EXISTENCE OF LOG CANONICAL FLIPS AND A SPECIAL LMMP

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ABSTRACT

Let $(X/Z, B + A)$ be a $\mathbf{Q}$-factorial dlt pair where $B, A \geq 0$ are $\mathbf{Q}$-divisors and $K_X + B + A \sim_{\mathbf{Q}} 0/Z$. We prove that any LMMP/Z on $K_X + B$ with scaling of an ample/Z divisor terminates with a good log minimal model or a Mori fibre space. We show that a more general statement follows from the ACC for lc thresholds. An immediate corollary of these results is that log flips exist for log canonical pairs.

1. Introduction

We work over $\mathbf{C}$. Extending results from the klt case to the lc case is often not as easy as it may sound. For example, it took some hard work to prove the cone and contraction theorem for lc pairs as done by Ambro [2] and Fujino [14]. Another major example is the finite generation of log canonical rings: the klt case was proved in [9] but extending this to the lc case is essentially equivalent to proving the abundance conjecture.

It is well-known that log flips exist for klt pairs [9]. In this paper we study the existence of log flips for lc pairs. Along the way, we came across the following more general statement which is more suitable for induction, and it is one of the main results of this paper.

Theorem **1.1.** — *Let* $(X/Z, B + A)$ *be a lc pair where* $B, A \geq 0$ *are* $\mathbf{Q}$*-divisors,* A *is* $\mathbf{Q}$*-Cartier, and the given morphism* $f: X \to Z$ *is surjective. Assume further that* $K_X + B + A \sim_{\mathbf{Q}} 0/Z$. *Then,*

(1) $(X/Z, B)$ *has a Mori fibre space or a log minimal model* $(Y/Z, B_Y)$,
(2) *if* $K_Y + B_Y$ *is nef/Z, then it is semi-ample/Z,*
(3) *if* $(X/Z, B)$ *is* $\mathbf{Q}$*-factorial dlt, then any LMMP/Z on* $K_X + B$ *with scaling of an ample/Z divisor terminates.*

In the theorem and throughout the paper Mori fibre spaces and log minimal models are meant as in Definitions 2.1 and 2.2 which are *slightly different* from the traditional definitions. The surjectivity of f is obviously not necessary but we have put there for practical convenience. Sketch of the main ideas of the proof of the theorem is given at the beginning of Section 6 and the full proof is given in the same section. A weaker form of the theorem was conjectured by Kollár [23, Conjecture 4.1], in the context of studying singularities.

As mentioned earlier, an immediate consequence of the above theorem concerns the existence of log flips for lc pairs.

Corollary **1.2.** — *Let* $(X/Z, B)$ *be a lc pair and* $X \to Y/Z$ *an extremal* $(K_X + B)$-*negative flipping contraction. Then, the* $(K_X + B)$-*flip of* $X \to Y$ *exists.*

The proof of the corollary is given at the end of Section 6. In view of Ambro [2] and Fujino [14], and the previous result, we can run the LMMP on any lc pair. Termination of such an LMMP can be reduced to the **Q**-factorial dlt case as in Remark 2.9.

Arguments very similar to those of the proof of Theorem 1.1 also work to prove a more general statement if we assume the ACC conjecture for lc thresholds.

Conjecture **1.3** (*ACC for lc thresholds*). — *Suppose that* $\Gamma \subseteq [0, 1]$ *and* $S \subseteq \mathbf{Q}$ *are finite sets of rational numbers, and* d *is a natural number. Then, the set*

$$\left\{ \mathrm{lct}(M, X, B) \mid (X, B) \text{ is lc of dimension} \leq d \right\}$$

satisfies the ACC where the coefficients of B *belong to* Γ, M *is a* **Q**-*Cartier divisor with coefficients in* S, *and* $\mathrm{lct}(M, X, B)$ *is the lc threshold of* M *with respect to* (X, B).

The conjecture is often formulated with Γ, S being DCC sets but we only need the finite case. The conjecture was proved for smooth varieties by de Fernex, Ein and Mustaţă [10] who used some of the ideas of Kollár [24]. Hacon, McKernan, and Xu have announced that they have solved the conjecture.

Theorem **1.4.** — *Assume Conjecture 1.3 in dimension* d. *Let* $(X/Z, B)$ *be a lc pair of dimension* d *where* B *is a* **Q**-*divisor and the given morphism* $f : X \to Z$ *is surjective. Assume further that* $K_X + B \sim_{\mathbf{Q}} 0$ *over some non-empty open subset* $U \subseteq Z$, *and if* η *is the generic point of a lc centre of* $(X/Z, B)$, *then* $f(\eta) \in U$. *Then,*

(1) $(X/Z, B)$ *has a log minimal model* $(Y/Z, B_Y)$,
(2) $K_Y + B_Y$ *is semi-ample*/Z,
(3) *if* $(X/Z, B)$ *is* **Q**-*factorial dlt, then any LMMP*/Z *on* $K_X + B$ *with scaling of an ample*/Z *divisor terminates.*

The proof is given in Section 7 using arguments quite similar to those of Section 6. In Section 5, we show that under finite generation a stronger statement holds:

Theorem **1.5.** — *Let* $(X/Z, B)$ *be a* **Q**-*factorial dlt pair where* B *is a* **Q**-*divisor and* $f : X \to Z$ *is surjective. Assume further that* $\mathcal{R}(X/Z, K_X + B)$ *is a finitely generated* $\mathcal{O}_Z$-*algebra, and that* $(K_X + B)|_{X_\eta} \sim_{\mathbf{Q}} 0$ *where* X_η *is the generic fibre of* f. *Then, any LMMP*/Z *on* $K_X + B$ *with scaling of an ample*/Z *divisor terminates with a good log minimal model.*

This in particular takes care of the klt case of Theorems 1.1 and 1.4.

Remark **1.6.** — Hacon and Xu [19] have independently obtained more general forms of some of our results using a circle of similar ideas. Compare 1.1, 1.2, 1.5, and 1.4 with [19, 1.6, 1.8, 2.11, 1.1].

