor}^{S}_{1}(l,l)\rangle$, and (2) yield (i) ⇔ (iii). □

Vanishing of $\pi_{\geqslant 2}(R)$ characterizes regularity, cf. Example 3.2, so we get

4.4. COROLLARY. – *If the ring R is regular, then φ is almost small. Conversely, if the canonical surjection $\varepsilon:R\to k$ is almost small, then R is regular.*

Using the functoriality of $\pi_{\bullet}(\ )$, we see that the proposition also implies

4.5. COROLLARY. – *Let $\psi:Q\to R$ and $\varphi:R\to S$ be local homomorphisms.*

(a) *If ψ and φ are almost small, then $\varphi\circ\psi$ is almost small.*
(b) *If $\varphi\circ\psi$ is almost small, then ψ is almost small.*
(c) *If $\varphi\circ\psi$ is almost small and $\pi_{\geqslant 2}(\psi)$ is bijective, then φ is almost small.*

As a further corollary, we get another example of almost small homomorphisms.

4.6. *Example.* – If φ is flat and $\operatorname{char}(l)=2$, then φ is almost small by André [5].

Here is what is known for flat homomorphisms in general.

4.7. *Remark.* – If $\varphi:(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ is a flat local homomorphism, then by [7, (1.1)] for every $i\geqslant 1$ there exists an exact sequence of l-vector spaces

$$0 \longrightarrow \pi_{2i}(R)\otimes_k l \xrightarrow{\pi_{2i}(\varphi)} \pi_{2i}(S) \longrightarrow \pi_{2i}(S/\mathfrak{m}S) \xrightarrow{\eth_{2i}}$$
$$\longrightarrow \pi_{2i-1}(R)\otimes_k l \xrightarrow{\pi_{2i-1}(\varphi)} \pi_{2i-1}(S) \longrightarrow \pi_{2i-1}(S/\mathfrak{m}S) \longrightarrow 0$$

André [6] proved $\sum_{i=1}^{\infty}\operatorname{rank}_l(\eth_{2i})\leqslant \operatorname{edim}(S/\mathfrak{m}S)-\operatorname{depth}(S/\mathfrak{m}S)$ and conjectured that $\pi_{2i-1}(\varphi)$ is injective for all $i\geqslant 2$. In view of Proposition 4.3, the conjecture can be restated to say that every flat homomorphism is almost small.

The next proposition fails for small homomorphisms, and presents one of the main technical reasons for working with almost small homomorphisms.

4.8. PROPOSITION. – *Let $\varphi:(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism. The following maps are almost small simultaneously*: φ, $\grave{\varphi}:R\to\widehat{S}$, $\hat{\varphi}:\widehat{R}\to\widehat{S}$, *and* $\rho:R'\to\widehat{S}$, *where* $R\xrightarrow{\dot{\varphi}}R'\xrightarrow{\rho}\widehat{S}$ *is a regular factorization of* φ.

Proof. – The map $\grave{\varphi}:R\to\widehat{S}$ is the composition of φ with the completion map $S\to\widehat{S}$, and also the composition of the completion map $R\to\widehat{R}$ with $\hat{\varphi}$. As $\pi_\bullet(\ )$ applied to either completion map yields an isomorphism, Corollary 4.5 shows that φ, $\hat{\varphi}$, and $\grave{\varphi}$ are almost small simultaneously. Finally, $\pi_{\geqslant 2}(R'/\mathfrak{m}R')=0$ because $R'/\mathfrak{m}R'$ is regular, cf. Example 3.2, so $\pi_{\geqslant 2}(\dot{\varphi})$ is bijective by the exact sequence of Remark 4.7. Thus, $\hat{\varphi}$ and $\grave{\varphi}$ are almost small simultaneously by Corollary 4.5.3. □

As an application, we show how to obtain almost small homomorphisms by factoring complete intersection homomorphisms.

4.9. COROLLARY. – *Let $\varphi:R\to(S,\mathfrak{n},l)$ be a local homomorphism. If there exists a local homomorphism $\xi:S\to(S',\mathfrak{n}',l')$ such that $\xi\circ\varphi$ is c.i. at $\mathfrak{n}'$, then φ is almost small. In particular, if φ is c.i. at $\mathfrak{n}$, then it is almost small.*

Proof. – Let $R\to R'\xrightarrow{\varphi'}\widehat{S}'$ be a reduced Cohen factorization of the composition

$$\widehat{\xi}\circ\grave{\varphi}:R\to\widehat{S}'.$$

By hypothesis, $\operatorname{Ker}(\varphi')$ is generated by an R'-regular sequence, so [19, (3.4.1)] shows that $\pi_n(\varphi')$ is injective for $n=2$ and bijective for $n\geqslant 3$. By Proposition 4.3, the map φ' is almost small, which implies, by Proposition 4.8, that $\xi\circ\varphi$ is almost small as well. It remains to invoke Corollary 4.5(b). □

Extending Example 4.2, we provide a numerical test for almost smallness.

4.10. PROPOSITION. – *Let $\varphi:(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism. If*

$$\operatorname{length}_S(S/\mathfrak{n}^s)=\sum_{i=0}^{s-1}\binom{e+i-1}{i}\operatorname{length}_R(R/\mathfrak{m}^{s-i})$$

for $e=\operatorname{edim}(S/\mathfrak{m}S)$ and $s=2+\operatorname{pol\,reg}R$, then φ is almost small.

Proof. – First we note that for every integer r there is an inequality

$$\operatorname{length}_S\bigl(S/\mathfrak{n}^r\bigr) \leqslant \sum_{i=0}^{r-1} \binom{e+i-1}{i} \operatorname{length}_R\bigl(R/\mathfrak{m}^{r-i}\bigr).$$

Indeed, if $R \to (R', \mathfrak{m}', l) \xrightarrow{\varphi'} \widehat{S}$ is a reduced Cohen factorization, then the right hand side of the formula above is equal to $\operatorname{length}_{R'}(R'/\mathfrak{m}'^r)$. The surjective homomorphism

$$\varphi'_r : R'/\mathfrak{m}'^r \to S/\mathfrak{n}^r$$

induced by $R' \to \widehat{S}$ yields the inequality above, and φ'_r is bijective if and only if equality holds, that is, if and only if $\operatorname{Ker}(\varphi') \subseteq \mathfrak{m}'^r$.

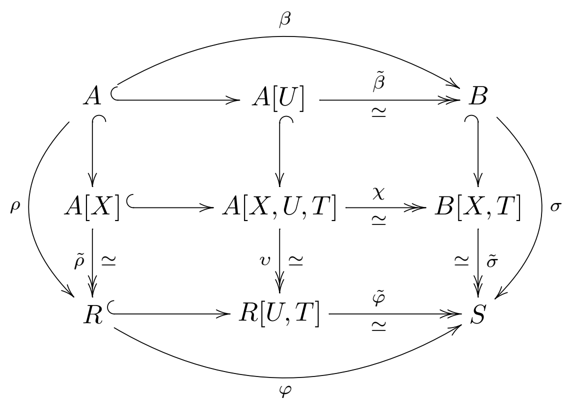
By the preceding argument, our hypothesis implies $\operatorname{Ker}(\varphi') \subseteq \mathfrak{m}'^s$ with $s = 2 + \operatorname{pol\,reg} R$. On the other hand, the associated graded rings of R' and R are linked by an isomorphism of graded l-algebras

$$\operatorname{gr}_{\mathfrak{m}'}(R') \cong l \otimes_k \operatorname{gr}_{\mathfrak{m}}(R)[x_1, \ldots, x_e]$$

where $x_1, \ldots, x_e$ are indeterminates. It follows that $\operatorname{pol\,reg}(R') = \operatorname{pol\,reg}(R)$. Example 4.2 now shows that the homomorphism φ' is small. In view of Proposition 4.8, it follows that the homomorphism φ is almost small. □

The next result is a structure theorem for morphisms of minimal models over certain almost small homomorphisms. A key ingredient of the proof is the general result on the map $\pi_\bullet(\varphi)$ established in Theorem 3.4.

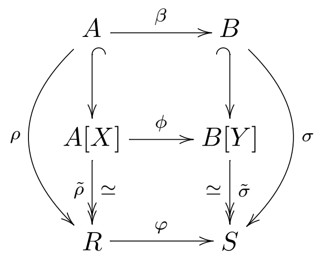
4.11. THEOREM. – *Let $\rho : (A, \mathfrak{p}, k) \to R$ and $\varphi : (R, \mathfrak{m}, k) \to S$ be surjective homomorphisms of local rings, with A regular and $\operatorname{Ker}(\rho) \subseteq \mathfrak{p}^2$. If φ is almost small, then there exists a commutative diagram of morphisms of DG algebras*



where $(B, \mathfrak{q}, k)$ is a regular local ring, β and σ are surjective homomorphisms, $\operatorname{Ker}(\sigma)$ is contained in $\mathfrak{q}^2$, $U = U_1$, the external rows and columns are minimal models, χ and υ are surjective quasiisomorphisms.

Proof. – Choose a subset $\boldsymbol{a} \subset A$ mapping to a basis of $(\operatorname{Ker}(\varphi) + \mathfrak{m}^2)/\mathfrak{m}^2$. As $\boldsymbol{a}$ is part of a regular system of parameters for A, the local ring $(B, \mathfrak{q}, k) = (A/(\boldsymbol{a}), \mathfrak{p}/(\boldsymbol{a}), k)$ is regular. Since $\boldsymbol{a}$ is contained in the kernel of $\varphi \circ \rho$, this map factors as a composition of surjective homomorphisms $\beta\colon A\to B$ and $\sigma\colon B\to S$. The choice of $\boldsymbol{a}$ ensures that $\operatorname{Ker}(\sigma)$ is contained in $\mathfrak{q}^2$.

Using 2.3, we form a commutative diagram of morphisms of DG algebras



It induces morphisms of DG algebras

$$\phi'\colon B[X]=A[X]\otimes_A B \xrightarrow{\phi\otimes_A B} B[Y]\otimes_A B = B[Y];$$
$$\overline{\phi}\colon k[X]=A[X]\otimes_A k \xrightarrow{\phi\otimes_A k} B[Y]\otimes_A k = k[Y].$$

As φ is almost small, Theorem 3.4 shows that the k-linear map

$$\operatorname{ind}(\overline{\phi})\colon \operatorname{ind}(k[X])\longrightarrow \operatorname{ind}(k[Y])$$

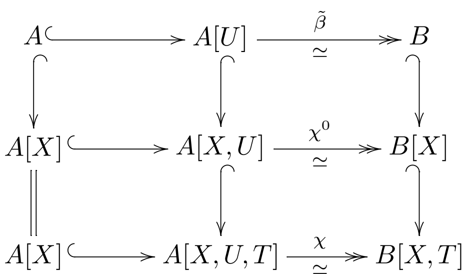
is injective. Thus, the set $\operatorname{ind}(\overline{\phi})(X)$ is linearly independent in $\operatorname{ind}(k[Y])$.

Choose a subset T in $B[Y]$ whose image in $\operatorname{ind}(k[Y])$ extends $\operatorname{ind}(\overline{\phi})(X)$ to a basis. It follows that T is a set of free variables over $k[X]$, and so the map $\overline{\phi}$ is injective. This map is equal to $\phi'\otimes_B k$, and ϕ' is a map of graded free B-modules, so we conclude by Nakayama's Lemma that ϕ' is injective and T is a set of free variables generating $B[Y]$ over $\phi'(B[X])$. Changing variables in $B[Y]$, we replace $B[Y]$ by $B[X,T]$ and ϕ' by the canonical inclusion $B[X]\hookrightarrow B[X,T]$.

Let $A[U]$ be the Koszul complex with $U=U_1$ and $\partial(U)=\boldsymbol{a}$, and set

$$A[X,U]=A[X]\otimes_A A[U].$$

These algebras appear in a commutative diagram of DG algebras



where $\tilde{\beta}$ is the canonical augmentation and $\chi^0 = A[X]\otimes_A \tilde{\beta}$. Since $\boldsymbol{a}$ is an A-regular sequence, $\tilde{\beta}$ is a quasiisomorphism; by 1.2, χ^0 is a quasiisomorphism as well.

The map χ is built inductively, starting with χ^0. Using the inclusions $\partial(T_1)\subseteq \operatorname{Ker}(\sigma)\subseteq \mathfrak{q}^2$, pick for each $t\in T_1$ an element $p_t\in\mathfrak{p}^2$ with $\beta(p_t)=\partial(t)$. Let $A[T_1]$ be the Koszul complex with $\partial(t) = p_t$ and χ^1 the morphism of DG algebras

$$A[X,U,T_1] = A[X,U] \otimes_A A[T_1] \xrightarrow{\chi^0 \otimes_A A[T_1]} B[X] \otimes_A A[T_1] = B[X,T_1].$$

By 1.2, χ^1 is a surjective quasiisomorphism. Assume next that a surjective quasiisomorphism $\chi^n : A[X,U,T_{\leqslant n}] \to B[X,T_{\leqslant n}]$ is available for some $n \geqslant 1$. For each $t \in T_{n+1}$ we pick a cycle $z_t \in A[X,U,T_{\leqslant n}]_n$ such that $\chi^n(z_t) = \partial(t)$, then we set

$$A[X,U,T_{\leqslant n+1}] = A[X,U,T_{\leqslant n}]\big[T_{n+1} \mid \partial(t) = z_t\big]$$

and define χ^{n+1} to be the extension of χ^n satisfying $\chi^{n+1}(t) = t$ for all $t \in T_{n+1}$. It is easy to verify that this map is a surjective quasiisomorphism, cf. also [9, (7.2.10)]. Taking direct limits, we obtain the surjective quasiisomorphism χ displayed in the diagram.

The diagram above provides the two upper squares of the diagram in the theorem. Its lower left square is obtained by base change along $\tilde{\rho}$. For its lower right square, we factor $\tilde{\sigma} \circ \chi$ through $\upsilon = \tilde{\rho} \otimes_{A[X]} A[X,U,T]$ to get a surjection $\widetilde{\varphi} : R[U,T] \to S$.

The top row and two side columns of the diagram are minimal models by construction. Furthermore, χ, υ, and $\widetilde{\varphi}$ are surjective quasiisomorphisms: the first by construction, the second by 1.2, and the third due to the commutativity of the diagram. Since the differential of $R[U,T]$ is induced by that of $A[X,U,T]$, to prove that the lower row is a minimal model it suffices to establish that the differential on $A[X,U,T]$ is decomposable.