An important ingredient of the proof of Theorems 1.1 and 1.4 is the following result which naturally appears when one tries to reduce the semi-ampleness of a lc divisor $K_X + B$ to semi-ampleness in the klt case (cf. [22]). It is a consequence of Kollár's injectivity theorem and semi-ampleness on semi-lc pairs. When Z is projective, it follows from Fujino-Gongyo [16]; the general case was proved by Hacon and Xu [20].

Theorem **1.7** ([16], [20]). — *Let* $(X/Z, B)$ *be a* **Q**-*factorial dlt pair and* $T := \lfloor B \rfloor$ *where* B *is a* **Q**-*divisor. Suppose that*

- $K_X + B$ *is nef*/Z,
- $(K_X + B)|_S$ *is semi-ample*/Z *for each component* S *of* T,
- $K_X + B - \epsilon P$ *is semi-ample*/Z *for some* **Q**-*divisor* $P \geq 0$ *with* $\operatorname{Supp} P = T$ *and for any sufficiently small rational number* $\epsilon > 0$.

Then, $K_X + B$ *is semi-ample*/Z.

We will now state some of the other results of this paper that are of independent interest. In Section 3, we prove the following which is similar to a result of Fujino [13].

Theorem **1.8.** — *Let* $(X/Z, B)$ *be a* **Q**-*factorial dlt pair with* $K_X + B \sim_{\mathbf{R}} M/Z$ *where* $M \geq 0$ *is very exceptional*/Z. *Then, any LMMP*/Z *on* $K_X + B$ *with scaling of an ample*/Z *divisor terminates with a model* Y *on which we have* $K_Y + B_Y \sim_{\mathbf{R}} M_Y = 0/Z$.

See Definition 3.1 for the notion of very exceptional divisors.

In Section 4, we prove a stronger version of [6, Theorem 1.5] which is repeatedly used in the subsequent sections:

Theorem **1.9.** — *Let* $(X/Z, B + C)$ *be a lc pair of dimension* d *such that* $K_X + B + C$ *is nef*/Z, $B, C \geq 0$ *and* C *is* **R**-*Cartier. Assume that we are given an LMMP*/Z *on* $K_X + B$ *with scaling of* C *as in Definition 2.4 with* λ_i *the corresponding numbers, and* $\lambda := \lim_{i \to \infty} \lambda_i$. *Then, the LMMP terminates in the following cases:*

(i) $(X/Z, B)$ *is* **Q**-*factorial dlt,* $B \geq H \geq 0$ *for some ample*/Z **R**-*divisor* H,
(ii) $(X/Z, B)$ *is* **Q**-*factorial dlt,* $C \geq H \geq 0$ *for some ample*/Z **R**-*divisor* H, *and* $\lambda > 0$,
(iii) $(X/Z, B + \lambda C)$ *has a log minimal model, and* $\lambda \neq \lambda_j$ *for any* j.

Finally, we briefly mention some previous works on flips. Mori proved the existence of flips for 3-folds with terminal singularities [26]. Shokurov proved it in full generality in dimension 3 [29], [30], in dimension 4 in the klt case [31], and also worked

out a significant proportion of what we know about flips in every dimension. Hacon-McKernan [17] filled in the missing parts of Shokurov's program on pl flips. Birkar-Cascini-Hacon-McKernan [9] (together with Hacon-McKernan [18]) proved the problem in the klt case in all dimensions (see also [8]). Fujino [12] proved it for lc pairs in dimension 4. Alexeev-Hacon-Kawamata [1] and Birkar [5] proved it in dimension 5 in the klt case.

2. Preliminaries

Notation and basic definitions. — We work over $k = \mathbf{C}$. A *pair* $(X/Z, B)$ consists of normal quasi-projective varieties X, Z over k, an $\mathbf{R}$-divisor B on X with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbf{R}$-Cartier, and a projective morphism $X \to Z$. For a prime divisor D on some birational model of X with a nonempty centre on X, $a(D, X, B)$ denotes the log discrepancy.

Let $(X/Z, B)$ be a lc pair. By a $K_X + B$-*flip*/Z we mean the flip of a $(K_X + B)$-negative extremal flipping contraction/Z. A *sequence of log flips*/Z *starting with* $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \to Z_i \leftarrow X_{i+1}$ is a $(K_{X_i} + B_i)$-flip/Z, B_i is the birational transform of B_1 on X_1, and $(X_1/Z, B_1) = (X/Z, B)$. *Special termination* means termination near $\lfloor B \rfloor$.

Definition **2.1** *(Weak lc and log minimal models).* — *A pair* $(Y/Z, B_Y)$ *is a* log birational model *of* $(X/Z, B)$ *if we are given a birational map* $\phi \colon X \dashrightarrow Y/Z$ *and* $B_Y = B^{\sim} + E$ *where* $B^{\sim}$ *is the birational transform of* B *and* E *is the reduced exceptional divisor of* ϕ^{-1}*, that is,* $E = \sum E_j$ *where* E_j *are the exceptional/X prime divisors on* Y*. A log birational model* $(Y/Z, B_Y)$ *is a* weak lc model *of* $(X/Z, B)$ *if*

- $K_Y + B_Y$ *is nef/Z, and*
- *for any prime divisor* D *on* X *which is exceptional/Y, we have*

$$a(D, X, B) \leq a(D, Y, B_Y)$$

A weak lc model $(Y/Z, B_Y)$ *is a* log minimal model *of* $(X/Z, B)$ *if*

- $(Y/Z, B_Y)$ *is* $\mathbf{Q}$*-factorial dlt,*
- *the above inequality on log discrepancies is strict.*

A log minimal model $(Y/Z, B_Y)$ *is* good *if* $K_Y + B_Y$ *is semi-ample/Z.*

Definition **2.2** *(Mori fibre space).* — *A log birational model* $(Y/Z, B_Y)$ *of a pair* $(X/Z, B)$ *is called a* Mori fibre space *if* $(Y/Z, B_Y)$ *is* $\mathbf{Q}$*-factorial dlt, there is a* $(K_Y + B_Y)$*-negative extremal contraction* $Y \to T/Z$ *with* $\dim Y > \dim T$*, and*

$$a(D, X, B) \leq a(D, Y, B_Y)$$

for any prime divisor D *(on birational models of* X*) and strict inequality holds if* D *is on* X *and contracted*$/Y$.

Definition **2.3** *(Log smooth model).* — *We need to define various kinds of log smooth models that in many situations allow us to replace a lc pair with a log smooth one. Let* $(X/Z, B)$ *be a lc pair, and let* $f: W \to X$ *be a log resolution. Let* $B_W \geq 0$ *be a boundary on* W *so that* $K_W + B_W = f^*(K_X + B) + E$ *where* $E \geq 0$ *is exceptional*$/X$ *and the support of* E *contains each prime exceptional*$/X$ *divisor* D *on* W *if* $a(D, X, B) > 0$. *We call* $(W/Z, B_W)$ *a* log smooth model *of* $(X/Z, B)$. *However, in practice we usually need further assumptions. We list the ones we will need:*

Type (1): We take B_W *to be the birational transform of* B *plus the reduced exceptional divisor of* f, *that is, we assume that* $a(D, W, B_W) = 0$ *for each prime exceptional*$/X$ *divisor* D *on* W.
Type (2): We assume that $a(D, W, B_W) > 0$ *if* $a(D, X, B) > 0$, *for each prime exceptional*$/X$ *divisor* D *on* W.

Note that if $(X/Z, B)$ *is klt and* $(W/Z, B_W)$ *is of type (2), then* $(W/Z, B_W)$ *is also klt.*

Definition **2.4** *(LMMP with scaling).* — *Let* $(X_1/Z, B_1 + C_1)$ *be a lc pair such that* $K_{X_1} + B_1 + C_1$ *is nef*$/Z$, $B_1 \geq 0$, *and* $C_1 \geq 0$ *is* **R***-Cartier. Suppose that either* $K_{X_1} + B_1$ *is nef*$/Z$ *or there is an extremal ray* R_1/Z *such that* $(K_{X_1} + B_1) \cdot R_1 < 0$ *and* $(K_{X_1} + B_1 + \lambda_1 C_1) \cdot R_1 = 0$ *where*

$$\lambda_1 := \inf\{t \geq 0 \mid K_{X_1} + B_1 + tC_1 \text{ is nef}/Z\}$$

Now, if $K_{X_1} + B_1$ *is nef*$/Z$ *or if* R_1 *defines a Mori fibre structure, we stop. Otherwise assume that* R_1 *gives a divisorial contraction or a log flip* $X_1 \dashrightarrow X_2$. *We can now consider* $(X_2/Z, B_2 + \lambda_1 C_2)$ *where* $B_2 + \lambda_1 C_2$ *is the birational transform of* $B_1 + \lambda_1 C_1$ *and continue. That is, suppose that either* $K_{X_2} + B_2$ *is nef*$/Z$ *or there is an extremal ray* R_2/Z *such that* $(K_{X_2} + B_2) \cdot R_2 < 0$ *and* $(K_{X_2} + B_2 + \lambda_2 C_2) \cdot R_2 = 0$ *where*

$$\lambda_2 := \inf\{t \geq 0 \mid K_{X_2} + B_2 + tC_2 \text{ is nef}/Z\}$$

By continuing this process, we obtain a sequence of numbers λ_i *and a special kind of LMMP*$/Z$ *which is called the* LMMP$/Z$ *on* $K_{X_1} + B_1$ *with scaling of* C_1. *Note that by definition* $\lambda_i \geq \lambda_{i+1}$ *for every* i, *and we usually put* $\lambda = \lim_{i \to \infty} \lambda_i$. *When we refer to* termination with scaling *we mean termination of such an LMMP.*

When we have a lc pair $(X/Z, B)$, *we can always find an ample*$/Z$ **R***-Cartier divisor* $C \geq 0$ *such that* $K_X + B + C$ *is lc and nef*$/Z$, *so we can run the LMMP*$/Z$ *with scaling assuming that all the necessary ingredients exist, e.g. extremal rays, log flips. In particular, when* $(X/Z, B)$ *is klt or* **Q***-factorial dlt the required extremal rays and log flips exist by [9] and [5, Lemma 3.1].*

Definition **2.5.** — *Let* $f: X \to Z$ *be a projective morphism of quasi-projective varieties with* X *normal, and* D *an* **R***-divisor on* X. *Define the divisorial sheaf algebra of* D *as*

$$\mathcal{R}(X/Z, D) = \bigoplus_{m \geq 0} f_* \mathcal{O}_X\big(\lfloor mD \rfloor\big)$$

where $m \in \mathbf{Z}$. If Z is affine, we take $R(X/Z, D)$ to be the global sections of $\mathcal{R}(X/Z, D)$.

By a *contraction* $f \colon X \to Z$ we mean a projective morphism of quasi-projective varieties with $f_* \mathcal{O}_X = \mathcal{O}_Z$.

Some basic facts about log minimal models and the LMMP. — We collect some basic properties of log minimal models and the LMMP that will be used in this paper.

Remark **2.6** *(Weak lc model).* — Let $(Y/Z, B_Y)$ be a weak lc model of a lc pair $(X/Z, B)$ and $\phi \colon X \dashrightarrow Y$ the corresponding birational map. Let $f \colon W \to X$ and $g \colon W \to Y$ be a common log resolution of $(X/Z, B)$ and $(Y/Z, B_Y)$. Let

$$E := f^*(K_X + B) - g^*(K_Y + B_Y) = \sum_D \big(a(D, Y, B_Y) - a(D, X, B)\big)D$$

where D runs over the prime divisors on W. We will show that E is effective and exceptional$/Y$. From the definition of weak lc models we get $f_* E \geq 0$, and since E is antinef$/X$, applying the negativity lemma implies that $E \geq 0$. Now assume that D is a component of E which is not exceptional$/Y$. Then, D is exceptional$/X$ otherwise $a(D, X, B) = a(D, Y, B_Y)$ and D could not be a component of E. By definition of weak lc models we get $a(D, Y, B_Y) = 0$, so by effectiveness of E we have $a(D, X, B) = 0$ which again shows that D cannot be a component of E. Therefore, E is exceptional$/Y$.