For any $y \in A[X,U,T]$ with $|y| = n+1 \geqslant 2$, write $\partial(y)$ in the form

$$\partial(y) = \sum_{x \in X_n} c_x x + \sum_{u \in U_n} b_u u + \sum_{t \in T_n} a_t t + w \in A[X,U,T]$$

with $a_t, b_u, c_x \in A$ and $w \in (X,U,T)^2 A[X,U,T]$. In the resulting equality

$$\partial\big(\chi(y)\big) = \sum_{x \in X_n} \beta(c_x) x + \sum_{t \in T_n} \beta(a_t) t + \chi(w) \in B[X,T]$$

we have $\chi(w) \in (X,T)^2 B[X,T]$. The differential of $B[X,T]$ is decomposable, so for all $t \in T_n$ and $x \in X_n$ we obtain $\beta(a_t), \beta(c_x) \in \mathfrak{q}$, that is, $a_t, c_x \in \mathfrak{p}$. Since $U = U_1$, we have $b_u = 0$ unless $n = 1$. If $n = 1$, then $w = 0$, so the equality $\partial^2(y) = 0$ yields

$$\sum_{u \in U_1} b_u \partial(u) = -\sum_{x \in X_1} c_x \partial(x) - \sum_{t \in T_1} a_t \partial(t).$$

By construction, we have $\partial(x) \in \operatorname{Ker}(\rho) \subseteq \mathfrak{p}^2$ for all $x \in X_1$ and $\partial(t) = p_t \in \mathfrak{p}^2$ for all $t \in T_1$, so the last equality yields $\sum_{u \in U_1} b_u \partial(u) \in \mathfrak{p}^2$. As $\partial(U_1) = \boldsymbol{a}$ is part of a regular system of parameters, this implies $b_u \in \mathfrak{p}$ for all $u \in U_1$, so the differential of $A[X,U,T]$ is decomposable. □

5. Weak category of a local homomorphism

We introduce a notion motivated by Félix and Halperin's [17, (4.3)] definition of rational Lusternik–Schnirelmann category $\operatorname{cat}_0(X)$ of a simply connected CW complex X of finite type. Weak category captures a Looking Glass [12] image of a corollary of the Mapping

Theorem: If $\mathrm{cat}_0(X) \leqslant s$, then by [17, (5.1)] for each $n \geqslant 2$ the n-connected cover X_n of X satisfies $\mathrm{cat}_0(X_n) \leqslant s$, hence by [17, (4.10)] the product of any $(s+1)$ cohomology classes in $\mathrm{H}_{\geqslant 1}(X_n;\mathbb{Q})$ is equal to 0.

5.1. DEFINITION. – If $(B,\mathfrak{q},k)$ is a local ring and $B[V]$ is a semifree extension with decomposable differential, then we define a notion of *weak category* by the formula

$$\mathrm{wcat}(B[V]) = \inf\left\{ s\in\mathbb{N} \;\middle|\; \begin{array}{l}\text{for each } n\geqslant 2 \text{ the product of any}\\ (s+1) \text{ elements of positive degree}\\ \text{in } \mathrm{H}(B[V]/(\mathfrak{q},V_{<n})) \text{ is equal to } 0\end{array}\right\}.$$

Finite weak category can often be detected by using a variant of [13, (1.2)]:

5.2. PROPOSITION. – *If $(B,\mathfrak{q},k)$ is a local ring, $B[V]$ a DG algebra with decomposable differential, and $B[V]\to S[W]$ a surjective morphism of DG algebras, then*

$$\mathrm{wcat}\big(S[W]\big) \leqslant \sup\big\{s\in\mathbb{N} \mid \mathrm{H}_s\big(B[V]/\mathfrak{q}B[V]\big)\neq 0\big\}.$$

Proof. – Set

$$k[V] = B[V]/\mathfrak{q}B[V] \quad \text{and} \quad k[W] = S[W]/\mathfrak{q}S[W].$$

All DG algebras under consideration are images of $B[V]$, so their differentials are decomposable. We have $\mathrm{wcat}(S[W]) = \mathrm{wcat}(k[W])$ by definition. As the induced morphism $k[V]\to k[W]$ is surjective, we get $\mathrm{wcat}(k[W]) \leqslant \sup\{s\in\mathbb{N}\mid \mathrm{H}_s(k[V])\neq 0\}$ from [13, (1.2)]. □

5.3. DEFINITION. – If $\varphi: R\to S$ is a local homomorphism and $R\to R'[U]\to \widehat{S}$ is a minimal model of $\grave{\varphi}$, then we define the *weak category* of φ by the equality

$$\mathrm{wcat}(\varphi) = \mathrm{wcat}\big(R'[U]\big).$$

Proposition 2.4 shows that it does not depend on the choice of minimal model.

By [10, §3], the sequence $(\varepsilon_n(\varphi))$ is positive and grows exponentially when φ is not c.i. and $\mathrm{fd}_R S$ is finite. A close reading of the proofs shows the last condition is used only to ensure $\mathrm{wcat}(\varphi)<\infty$, cf. Theorem 5.7, so at no further expense we get

5.4. THEOREM. – *Let $\varphi: R\to (S,\mathfrak{n},l)$ be a local homomorphism.*
If $\mathrm{wcat}(\varphi)$ is finite, then the following conditions are equivalent.
(i) *φ is not complete intersection at $\mathfrak{n}$.*
(ii) *$\varepsilon_n(\varphi)>0$ for all $n\geqslant 2$.*
(iii) *$\limsup_n \sqrt[n]{\varepsilon_n(\varphi)}>1$.*
(iv) *There exist a real number $c>1$ and a sequence of integers s_j with*

$$0<2s_j\leqslant s_{j+1}\leqslant \big(\mathrm{wcat}(\varphi)+1\big)s_j \quad \textit{and} \quad \varepsilon_{s_j}(\varphi)>c^{s_j} \quad \textit{for all } j\geqslant 1.$$

Proof. – Proposition 2.8 shows that (ii) or (iii) implies (i).
If (i) holds, then $\varepsilon_n(\varphi)\neq 0$ for $n=2,3$ by Propositions 2.6 and 2.8, and $\varepsilon_n(\varphi)\neq 0$ $n\geqslant 4$ by the *proof* of [10, (3.4)]; thus, (i) implies (ii).
The *proof* of [10, (3.10)] shows that (i) implies (iv). □

5.5. COROLLARY. – *The following conditions are equivalent.*

(i) φ *is complete intersection at* $\mathfrak{n}$.
(ii) $\operatorname{wcat}(\varphi) < \infty$ *and* $\varepsilon_n(\varphi) = 0$ *for some* $n \geqslant 2$.
(iii) $\operatorname{wcat}(\varphi) < \infty$ *and* $\limsup_n \sqrt[n]{\varepsilon_n(\varphi)} \leqslant 1$.
(iv) $\operatorname{wcat}(\varphi) = 0$.

Proof. – The theorem shows that (i) follows from either (ii) or (iii).

Let $R \to R'[U] \to \widehat{S}$ be a minimal model of $\dot{\varphi} : R \to \widehat{S}$. Proposition 2.8 shows that φ is c.i. at $\mathfrak{n}$ if and only if $U = U_1$. Thus, (i) implies (iv) by definition of $\operatorname{wcat}(\varphi)$. Conversely, if (iv) holds, then $U = U_1$ by 2.2, hence (i) holds.

Finally, conditions (i) and (iv) imply (ii) and (iii) by Proposition 2.8. □

Next we establish a most important property of almost small homomorphisms.

5.6. THEOREM. – *If a local homomorphism* $\varphi : R \to S$ *is almost small, then*

$$\operatorname{wcat}(\varphi) \leqslant \operatorname{edim} S - \operatorname{depth} S.$$

Proof. – Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ be a Cohen presentation of $\dot{\varphi}$. As $R' \xrightarrow{=} R' \xrightarrow{\varphi'} \widehat{S}$ is a Cohen presentation of $\dot{\varphi}' = \varphi'$, we have $\operatorname{wcat}(\varphi) = \operatorname{wcat}(\varphi')$. On the other hand, the map φ' is almost small by Proposition 4.8. Furthermore, $\operatorname{edim} \widehat{S} = \operatorname{edim} S$ and $\operatorname{depth} \widehat{S} = \operatorname{depth} S$. Thus, we may assume that R and S are complete and that the local homomorphism $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, k)$ is surjective and almost small.