Remark **2.7** *(Two weak lc models).* — Let $(Y_1/Z, B_{Y_1})$ and $(Y_2/Z, B_{Y_2})$ be two weak lc models of a lc pair $(X/Z, B)$. Let $f \colon W \to X$ and $g_i \colon W \to Y_i$ be a common resolution, and put

$$E_i := f^*(K_X + B) - g_i^*(K_{Y_i} + B_{Y_i})$$

Then, by Remark 2.6, E_i is effective and exceptional$/Y_i$. Since $g_{2*}(E_1 - E_2) \geq 0$ and $E_1 - E_2$ is anti-nef$/Y_2$, by the negativity lemma, $E_1 - E_2 \geq 0$. Similarly, $E_2 - E_1 \geq 0$. Therefore,

$$g_1^*(K_{Y_1} + B_{Y_1}) = g_2^*(K_{Y_2} + B_{Y_2})$$

In particular, if $K_{Y_1} + B_{Y_1}$ is ample$/Z$, then $K_{Y_2} + B_{Y_2}$ is semi-ample$/Z$ and the birational map $Y_2 \dashrightarrow Y_1$ is actually a morphism which pulls back $K_{Y_1} + B_{Y_1}$ to $K_{Y_2} + B_{Y_2}$.

Remark **2.8** *(Log smooth models).* — Log smooth models satisfy certain nice properties besides being simple in terms of singularities. Let $(W/Z, B_W)$ be a log smooth model of a

lc pair $(X/Z, B)$. Let D be a prime divisor on W. Then, $a(D, X, B) \geq a(D, W, B_W)$ with strict inequality iff D is exceptional$/X$ and $a(D, X, B) > 0$.

Another basic property of $(W/Z, B_W)$ is that any log minimal model of $(W/Z, B_W)$ is also a log minimal model of $(X/Z, B)$. Indeed let $(Y/Z, B_Y)$ be a log minimal model of $(W/Z, B_W)$. Let $e \colon V \to W$ and $h \colon V \to Y$ be a common resolution. Then,

$$e^*(K_W + B_W) = h^*(K_Y + B_Y) + G$$

where $G \geq 0$ is exceptional$/Y$ by Remark 2.6. Thus, using

$$K_W + B_W = f^*(K_X + B) + E$$

we get

$$e^* f^*(K_X + B) = h^*(K_Y + B_Y) + G - e^* E$$

Since $G - e^* E$ is antinef$/X$, and $f_* e_*(G - e^* E) \geq 0$, by the negativity lemma, $G - e^* E \geq 0$. In particular, this means that $a(D, X, B) \leq a(D, Y, B_Y)$ for any prime divisor D on V. Next we will compare log discrepancies of prime divisors on X and Y.

If D is a prime divisor on X which is exceptional$/Y$, then

$$a(D, X, B) = a(D, W, B_W) < a(D, Y, B_Y)$$

On the other hand, let D be a prime divisor on Y which is exceptional$/X$. If D is exceptional$/W$, then by definition of log minimal models, $a(D, Y, B_Y) = 0$. Assume that D is not exceptional$/W$. Then, $a(D, W, B_W) = a(D, Y, B_Y)$ which implies that $a(D, X, B) \leq a(D, W, B_W)$ because of the relation $a(D, X, B) \leq a(D, Y, B_Y)$ obtained above. This is possible only if

$$a(D, X, B) = a(D, W, B_W) = 0$$

hence again $a(D, Y, B_Y) = 0$. Therefore, the prime exceptional divisors of $Y \dashrightarrow X$ appear with coefficient one in B_Y. Finally, if D is a prime divisor on Y which is not exceptional$/X$, or if D is a prime divisor on X which is not exceptional$/Y$, then

$$a(D, X, B) = a(D, W, B_W) = a(D, Y, B_Y)$$

So, B_Y is the birational transform of B plus the reduced exceptional divisor of $Y \dashrightarrow X$ hence $(Y/Z, B_Y)$ is a log minimal model of $(X/Z, B)$.

Remark **2.9** *(Lifting a sequence of log flips with scaling).* — (1) Assume that we are given an LMMP with scaling as in Definition 2.4 which consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips. Let $(X_1'/Z, B_1')$ be a **Q**-factorial dlt blowup of $(X_1/Z, B_1)$ and C_1' the pullback of C_1 (this exists by Corollary 3.6 below). Since $K_{X_1} + B_1 + \lambda_1 C_1 \equiv 0/Z_1$, we get $K_{X_1'} + B_1' + \lambda_1 C_1' \equiv 0/Z_1$.

Run an LMMP/Z_1 on $K_{X'_1} + B'_1$ with scaling of some ample/Z_1 divisor which is automatically also an LMMP/Z_1 on $K_{X'_1} + B'_1$ with scaling of $\lambda_1 C'_1$. Assume that this LMMP terminates with a log minimal model $(X'_2/Z_1, B'_2)$. By construction, $(X_2/Z_1, B_2)$ and $(X'_2/Z_1, B'_2)$ are both weak lc models of $(X'_1/Z_1, B'_1)$. By Remark 2.7, X'_2 maps to X_2 and $K_{X'_2} + B'_2$ is the pullback of $K_{X_2} + B_2$. Thus, $(X'_2/Z, B'_2)$ is a **Q**-factorial dlt blowup of $(X_2/Z, B_2)$. Now, $K_{X'_2} + B'_2 + \lambda_1 C'_2 \equiv 0/Z_1$ where C'_2 is the birational transform of C'_1 and actually the pullback of C_2. We can continue this process: that is use the fact that $K_{X_2} + B_2 + \lambda_2 C_2 \equiv 0/Z_2$ and $K_{X'_2} + B'_2 + \lambda_2 C'_2 \equiv 0/Z_2$ and run an LMMP/Z_2 on $K_{X'_2} + B'_2$, etc.

We have shown that we can lift the original sequence to an LMMP/Z on $K_{X'_1} + B'_1$ with scaling of C'_1 assuming that the following statement holds for each i:

(∗) some LMMP/Z_i on $K_{X'_i} + B'_i$ with scaling of some ample/Z_i divisor terminates where $(X'_i/Z, B'_i)$ is a **Q**-factorial dlt blowup of $(X_i/Z, B_i)$.