Choose a regular local ring $(A, \mathfrak{p}, k)$ and a surjective homomorphism $\rho : A \to R$ with $\operatorname{Ker}(\rho) \subseteq \mathfrak{p}^2$. From Theorem 4.11 we get a minimal model $R \to R[U, T] \to S$ where $U = U_1$, together with a minimal model $B \to B[X, T] \xrightarrow{\tilde{\sigma}} S$ where $(B, \mathfrak{q}, k)$ is a regular local ring and $\operatorname{Ker}(\tilde{\sigma}_0) \subseteq \mathfrak{q}^2$, linked for each $n \geqslant 2$ by isomorphisms

$$\frac{R[U,T]}{(\mathfrak{m}, U, T_{<n})R[U,T]} \cong \frac{A[X,U,T]}{(\mathfrak{p}, X, U, T_{<n})A[X,U,T]} \cong \frac{B[X,T]}{(\mathfrak{q}, X, T_{<n})B[X,T]}.$$

We also have $\mathrm{H}_i(B[X,T]/\mathfrak{q}B[X,T]) = \operatorname{Tor}_i^B(S,k)$ by definition, and $\operatorname{Tor}_i^B(S,k) = 0$ for $i > \dim B - \operatorname{depth} S$ by the Auslander–Buchsbaum Equality. Thus, Proposition 5.2 yields $\operatorname{wcat}(\varphi) = \operatorname{wcat}(R[U,T]) \leqslant \dim B - \operatorname{depth} S$. It remains to note that $\dim B = \operatorname{edim} S$ because $\operatorname{Ker}(\widetilde{\sigma}_0)$ is contained in $\mathfrak{q}^2$. □

For completeness, we deduce [10, (3.8)] from Proposition 5.2.

5.7. THEOREM. – *If* $\varphi : (R, \mathfrak{m}, k) \to S$ *is a local homomorphism, then*

$$\operatorname{wcat}(\varphi) \leqslant \operatorname{edim}(S/\mathfrak{m}S) + \operatorname{fd}_R S.$$

Proof. – There is nothing to prove unless $f = \operatorname{fd}_R S$ is finite. Let $R \to R'[U] \to \widehat{S}$ be a minimal model of $\dot{\varphi}$ with $\operatorname{edim} R'/\mathfrak{m}R' = \operatorname{edim} S/\mathfrak{m}S$; call this number e. By Proposition 5.2, it suffices to show $\mathrm{H}_i(R'[U]/\mathfrak{m}'R'[U]) = 0$ for $i > e + f$.

Since $R'[U]$ is a flat resolution of $\widehat{S}$ over R, we have $\mathrm{H}_i(F) = \operatorname{Tor}_i^R(k, S) = 0$ for $i > f$ and $F = R'[U]/\mathfrak{m}R'[U]$. Now note that F is a complex of free modules over the regular local ring $\overline{R} = R'/\mathfrak{m}R'$, and that $\dim \overline{R} = e$ by the minimality of the Cohen factorization. For $i = 0, \ldots, e-1$ form exact sequences of complexes

$$0 \longrightarrow F/(x_1, \ldots, x_i) \xrightarrow{x_{i+1}} F/(x_1, \ldots, x_i) \longrightarrow F/(x_1, \ldots, x_{i+1}) \longrightarrow 0$$

where $x_1, \ldots, x_e$ is a regular system of parameters of $\overline{R}$. From their homology exact sequences, one sees by induction on i that the homology of the complex $F/(x_1, \ldots, x_e) = R'[U]/\mathfrak{m}'R'[U]$ vanishes in degrees greater than $e + f$. □

6. André–Quillen homology of local homomorphisms

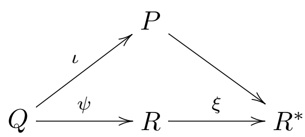
In this section we prove local versions of our results on André–Quillen homology. When using the general properties of the theory we take André's monograph [2] as standard reference. In addition, we heavily draw on some results from [10], *verbatim* or in variants. We recall them below.

6.1. For each local homomorphism $\psi : Q \to (R, \mathfrak{m}, k)$, by [10, (4.3)] one has

$$\operatorname{rank}_k \mathrm{D}_n(R \mid Q; k) = \varepsilon_{n+1}(\psi) \text{ for } \begin{cases} 2 \leqslant n < \infty & \text{if } \operatorname{char} k = 0; \\ 2 \leqslant n \leqslant 2p-1 & \text{if } \operatorname{char} k = p > 0. \end{cases}$$

6.2. For each local homomorphism $\psi : Q \to (R, \mathfrak{m}, k)$ the following are equivalent.
(i) $\mathrm{D}_n(R \mid Q; k) = 0$ for all $n \gg 0$, and $\operatorname{fd}_Q R < \infty$.
(ii) $\mathrm{D}_n(R \mid Q; k) = 0$ for all $n \geqslant 2$.
(iii) $\mathrm{D}_2(R \mid Q; k) = 0$.
(iv) $\mathrm{D}_n(R \mid Q; k) = 0$ for some $n \geqslant 2$ such that $\lfloor \frac{n}{2} \rfloor ! \neq 0 \in k$, and $\operatorname{fd}_Q R < \infty$.
(v) ψ is complete intersection at $\mathfrak{m}$.
Indeed, (ii), (iii), and (v) are equivalent by [10, (1.8)]. If ψ is c.i. at $\mathfrak{m}$, then $\operatorname{fd}_Q R$ is finite, cf. [11, (3.2)], so (ii) and (v) imply (i) and (iv). Conversely, (i) implies (v) by [10, (4.4)], while (iv) implies (v) by [10, (3.4)] via the equalities in 6.1.

6.3. Let $\xi : (R, \mathfrak{m}, k) \to (R^*, \mathfrak{m}^*, k^*)$ be a flat local homomorphism such that $\mathfrak{m}^* = \mathfrak{m}R^*$. If in a commutative square of local homomorphisms



the upper path is a regular factorization of the composition $\xi \circ \psi$, then the canonical maps $\mathrm{D}_n(\xi \mid \iota; k^*) : \mathrm{D}_n(R \mid Q; k^*) \to \mathrm{D}_n(R^* \mid P; k^*)$ are bijective for all $n \geqslant 2$.

Indeed, the argument for [10, (1.7)] carries over with only notational changes.

We now present our main local result, describing c.i. homomorphisms in terms parallel to those in 6.2, but *without* the hypothesis of finite flat dimension.

6.4. THEOREM. – *If $\psi : Q \to (R, \mathfrak{m}, k)$ is a local homomorphism, then the following conditions are equivalent.*
(i) $\mathrm{D}_n(R \mid Q; k) = 0$ *for all* $n \gg 0$, *and* ψ *is almost small.*
(ii) $\mathrm{D}_n(R \mid Q; k) = 0$ *for all* $n \gg 0$ *and* $\operatorname{wcat}(\psi) < \infty$.
(iii) $\mathrm{D}_n(R \mid Q; k) = 0$ *for some* $n \geqslant 2$ *with* $\lfloor \frac{n}{2} \rfloor ! \neq 0 \in k$, *and* ψ *is almost small.*
(iv) $\mathrm{D}_n(R \mid Q; k) = 0$ *for some* $n \geqslant 2$ *with* $\lfloor \frac{n}{2} \rfloor ! \neq 0 \in k$, *and* $\operatorname{wcat}(\psi) < \infty$.
(v) ψ *is complete intersection at* $\mathfrak{m}$.