Note however that each $X'_i \dashrightarrow X'_{i+1}$ is a sequence of log flips and divisorial contractions (not necessarily just one log flip).

(2) We will show that (∗) holds for i if $(X_i/Z, tB_i)$ is klt for some $t \in [0, 1]$ (in particular, this holds if $(X_i/Z, B_i)$ is klt or **Q**-factorial dlt). Pick an ample/Z_i **R**-divisor $H \geq 0$ on X_i so that

$$K_{X_i} + B_i + H \sim_{\mathbf{R}} 0/Z_i$$

and $(X_i/Z, B_i + H)$ is lc. Now for a sufficiently small $\epsilon > 0$, there is a small $\delta > 0$ and Δ_i such that

$$(1 - \delta)B_i \leq \Delta_i \sim_{\mathbf{R}} B_i + \epsilon H/Z_i$$

and that $(X_i/Z, \Delta_i)$ is klt: if $t = 1$, then we can just take $\Delta_i = B_i + \epsilon H$; but if $t < 1$, then B_i is **R**-Cartier and we take $\delta > 0$ small enough so that $\delta B_i + \epsilon H$ is ample/Z_i, and we put $\Delta_i = (1 - \delta)B_i + H'$ for a general $H' \sim_{\mathbf{R}} \delta B_i + \epsilon H/Z_i$. Now let $K_{X'_i} + \Delta'_i$ be the pullback of $K_{X_i} + \Delta_i$. We can assume that $\Delta'_i \geq 0$ by choosing δ small enough. Now run an LMMP/Z_i on $K_{X'_i} + \Delta'_i$ with scaling of some ample/Z_i divisor. By [9], the LMMP terminates. By construction,

$$K_{X'_i} + \Delta'_i \sim_{\mathbf{R}} (1 - \epsilon)\big(K_{X'_i} + B'_i\big)/Z_i$$

so the LMMP is also an LMMP/Z_i on $K_{X'_i} + B'_i$.

Remark **2.10** *(On special termination).* — Assume that we are given an LMMP with scaling as in Definition 2.4 which consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips, and that $(X_1/Z, B_1)$ is **Q**-factorial dlt. Assume $\lfloor B_1 \rfloor \neq 0$ and pick a component S_1 of $\lfloor B_1 \rfloor$. Let $S_i \subset X_i$ be the birational transform of S_1 and T_i the normalisation of the image of S_i in Z_i. Using standard special termination arguments, we will see that termination

of the LMMP near S_1 is reduced to termination in lower dimensions. It is well-known that the induced map $S_i \dashrightarrow S_{i+1}/T_i$ is an isomorphism in codimension one if $i \gg 0$ (cf. [11]). So, we could assume that these maps are all isomorphisms in codimension one. Put $K_{S_i} + B_{S_i} := (K_{X_i} + B_i)|_{S_i}$. In general, $S_i \dashrightarrow S_{i+1}/T_i$ is not a $K_{S_i} + B_{S_i}$-flip. To apply induction, we need to simplify the situation as follows.

Assume that $(S_1', B_{S_1'})$ is a **Q**-factorial dlt blowup of (S_1, B_{S_1}) (this exists by Corollary 3.6 below). The idea is to use the sequence $S_i \dashrightarrow S_{i+1}/T_i$ to construct an LMMP/T on $K_{S_1'} + B_{S_1'}$ with scaling of $C_{S_1'}$ where T is the normalisation of the image of S_1 in Z and $C_{S_1'}$ is the pullback of C_1. This can be done similar to Remark 2.9 assuming that something like (∗) is satisfied (in practice, this is satisfied by induction; it also can be derived from Theorem 4.1 and this remark is applied only after 4.1). More precisely, we first run an LMMP/T_1 on $K_{S_1'} + B_{S_1'}$. This is also an LMMP/T_1 on $K_{S_1'} + B_{S_1'}$ with scaling of $\lambda_1 C_{S_1'}$ because $K_{S_1'} + B_{S_1'} + \lambda_1 C_{S_1'} \equiv 0/T_1$. Assuming this terminates, we get a model S_2' on which $K_{S_2'} + B_{S_2'}$ is nef/T_1. Since $S_1 \dashrightarrow S_2$ is an isomorphism in codimension one and $K_{S_2} + B_{S_2}$ is ample/T_1, $(S_2/T_1, B_{S_2})$ is the lc model of all the pairs $(S_1/T_1, B_{S_1})$, $(S_1'/T_1, B_{S_1'})$, and $(S_2'/T_1, B_{S_2'})$. Thus, $K_{S_2'} + B_{S_2'}$ is semi-ample/T_1 and the map $S_2' \dashrightarrow S_2$ is a morphism which pulls back $K_{S_2} + B_{S_2}$ to $K_{S_2'} + B_{S_2'}$. We continue the process by running an LMMP/T_2 on $K_{S_2'} + B_{S_2'}$ and so on. So, we get an LMMP/T on $K_{S_1'} + B_{S_1'}$ with scaling of $C_{S_1'}$.

If the LMMP/T on $K_{S_1'} + B_{S_1'}$ terminates, then the original LMMP terminates near S_1. One usually applies this argument to every component of $\lfloor B_1 \rfloor$ to derive termination near $\lfloor B_1 \rfloor$.

3. LMMP on very exceptional divisors

Let $(X/Z, B)$ be a **Q**-factorial dlt pair such that $X \to Z$ is birational and $K_X + B \sim_{\mathbf{R}} M/Z$ with $M \geq 0$ exceptional/Z. Run an LMMP/Z on $K_X + B$ with scaling of an ample divisor. In some step of the LMMP, $K_X + B$ becomes nef along very general curves of D/Z for any prime divisor D. Since M is exceptional/Z, this is possible only if M is contracted in the process (this follows from a general form of the negativity lemma that is discussed below). So, the LMMP terminates. This is useful in many situations, for example, to construct a dlt blowup of a lc pair. It also plays a crucial role in the proof of Theorem 1.5.