Proof. – If ψ is c.i. at $\mathfrak{m}$, then $\mathrm{D}_n(R \mid Q; k) = 0$ for all $n \geqslant 2$ by 6.2, and ψ is almost small by Corollary 4.9, so (v) implies (i) and (iii). If ψ is almost small, then it has finite weak category by Theorem 5.6, so (i) implies (ii) and (iii) implies (iv). If (ii) holds, then the *proof* of [10, (4.4)] shows that $\limsup_n \sqrt[n]{\varepsilon_n(\psi)} \leqslant 1$, so (v) holds by Corollary 5.5. In view of 6.1, the same corollary shows that (iv) implies (v). □

To continue, we recall some general computations of André–Quillen homology.

6.5. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings, and N an S-module. André [1, (16.1)] constructs a universal coefficients spectral sequence

$$^2\,\mathrm{E}_{p,q} = \mathrm{Tor}_p^S\big(\mathrm{D}_q(S\mid R;S), N\big) \Rightarrow \mathrm{D}_{p+q}(S\mid R;N).$$

Thus, if for some $m \in \mathbb{Z}$ the S-module $\mathrm{D}_n(S\mid R;S)$ is flat for all $n \leqslant m$, then

$$\mathrm{D}_n(S\mid R;N) \cong \mathrm{D}_n(S\mid R;S)\otimes_S N \quad \text{for all } n \leqslant m+1. \tag{10}$$

Let φ be surjective, and set $\mathfrak{a} = \mathrm{Ker}(\varphi)$. There are isomorphisms of S-modules

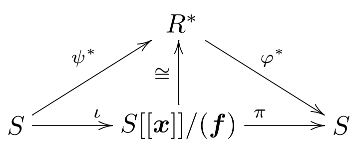
$$\mathrm{D}_1(S\mid R;N) \cong \mathfrak{a}/\mathfrak{a}^2 \otimes_R N \cong \mathrm{Tor}_1^R(S,N); \tag{11}$$

$$\mathrm{D}_2(S\mid R;S) \cong \frac{\mathrm{Tor}_2^R(S,S)}{\mathrm{Tor}_1^R(S,S)\cdot \mathrm{Tor}_1^R(S,S)}, \tag{12}$$

where the second one is elementary, and the other two come from [2, (6.1), (15.8)].

We give "concrete" descriptions of split c.i. local homomorphisms.

6.6. Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},k)$ be a local homomorphism, set $\mathrm{Ker}(\varphi) = \mathfrak{a}$, and let asterisks * denote $\mathfrak{a}$-adic completion. If $\psi\colon S \to R$ is a section of φ, then there exist a set $\boldsymbol{x}$ of formal indeterminates over S and a commutative diagram



of homomorphisms of rings, where the isomorphism is induced by a surjective map

$$\rho\colon S[[\boldsymbol{x}]] \to R^*$$

with $\mathrm{Ker}(\rho) \subseteq \mathfrak{n}(\boldsymbol{x}) + (\boldsymbol{x})^2$, $\boldsymbol{f}$ is a minimal set of generators of $\mathrm{Ker}(\rho)$, π is the surjection with $\mathrm{Ker}(\pi) = (\boldsymbol{x})$, and ι is the natural injection.

Indeed, as S is discrete in the $\mathfrak{a}$-adic topology, completion yields homomorphisms of rings $\psi^*\colon S \to R^*$ and $\varphi^*\colon R^* \to S$ whose composition is the identity map of S. Pick a set

$$\boldsymbol{a} = \{a_1,\dots,a_e\} \subset R^*$$

minimally generating $\mathrm{Ker}\,\varphi^*$, let $\boldsymbol{x} = \{x_1,\dots,x_e\}$ be a set of formal indeterminates, and let $\rho\colon S[[\boldsymbol{x}]] \to R^*$ be the unique homomorphism of S-algebras mapping x_i to a_i for each i. It is necessarily surjective, and in view of the choice of $\boldsymbol{a}$ we have

$$\mathrm{Ker}(\rho) \subseteq \big(\mathfrak{n} + (\boldsymbol{x})\big)^2 \cap (\boldsymbol{x}) = \mathfrak{n}(\boldsymbol{x}) + (\boldsymbol{x})^2.$$

6.7. PROPOSITION. – *In the notation of* 6.6 *the following hold.*

(a) *$\boldsymbol{f}$ is a regular sequence if and only if ψ is c.i. at $\mathfrak{m}$.*

(b) *$\boldsymbol{f}$ is a regular sequence contained in $(\boldsymbol{x})^2$ if and only if the S-modules $\mathrm{D}_1(S \mid R; S)$ and $\mathrm{D}_2(S \mid R; S)$ are free, and $\mathrm{D}_3(S \mid R; S) = 0$.*

(c) *$\boldsymbol{f} = \emptyset$ if and only if the S-module $\mathrm{D}_1(S \mid R; S)$ is free, $\mathrm{D}_2(S \mid R; S) = 0$, and $\mathrm{D}_3(S \mid R; S) = 0$.*

Proof. – Set $P = S[[\boldsymbol{x}]]$. The ring R^* is local with maximal ideal $\mathfrak{m}R^*$ and residue field $R^*/\mathfrak{m}R^* \cong k$, and R^* is flat over R. Thus, [2, (4.54)] yields $\mathrm{D}_n(S \mid R; S) \cong \mathrm{D}_n(S \mid R^*; S)$ for all $n \in \mathbb{Z}$, so for the rest of the proof we may assume $R = R^*$.

As $\mathrm{D}_2(R \mid S; k) \cong \mathrm{D}_2(R \mid P; k)$ by 6.3, (a) follows from 6.2.

Since $\boldsymbol{x}$ is P-regular, $\mathrm{D}_n(S \mid P; -) = 0$ for $n \geqslant 2$, so the Jacobi–Zariski exact sequence [2, (5.1)] generated by $P \to R \to S$ yields isomorphisms of functors

$$\mathrm{D}_n(S \mid R; -) \cong \mathrm{D}_{n-1}(R \mid P; -) \quad \text{for } n \geqslant 3 \tag{13}$$

and, in view of the isomorphism (11), also an exact sequence

$$0 \longrightarrow \mathrm{D}_2(S \mid R; S) \longrightarrow \big((\boldsymbol{f})/(\boldsymbol{f})^2\big) \otimes_R S \xrightarrow{\delta} (\boldsymbol{x})/(\boldsymbol{x})^2 \longrightarrow \mathrm{D}_1(S \mid R; S) \longrightarrow 0 \tag{14}$$

of S-modules. Note that the S-module $(\boldsymbol{x})/(\boldsymbol{x})^2$ is free, and

$$\mathrm{Im}(\delta) = \big((\boldsymbol{f}) + (\boldsymbol{x})^2\big)/(\boldsymbol{x})^2 \subseteq \big(\mathfrak{n}(\boldsymbol{x}) + (\boldsymbol{x})^2\big)/(\boldsymbol{x})^2 = \mathfrak{n}\big((\boldsymbol{x})/(\boldsymbol{x})^2\big). \tag{15}$$

If $\boldsymbol{f}$ is a regular sequence, then $\mathrm{D}_2(R \mid P; S) = 0$ and $(\boldsymbol{f})/(\boldsymbol{f})^2$ is free over R, so the "only if" parts of (b) and (c) are now clear. To obtain converses, we assume that $\mathrm{D}_n(S \mid R; S)$ is free for $n \leqslant 2$, and $\mathrm{D}_3(S \mid R; S) = 0$. From (13) and (10) we get

$$\mathrm{D}_2(R \mid P; k) \cong \mathrm{D}_3(S \mid R; k) \cong \mathrm{D}_3(S \mid R; S) \otimes_S k = 0,$$

from where we conclude that $\boldsymbol{f}$ is a regular sequence, cf. 6.2. Due to (14) and (15), the freeness of $\mathrm{D}_1(S \mid R; S)$ implies $\delta = 0$, that is, $\boldsymbol{f} \subseteq (\boldsymbol{x})^2$ and $\mathrm{D}_2(S \mid R; S) \cong (\boldsymbol{f})/(\boldsymbol{f})^2$. It is now clear that the "if" parts of (b) and (c) hold as well. □

The next theorem is a local version of Theorem II. The proof draws on results obtained above and on earlier results from [13]. One of them is for a homomorphism

$$\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, k)$$

of local rings that is *large* in the sense of Levin [23], meaning that the map

$$\mathrm{Tor}^{\varphi}_{\bullet}(\overline{\varphi}, k) : \mathrm{Tor}^R_{\bullet}(k, k) \to \mathrm{Tor}^S_{\bullet}(k, k)$$

is surjective.

6.8. THEOREM. – *For an algebra retract $S \xrightarrow{\psi} (R, \mathfrak{m}, k) \xrightarrow{\varphi} S$ of local rings the following conditions are equivalent.*

(i) *The S-algebra $\mathrm{Tor}^R_{\bullet}(S, S)$ is finitely generated.*

(ii) *For every S-algebra T there is an isomorphism of graded T-algebras*

$$\mathrm{Tor}^R_{\bullet}(S, T) \cong \Big(\bigwedge_S D_1 \otimes_S \mathrm{Sym}_S\, D_2\Big) \otimes_S T$$

where D_1 *and* D_2 *are free* S*-modules concentrated in degrees* 1 *and* 2*, respectively. Moreover, if* $\operatorname{char}(k) > 0$*, then* $D_2 = 0$.

(iii) *The* S*-modules* $\mathrm{D}_1(S \mid R; S)$ *and* $\mathrm{D}_2(S \mid R; S)$ *are free, and* $\mathrm{D}_3(S \mid R; S) = 0$*. Moreover, if* $\operatorname{char}(k) > 0$*, then* $\mathrm{D}_2(S \mid R; S) = 0$.

(iv) *The sequence* $\boldsymbol{f}$ *in the commutative diagram constructed in* 6.6 *is contained in* $(\boldsymbol{x})^2$*. Moreover, if* $\operatorname{char}(k) > 0$*, then* $\boldsymbol{f} = \emptyset$.

Proof. – Suppose that $\operatorname{char}(k) > 0$. Conditions (iii) and (iv) are then equivalent by Proposition 6.7(c). Functoriality in the ring argument shows that φ is large. For every large homomorphism, conditions (i), (ii) and (iv) are equivalent by [13, (3.1)].

Now consider the case when $\operatorname{char}(k) = 0$. Conditions (iii) and (iv) are equivalent by Proposition 6.7(b). The equivalence of (i), (ii), and (iv) is established in [13, (4.1)] under an additional hypothesis, R has finite flat dimension over S, which is used only once, to conclude that $\mathrm{D}_n(R^* \mid S[[\boldsymbol{x}]]; k) = 0$ for all $n \gg 0$ implies that $\boldsymbol{f}$ is regular (cf. [13, p. 163]). By Theorem 6.4, the same conclusion holds when the map $\rho : S[[\boldsymbol{x}]] \to R^*$ is almost small. To see that it is, note that the composition $S[[\boldsymbol{x}]] \xrightarrow{\rho} R^* \xrightarrow{\psi^*} S$ is complete intersection at $\mathfrak{n}$, and apply Corollary 4.9. □

The proof above raises the question whether the first three conditions are equivalent for large homomorphisms in characteristic 0. Here is what we know.

6.9. Let $\varphi : R \to (S, \mathfrak{n}, k)$ be a large homomorphism, where $\operatorname{char}(k) = 0$. If $R \to R[U] \to S$ is a minimal model of φ, then results of Quillen, [25, (9.5), (10.1)], imply

$$\sup\bigl\{n \in \mathbb{N} \mid \mathrm{D}_n(S \mid R; k) \neq 0\bigr\} = \sup\{n \in \mathbb{N} \mid U_n \neq \emptyset\}.$$

Thus, [13, (3.2)] shows that if $\operatorname{Tor}^R_\bullet(S, S)$ is finitely generated, then $\text{AQ-}\dim_S R$ is finite, and [21, (3.1)] proves that in this case $\text{AQ-}\dim_S R$ is equal to 1 or is even.

7. André–Quillen dimension

In this section we prove the theorems stated in the introduction. The local case was essentially settled in Section 6, but reduction to that case needs some attention, as weak category and almost small homomorphism are intrinsically local notions.

We start by recording a slight extension of a result of André [2]. For each $\mathfrak{n} \in \operatorname{Spec} S$, we let $k(\mathfrak{n})$ denote the residue field $S_\mathfrak{n}/\mathfrak{n}S_\mathfrak{n}$.

7.1. If $Q \to R \to S$ are homomorphisms of noetherian rings and j is a non-negative integer, then the following conditions are equivalent.

(i) $\mathrm{D}_n(R \mid Q; -) = 0$ on the category of S-modules for all $n \geqslant j$.

(ii) $\mathrm{D}_n(R_\mathfrak{m} \mid Q_{\mathfrak{n}\cap Q}; k(\mathfrak{m})) = 0$ for all $n \geqslant j$, all $\mathfrak{n} \in \operatorname{Spec} S$, and $\mathfrak{m} = \mathfrak{n} \cap R$.

Indeed, [2, (4.58), (5.27)] produce for each n canonical isomorphisms

$$\mathrm{D}_n(R \mid Q; k(\mathfrak{n})) \cong \mathrm{D}_n(R_\mathfrak{m} \mid Q; k(\mathfrak{m})) \otimes_{k(\mathfrak{m})} k(\mathfrak{n}) \cong \mathrm{D}_n(R_\mathfrak{m} \mid Q_{\mathfrak{n}\cap Q}; k(\mathfrak{m})) \otimes_{k(\mathfrak{m})} k(\mathfrak{n})$$

which show that (i) implies (ii). If $R \to S$ is the identity map, then (ii) implies (i) by [2, Suppl. Prop. 29]; the *proof* of that proposition applies *verbatim* to every homomorphism $R \to S$ in which the ring S is noetherian.

We recall from [10] the definition of l.c.i. homomorphism of noetherian rings.

7.2. DEFINITION. – A homomorphism of noetherian rings $\varphi : R \to S$ is called *locally complete intersection* (or *l.c.i.*) if for every $\mathfrak{n} \in \operatorname{Spec} S$ the induced local homomorphism $\varphi_{\mathfrak{n}} : R_{\mathfrak{n} \cap R} \to S_{\mathfrak{n}}$ is complete intersection at $\mathfrak{n} S_{\mathfrak{n}}$ in the sense of 2.7.

We return to the discussion, started in the Introduction, of criteria for l.c.i. homomorphisms in terms of André–Quillen homology. Recall that $\varphi : R \to S$ is said to be *locally of finite flat dimension* if $\operatorname{fd}_R(S_{\mathfrak{n}}) < \infty$ for every $\mathfrak{n} \in \operatorname{Spec} S$.