In this section, we generalise and make precise the above phenomenon. Many of the ideas in this section (and in the proof of Theorem 1.5) are actually explicit or implicit in Shokurov [31]. However, we would like to give full details here.

Definition **3.1** *(cf. Shokurov [31, Definition 3.2]). — Let $f: X \to Y$ be a contraction of normal varieties, D an **R**-divisor on X, and $V \subset X$ a closed subset. We say that V is vertical over Y if $f(V)$ is a proper subset of Y. We say that D is very exceptional/Y if D is vertical/Y and for any*

prime divisor P *on* Y *there is a prime divisor* Q *on* X *which is not a component of* D *but* $f(Q) = P$, *i.e. over the generic point of* P *we have:* $\operatorname{Supp} f^*P \nsubseteq \operatorname{Supp} D$.

If $\operatorname{codim} f(D) \geq 2$, then D is very exceptional. On the other hand, when f is birational, then exceptional and very exceptional are equivalent notions. The next lemma indicates where one can expect to find very exceptional divisors.

Lemma **3.2** *(cf. Shokurov [31, Lemma 3.19]). — Let* $f : X \to Y$ *be a contraction of normal varieties, projective over a normal affine variety* Z. *Let* A *be an ample/*Z *divisor on* Y *and* $F = f^*A$. *If* $E \geq 0$ *is a divisor on* X *which is vertical/*Y *and such that* $mE = \operatorname{Fix}(mF + mE)$ *for every integer* $m \gg 0$, *then* E *is very exceptional/*Y.

Proof. — We can assume that A is very ample/Z. Since E is effective, for each integer $l > 0$, we have the natural exact sequence $0 \to \mathscr{O}_X \to \mathscr{O}_X(lE)$ given by the inclusion $\mathscr{O}_X \subseteq \mathscr{O}_X(lE)$. Since $mF = \operatorname{Mov}(mF + lE)$ for each $m \gg 0$, the induced homomorphism

$$\bigoplus_{m \in \mathbf{Z}} H^0(X, mF) \to \bigoplus_{m \in \mathbf{Z}} H^0(X, mF + lE)$$

of $R(X/Z, F)$-modules is an isomorphism in large degrees. This in turn induces a homomorphism

$$\bigoplus_{m \in \mathbf{Z}} H^0(Y, mA) \to \bigoplus_{m \in \mathbf{Z}} H^0\big(Y, \big(f_*\mathscr{O}_X(lE)\big)(mA)\big)$$

of $R(Y/Z, A)$-modules which is an isomorphism in large degrees. Therefore, by [21, II, Proposition 5.15 and Exercise 5.9], the injective morphism $\mathscr{O}_Y = f_*\mathscr{O}_X \to f_*\mathscr{O}_X(lE)$ is also surjective hence we get $\mathscr{O}_Y = f_*\mathscr{O}_X = f_*\mathscr{O}_X(lE)$.

Assume that E is not very exceptional/Y. Then, there exist a prime divisor P on Y such that if Q is any prime divisor on X with $f(Q) = P$, then Q is a component of E. Let U be a smooth open subset of Y so that $\mathscr{O}_U \subsetneq \mathscr{O}_U(P|_U)$, $P|_U$ is Cartier, and each component of $f^*P|_U$ maps onto $P|_U$. Let $V = f^{-1}U$. Then, $\operatorname{Supp} f^*P|_U \subseteq \operatorname{Supp} E|_V$ and for some $l > 0, f^*P|_U \leq lE|_V$. But then,

$$\mathscr{O}_U \subsetneq \mathscr{O}_U(P|_U) \subseteq f_*\mathscr{O}_V(lE|_V) = \mathscr{O}_U$$

which is a contradiction. Therefore, E is very exceptional/Y. $\square$

Let $S \to Z$ be a projective morphism of varieties and M an **R**-Cartier divisor on S. We say that M is *nef on the very general curves of* S/Z if there is a countable union Λ of proper closed subsets of S such that $M \cdot C \geq 0$ for any curve C on S contracted over Z satisfying $C \nsubseteq \Lambda$. We need the following general negativity lemma of Shokurov [31, Lemma 3.22] (see also Prokhorov [28, Lemma 1.7]).

Lemma **3.3** *(Negativity).* — *Let* $f : X \to Z$ *be a contraction of normal varieties. Let* D *be an* **R**-*Cartier divisor on* X *written as* $D = D^+ - D^-$ *with* $D^+, D^- \geq 0$ *having no common components. Assume that* D^- *is very exceptional/Z, and that for each component* S *of* D^-, $-D|_S$ *is nef on the very general curves of* S/Z. *Then,* $D \geq 0$.

Proof. — Assume that $D^- \neq 0$ otherwise there is nothing to prove. By assumptions, there is a countable union Λ of codim ≥ 2 proper closed subsets of X such that $\Lambda \subset \operatorname{Supp} D^-$ and such that for any component S of D^- and any curve C of S/Z satisfying $C \nsubseteq \Lambda$, we have $-D \cdot C \geq 0$. Let $P = f(\operatorname{Supp} D^-)$. By shrinking Z if necessary we can assume that P is irreducible and that every component of D^- maps onto P. When $\dim P > 0$ we reduce the problem to lower dimension by taking a hyperplane section of Z. In contrast, when $\dim P = 0$ we reduce the problem to lower dimension by taking a hyperplane section of X except when $\dim Z = 1$ or $\dim Z = \dim X = 2$ which can be dealt with directly.

Assume that $\dim P > 0$. Let Z' be a very general hyperplane section of Z, $X' = f^*Z'$, and let f' be the induced contraction $X' \to Z'$. Since Z' is very general, it intersects P, and X' does not contain any component of D nor any "component" of Λ. So, $\Lambda \cap X'$ is again a countable union of codim ≥ 2 proper closed subsets of X'. Since X' is general, it does not contain any component of the singular locus of X hence $D|_{X'}$ is determined on the smooth locus of X. Similarly, the negative part of $D|_{X'}$ is given by $D^-|_{X'}$ defined on the smooth locus of X. Moreover, we can assume that X' does not contain any component of $S_1 \cap S_2$ if S_1, S_2 are prime divisors mapping onto P.