7.3. If $\varphi : R \to S$ is a homomorphism of noetherian rings, then the following conditions are equivalent.

(i) $\text{AQ-dim}_R S < \infty$, and φ is locally of finite flat dimension.
(ii) $\text{AQ-dim}_R S \leqslant 1$.
(iii) $\mathrm{D}_2(S \mid R; -) = 0$.
(iv) $\mathrm{D}_n(S \mid R; -) = 0$ for some $n \geqslant 2$ with $\lfloor \frac{n}{2} \rfloor !$ invertible in S, and φ is locally of finite flat dimension.
(v) φ is locally complete intersection.

Indeed, (v) is equivalent to (ii), (iii) by [10, (1.2)], and to (i), (iv) by [10, (1.5)].

A major difficulty in dealing with Quillen's Conjecture is that R-algebras S with $\text{AQ-dim}_R S \leqslant 2$ have not been described in structural terms. All known algebras satisfying this condition are constructed by factoring some l.c.i. homomorphism.

7.4. *Example*. – Let $Q \to S$ be an l.c.i. homomorphism from a noetherian ring Q (say, a surjection with kernel generated by a regular sequence $\boldsymbol{g}$), through some homomorphism $Q \to R$ that is c.i. at all primes of R contracted from S (say, a surjection with kernel generated by a regular sequence in $(\boldsymbol{g})$). The desired vanishing property follows from 7.3, via the Jacobi–Zariski exact sequence [2, (5.1)].

The construction described above is rather rigid. This is demonstrated by the next result, which could be compared with another factorization theorem for local homomorphisms from [10]: If $\varphi \circ \psi$ is l.c.i. and φ is locally of finite flat dimension, then φ is l.c.i. and ψ is complete intersection at $\mathfrak{n} \cap R$ for each $\mathfrak{n} \in \operatorname{Spec} S$.

7.5. THEOREM. – *If $\psi : Q \to R$ and $\varphi : R \to S$ are homomorphisms of noetherian rings such that $\varphi \circ \psi$ is l.c.i., then the following conditions are equivalent.*

(i) $\text{AQ-dim}_R S < \infty$.
(ii) $\text{AQ-dim}_R S \leqslant 2$.
(iii) $\mathrm{D}_3(S \mid R; -) = 0$.
(iv) $\mathrm{D}_n(S \mid R; -) = 0$ *for some $n \geqslant 3$ such that $\lfloor \frac{n-1}{2} \rfloor !$ is invertible in S.*
(v) *ψ is complete intersection at $\mathfrak{n} \cap R$ for each $\mathfrak{n} \in \operatorname{Spec} S$.*

Proof. – The Jacobi–Zariski exact sequence of André–Quillen homology, cf. [2, (5.1)], yields an exact sequence of functors on the category of S-modules

$$\cdots \to \mathrm{D}_{n+1}(S \mid Q; -) \to \mathrm{D}_{n+1}(S \mid R; -) \to \mathrm{D}_n(R \mid Q; -) \to \mathrm{D}_n(S \mid Q; -) \to \cdots.$$

Since $\varphi \circ \psi$ is l.c.i., we have $\mathrm{D}_n(S \mid Q; -) = 0$ for $n \geqslant 2$ by 7.3, so for $n \geqslant 2$ we get isomorphisms $\mathrm{D}_{n+1}(S \mid R; -) \cong \mathrm{D}_n(R \mid Q; -)$ of functors of S-modules. Thus, each one of conditions (i), (ii), and (iii) is equivalent to its primed version below:

(i′) $\mathrm{D}_n(R \mid Q; -) = 0$ on the category of S-modules for all $n \gg 0$.
(ii′) $\mathrm{D}_n(R \mid Q; -) = 0$ on the category of S-modules for all $n \geqslant 2$.
(iii′) $\mathrm{D}_2(R \mid Q; -) = 0$ on the category of S-modules.

(iv′) $\mathrm{D}_n(R \mid Q; -) = 0$ on the category of S-modules for some $n \geqslant 2$ such that $\lfloor \frac{n}{2} \rfloor!$ is invertible in S.

The equivalence of (ii′) and (v) results from 7.1 and 6.2. For each $\mathfrak{n} \in \operatorname{Spec} S$, the homomorphism $\psi_{\mathfrak{n}\cap R} : Q_{\mathfrak{n}\cap Q} \to R_{\mathfrak{n}\cap R}$ is almost small by Corollary 4.9. The equivalence of (i′), (iii′), (iv′), and (v) now comes from 7.1 and Theorem 6.4. □

The last result implies part of [10, (1.5)]: Quillen's Conjecture holds when the ring S is l.c.i., because the theorem then applies with $Q = \mathbb{Z}$. More to the point, it reduces the proof of Theorem I from the Introduction to a mere formality.

Proof of Theorem I. – Let $\psi : S \to R$ be any section of φ.

The map $\varphi \circ \psi = \mathrm{id}_S$ is obviously l.c.i., so Theorem 7.5 and Proposition 6.7(a) show that the conditions of Theorem I are equivalent. □

It remains to deduce Theorem II from its local version established in Section 6.

Proof of Theorem II. – The implication (ii) ⇒ (i) is clear. By (11) and (12), the S-modules $\mathrm{D}_1(R \mid S; S)$ and $\mathrm{D}_2(R \mid S; S)$ are finitely generated. Thus the implications (i) ⇒ (iii) ⇔ (iv) follow from Theorem 6.8 via the isomorphisms

$$\operatorname{Tor}^R_\bullet(S,S)_\mathfrak{n} \cong \operatorname{Tor}^{R_{\mathfrak{n}\cap R}}_\bullet(S_\mathfrak{n}, S_\mathfrak{n});$$
$$\mathrm{D}_\bullet(S \mid R; S)_\mathfrak{n} \cong \mathrm{D}_\bullet(S_\mathfrak{n} \mid R_{\mathfrak{n}\cap R}; S_\mathfrak{n}),$$

respectively of graded $S_\mathfrak{n}$-algebras and graded $S_\mathfrak{n}$-modules, cf. [2, (4.59), (5.27)].

It remains to prove (iii) ⇒ (ii). Let $D_\bullet$ denote the graded S-module with $D_n = \mathrm{D}_n(S \mid R; S)$ for $n = 1, 2$, and $D_n = 0$ otherwise. The isomorphisms (11) and (12) define a surjection $\tau : \operatorname{Tor}^R_\bullet(S,S) \to D_\bullet$ of graded S-modules. Since $D_\bullet$ is projective, we may choose an S-linear map $\theta : D_\bullet \to \operatorname{Tor}^R_\bullet(S,S)$ with $\tau \circ \theta = \mathrm{id}_{D_\bullet}$. It extends to a homomorphism of graded S-algebras

$$\vartheta : \bigwedge_S D_1 \otimes_S \operatorname{Sym}_S D_2 \longrightarrow \operatorname{Tor}^R_\bullet(S,S).$$

Theorem 6.8 and the isomorphisms above show that $\vartheta_\mathfrak{n}$ is bijective for every $\mathfrak{n} \in \operatorname{Max} S$, so ϑ is an isomorphism. It induces an isomorphism

$$\vartheta \otimes_S T : \Bigl(\bigwedge_S D_1 \otimes_S \operatorname{Sym}_S D_2\Bigr) \otimes_S T \xrightarrow{\cong} \operatorname{Tor}^R_\bullet(S,S) \otimes_S T$$

of graded T-algebras. As each S-module $\operatorname{Tor}^R_n(S,S)$ is projective, the Universal Coefficients Theorem shows that for every S-algebra T the homomorphism

$$\operatorname{Tor}^R_\bullet(S,S) \otimes_S T \xrightarrow{\cong} \operatorname{Tor}^R_\bullet(S,T)$$

of graded T-algebras given by the Künneth map is bijective. Composing the isomorphisms above we obtain the desired isomorphism of graded T-algebras. □

We finish the paper by revisiting the announcement in [14].