We show that the negative part of $D|_{X'}$, say $D|_{X'}^-$, is very exceptional/Z': if codim $P \geq 2$, the claim is trivial; if codim $P = 1$, we can assume that Z is smooth at every point of P and that every component of $D|_{X'}^-$ maps onto $P' := Z' \cap P$; now since D^- is very exceptional/Z, there is a component S of f^*P mapping onto P such that S is not a component of D^-; let Q be a component of $S \cap X'$ having codimension one in X' and mapping onto P'; then, by our choice of X', Q is not a component of $D|_{X'}^-$. Since Q and each component of $D|_{X'}^-$ maps onto P', and since Q is not a component of $D|_{X'}^-$, indeed $D|_{X'}^-$ is very exceptional/Z' as claimed. On the other hand, if T is any component of $D|_{X'}^-$, then T is a component of $L \cap X'$ for some component L of D^- hence $-D|_T$ is nef on the curves of T/Z' outside $\Lambda \cap X'$ (note that by construction $T \nsubseteq \Lambda$). By induction on dimension applied to $D|_{X'}$ and $X' \to Z'$, $D|_{X'}^- = 0$ which is a contradiction.

From now on we can assume that $\dim P = 0$. Assume $\dim Z = 1$. Then f^*P is a divisor which is numerically zero over Z. Let t be the smallest real number such that $D + tf^*P \geq 0$. Then, there is a component S of D^- which has coefficient zero in $D + tf^*P \geq 0$ but such that S intersects $\operatorname{Supp}(D + tf^*P)$. If C is any curve on S which is not inside $S \cap \operatorname{Supp}(D + tf^*P)$ but such that C intersects $\operatorname{Supp}(D + tf^*P)$, then $D \cdot C = (D + tf^*P) \cdot C > 0$. This contradicts our assumptions. On the other hand, if $\dim X = \dim Z = 2$, the lemma is quite well-known and elementary and it can be proved similarly. So, from now on we assume that $\dim Z \geq 2$, and $\dim X \geq 3$.

Let H be a very general hyperplane section of X and let $g\colon H \to G$ be the contraction given by the Stein factorisation of $H \to Z$. Since $P = f(\operatorname{Supp} D^-)$ is just a point and $\dim Z \geq 2$ and $\dim X \geq 3$, it is obvious that $D|_H^-$, the negative part of $D|_H$, is very exceptional/G. Moreover, arguments similar to the above show that if T is any component of $D|_H^-$, then $-D|_H$ is nef on the very general curves of T/G. So, we can apply induction to deduce that $D|_H^- = 0$ which gives a contradiction. $\square$

We now come to the LMMP that was mentioned at the beginning of this section. It is a simple consequence of the negativity lemma.

Theorem **3.4** *(= Theorem 1.8). — Let* $(X/Z, B)$ *be a* **Q**-*factorial dlt pair such that* $K_X + B \sim_{\mathbf{R}} M/Z$ *with* $M \geq 0$ *very exceptional/Z. Then, any LMMP/Z on $K_X + B$ with scaling of an ample/Z divisor terminates with a model Y on which $K_Y + B_Y \sim_{\mathbf{R}} M_Y = 0/Z$.*

Proof. — Assume that $C \geq 0$ is an ample/Z **R**-divisor such that $K_X + B + C$ is lc and nef/Z. Run the LMMP/Z on $K_X + B$ with scaling of C. The only divisors that can be contracted are the components of M hence M remains very exceptional/Z during the LMMP. We may assume that the LMMP consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of $K_X + B$-flips/Z with $X_1 = X$. If λ_i are the numbers appearing in the LMMP, and $\lambda := \lim_{i \to \infty} \lambda_i$, then by [9], we may assume that $\lambda = 0$. Since $\lambda = 0$, $K_X + B$ is a limit of movable/Z **R**-Cartier divisors hence for any prime divisor S on X, $(K_X + B) \cdot \Gamma = M \cdot \Gamma \geq 0$ for the very general curves Γ of S/Z. Now by assumptions M is very exceptional/Z hence by the negativity lemma (3.3), $M \leq 0$ which implies that $M = 0$. So, the LMMP terminates and contracts M. $\square$

The same arguments as in the previous theorem imply:

Theorem **3.5.** *— Let* $(X/Z, B)$ *be a* **Q**-*factorial dlt pair such that $X \to Z$ is birational, and $K_X + B \sim_{\mathbf{R}} M = M^+ - M^-/Z$ where $M^+, M^- \geq 0$ have no common components and M^+ is exceptional/Z. Then, any LMMP/Z on $K_X + B$ with scaling of an ample/Z divisor contracts M^+ after finitely many steps.*

The birationality condition on $X \to Z$ in the previous theorem is needed to make sure that M^+ remains very exceptional/Z during the LMMP.

An application of the above theorems is the existence of **Q**-factorial dlt blowups which is due to Hacon (cf. Fujino [13, Theorem 4.1]).

Corollary **3.6.** *— Let* $(X/Z, B)$ *be a lc pair. Then, there is a* **Q**-*factorial dlt blowup of* $(X/Z, B)$.

Proof. — Let $(W/Z, B_W)$ be a log smooth model of $(X/Z, B)$ of type (1) as in Definition 2.3 constructed via a log resolution $f\colon W \to X$. Then, $K_W + B_W = f^*(K_X + B) + E$

where $E \geq 0$ is exceptional/X. Now, by Theorem 3.4 some LMMP/X on $K_W + B_W$ with scaling of an ample divisor terminates with a model Y on which $K_Y + B_Y \sim_{\mathbf{R}} 0/X$. If $g \colon Y \to X$ is the induced morphism, then $K_Y + B_Y = g^*(K_X + B)$ because E is contracted over Y. Now $(Y/Z, B_Y)$ is the desired **Q**-factorial dlt blowup. □

Corollary **3.7.** — *Let* $(X/Z, B)$ *be a lc pair. If* $(X/Z, B)$ *has a weak lc model, then it has a log minimal model.*

Proof. — Assume that $(Y'/Z, B_{Y'})$ is a weak lc model of $(X/Z, B)$. Let $(W/Z, B_W)$ be a log smooth model of $(X/Z, B)$ of type (1) constructed via a log resolution $f \colon W \to X$. We can assume that the induced map $g \colon W \dashrightarrow Y'$ is a morphism. We can write

$$K_W + B_W = f^*(K_X + B) + E$$

and

$$f^*(K_X + B) = g^*(K_{Y'} + B_{Y'}) + G$$

where $E \geq 0$ is exceptional/X and $G \geq 0$ is exceptional/Y' (see Remark 2.6). Now, $K_W + B_W \sim_{\mathbf{R}} G + E/Y'$. Moreover, $G + E$ is exceptional/Y': G is exceptional/Y' as noted; on the other hand if D is a component of E, then $a(D, X, B) > 0$ which implies that D is exceptional/Y' otherwise $0 = a(D, Y', B_{Y'}) = a(D, X, B)$; so, E is also exceptional/Y'.