7.6. *Remark.* – In [14, (2.6)], which is an avatar of Theorem 7.5, instead of condition (7.5.i) one finds the seemingly weaker condition: $\text{AQ-dim}_R S_\mathfrak{n} < \infty$ for all $\mathfrak{n} \in \operatorname{Spec} S$. We show that this condition is in fact equivalent to those in Theorem 7.5.

Indeed, under the hypotheses of Theorem 7.5 (which coincide with those [14, (2.6)]) the composition of the maps $\psi : Q \to R$ and $R \to S_{\mathfrak{n}}$ is an l.c.i. homomorphism. Fixing a prime ideal $\mathfrak{n}$ of S, from Theorem 7.5 we see that $\mathrm{AQ}\text{-}\dim_R S_{\mathfrak{n}}$ is finite if and only if ψ is c.i. at $\mathfrak{q} \cap R$ for all $\mathfrak{q} \in \operatorname{Spec} S$ with $\mathfrak{q} \subseteq \mathfrak{n}$. Letting now $\mathfrak{n}$ range over $\operatorname{Spec} S$, we conclude that $\mathrm{AQ}\text{-}\dim_R S_{\mathfrak{n}}$ is finite for all $\mathfrak{n} \in \operatorname{Spec} S$ if and only if ψ is c.i. at $\mathfrak{q} \cap R$ for all $\mathfrak{q} \in \operatorname{Spec} S$. This is condition (7.5.v).

We take this opportunity to make two minor corrections to [14]:

– in [14, (2.6.i)] 'at each $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \supseteq \operatorname{Ker}(\varphi)$' should read 'at $\mathfrak{n} \cap R$ for each $\mathfrak{n} \in \operatorname{Spec} S$';
– in [14, line above (4.1)] 'in [13, (4.1.iii)]' should read 'in [13, Theorem II]'.

We thank the editors of the present article for pointing out these errors to us.

REFERENCES

[1] ANDRÉ M., *Méthode simpliciale en algèbre homologique et algèbre commutative*, in: Lecture Notes Math., Vol. **32**, Springer-Verlag, Berlin, 1967.
[2] ANDRÉ M., *Homologie des algèbres commutatives*, in: Grundlehren Math. Wiss., Vol. **206**, Springer-Verlag, Berlin, 1974.
[3] ANDRÉ M., Algèbres graduées associées et algèbres symétriques plates, *Comment. Math. Helv.* **49** (1974) 277–301.
[4] ANDRÉ M., Localisation de la lissité formelle, *Manuscripta Math.* **13** (1974) 297–307.
[5] ANDRÉ M., Algèbre homologique des anneaux locaux à corps résiduels de caractéristique deux, in: *Sém. d'Algèbre P. Dubreil (Paris, 1979)*, in: Lecture Notes Math., Vol. **740**, Springer-Verlag, Berlin, 1979, pp. 237–242.
[6] ANDRÉ M., Le caractère additif des déviations des anneaux locaux, *Comment. Math. Helv.* **57** (1982) 648–675.
[7] AVRAMOV L.L., Homology of local flat extensions and complete intersection defects, *Math. Ann.* **228** (1977) 27–37.
[8] AVRAMOV L.L., Small homomorphisms of local rings, *J. Algebra* **50** (1978) 400–453.
[9] AVRAMOV L.L., Infinite free resolutions, in: *Six Lectures in Commutative Algebra (Bellaterra, 1996)*, in: Progress in Math., Vol. **166**, Birkhäuser, Boston, 1998, pp. 1–118.
[10] AVRAMOV L.L., Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology, *Ann. of Math. (2)* **150** (1999) 455–487.
[11] AVRAMOV L.L., FOXBY H.-B., HERZOG B., Structure of local homomorphisms, *J. Algebra* **164** (1994) 124–145.
[12] AVRAMOV L.L., HALPERIN S., Through the looking glass: A dictionary between rational homotopy theory and local algebra, in: *Algebra, Algebraic Topology, and Their Interactions (Stockholm, 1983)*, in: Lecture Notes Math., Vol. **1183**, Springer-Verlag, Berlin, 1986, pp. 1–27.
[13] AVRAMOV L.L., IYENGAR S., Finite generation of Hochschild homology algebras, *Invent. Math.* **140** (2000) 143–170.
[14] AVRAMOV L.L., IYENGAR S., Homological criteria for regular homomorphisms and for locally complete intersection homomorphisms, in: *Algebra, Arithmetic and Geometry (Mumbay, 2000)*, in: T.I.F.R. Studies in Math. 16, Vol. **I**, Narosa, New Delhi, 2002, pp. 97–122.
[15] CARTAN H., *Algèbres d'Eilenberg–MacLane*, Exposés 2 à 11, Sém. H. Cartan, Éc. Normale Sup. (1954–1955), Secrétariat Math., Paris, 1956; reprinted in: *Œuvres, Vol. III*, Springer-Verlag, Berlin, 1979, pp. 1309–1394.
[16] CARTAN H., EILENBERG S., *Homological Algebra*, Princeton Univ. Press, Princeton, NJ, 1956.
[17] FÉLIX Y., HALPERIN S., Rational L.-S. category and its applications, *Trans. Amer. Math. Soc.* **273** (1982) 1–37.
[18] GULLIKSEN T.H., A proof of the existence of minimal algebra resolutions, *Acta Math.* **120** (1968) 53–58.
[19] GULLIKSEN T.H., LEVIN G., Homology of local rings, *Queen's Papers Pure Appl. Math., Queen's Univ., Kingston, Ont.* **20** (1969).

[20] HOCHSCHILD G., KOSTANT B., ROSENBERG A., Differential forms on a regular affine algebra, *Trans. Amer. Math. Soc.* **102** (1962) 383–408.
[21] IYENGAR S., Acyclicity of Tate constructions, *J. Pure Appl. Algebra* **163** (2001) 289–300.
[22] LEVIN G., *Homology of local rings*, Ph.D. Thesis, Univ. of Chicago, Chicago, IL, 1965.
[23] LEVIN G., Large homomorphisms of local rings, *Math. Scand.* **46** (1980) 209–215.
[24] MACLANE S., *Homology*, in: Grundlehren Math. Wiss., Vol. **114**, Springer-Verlag, Berlin, 1967.
[25] QUILLEN D., On the (co-)homology of commutative rings, in: *Applications of Categorical Algebra (New York, 1968)*, in: Proc. Symp. Pure Math., Vol. **17**, American Mathematical Society, Providence, RI, 1970, pp. 65–87.
[26] SCHOELLER C., Homologie des anneaux locaux noethériens, *C. R. Acad. Sci. Paris Sér. A* **265** (1967) 768–771.
[27] ŞEGA L.M., Homological properties of powers of the maximal ideal of a local ring, *J. Algebra* **241** (2001) 827–858.
[28] TURNER J., *Nilpotency in the homotopy of simplicial commutative algebras,* Preprint, 2001.

(Manuscrit reçu le 22 octobre 2001 ;
accepté, après révision, le 21 mai 2002.)

Luchezar L. AVRAMOV
Department of Mathematics,
Purdue University,
West Lafayette, IN 47907, USA
Current address:
Department of Mathematics,
University of Nebraska,
Lincoln, NE 68588, USA
E-mail: avramov@math.unl.edu

Srikanth IYENGAR
Pure Mathematics,
Hicks Building,
University of Sheffield,
Sheffield S3 7RH, UK
Current address:
Department of Mathematics,
University of Missouri,
Columbia, MO 65201, USA
E-mail: iyengar@math.missouri.edu