By Theorem 3.4, some LMMP/Y' on $K_W + B_W$ with scaling of an ample divisor terminates with a model Y on which $K_Y + B_Y \sim_{\mathbf{R}} 0/Y'$. This means that $K_Y + B_Y$ is the pullback of $K_{Y'} + B_{Y'}$ hence $K_Y + B_Y$ is nef/Z and $(Y/Z, B_Y)$ is a log minimal model of $(W/Z, B_W)$ hence of $(X/Z, B)$ by Remark 2.8. □

4. From log minimal models to termination with scaling

In this section we prove Theorem 1.9 which is essentially [6, Theorem 1.5]but with weaker assumptions. The theorem is of independent interest and it will be used repeatedly in subsequent sections. The proof follows the general idea of deriving termination with scaling from existence of log minimal models that was developed in [9] and [32]. In [9], finiteness of models is used to get termination. However, this works only in the klt case when the boundary is big, or in the dlt case when the boundary contains an ample divisor. The proof below is closer to [32] in spirit but we still make use of [9]. Much of the difficulties in the proof are caused by the presence of non-klt singularities.

Theorem **4.1** *(=Theorem 1.9).* — *Let* $(X/Z, B + C)$ *be a lc pair of dimension d such that* $K_X + B + C$ *is nef/Z*, $B, C \geq 0$ *and* C *is* **R**-*Cartier. Assume that we are given an LMMP/Z on* $K_X + B$ *with scaling of* C *as in Definition 2.4 with* λ_i *the corresponding numbers, and* $\lambda := \lim_{i \to \infty} \lambda_i$. *Then, the LMMP terminates in the following cases:*

(i) $(X/Z, B)$ *is* $\mathbf{Q}$*-factorial dlt,* $B \geq H \geq 0$ *for some ample*$/Z$ $\mathbf{R}$*-divisor* H,

(ii) $(X/Z, B)$ *is* $\mathbf{Q}$*-factorial dlt,* $C \geq H \geq 0$ *for some ample*$/Z$ $\mathbf{R}$*-divisor* H, *and* $\lambda > 0$,

(iii) $(X/Z, B + \lambda C)$ *has a log minimal model, and* $\lambda \neq \lambda_j$ *for any* j.

The following two lemmas essentially contain the main points of the proof of Theorem 4.1. We will reduce the proof of Theorem 4.1 to these lemmas. When $(X/Z, B + C)$ is $\mathbf{Q}$-factorial klt, then one can easily see that the assumptions of Lemma 4.3 are automatically satisfied, so in this case we do not need to go beyond the lemmas.

Lemma **4.2.** *— Theorem* 4.1(i) *and* (ii) *hold.*

Proof. — (i) Since H is ample$/Z$, we can perturb the coefficients of B hence assume that $(X/Z, B)$ is klt. If $\lambda_i < 1$ for some i, then $(X/Z, B + \lambda_i C)$ is klt and after finitely many steps we could replace C with $\lambda_i C$ hence we could assume that $(X/Z, B + C)$ is klt and B is big$/Z$. We can then apply [9]. Now, assume that $\lambda_i = 1$ for every i. Then, the LMMP is also an LMMP$/Z$ on $K_X + B + \frac{1}{2}C$ with scaling of $\frac{1}{2}C$. By replacing B with $B + \frac{1}{2}C$ and C with $\frac{1}{2}C$, we can assume that every lc centre of $(X/Z, B + C)$ is inside $\operatorname{Supp} B$. Now perturb B again so that $(X/Z, B + C)$ becomes klt and apply [9].

(ii) The LMMP is also an LMMP$/Z$ on $K_X + B + \frac{\lambda}{2}C$ with scaling of $(1 - \frac{\lambda}{2})C$. We can replace B with $B + \frac{\lambda}{2}C$, C with $(1 - \frac{\lambda}{2})C$, and λ_i with $\frac{\lambda_i - \frac{\lambda}{2}}{1 - \frac{\lambda}{2}}$. After this change, we can assume that $B \geq \frac{\lambda}{2}H$. Now use (i). $\qquad\square$

Lemma **4.3.** *— Theorem* 4.1(iii) *holds if there is a birational map* $\phi\colon X \dashrightarrow Y/Z$ *satisfying:*

- *the induced maps* $X_i \dashrightarrow Y$ *are isomorphisms in codimension one for every* $i \gg 0$ *where* X_i *is the variety corresponding to* λ_i,
- $(Y/Z, B_Y + \lambda C_Y)$ *is a log minimal model of* $(X/Z, B + \lambda C)$ *with respect to the given map* ϕ,
- *there is a reduced divisor* $A \geq 0$ *on* X *whose components are movable*$/Z$ *and they generate* $N^1(X/Z)$,
- $(X/Z, B + C + \epsilon A)$ *and* $(Y/Z, B_Y + \lambda C_Y + \delta C_Y + \epsilon A_Y)$ *are* $\mathbf{Q}$*-factorial dlt for some* $\delta, \epsilon > 0$.

(As usual, for a divisor D, D_Y *denotes the birational transform on* Y.)

Proof.

Step 1. Assume that the LMMP does not terminate. Pick $j \gg 0$ so that $\lambda_{j-1} > \lambda_j$. Then, $X \dashrightarrow X_j$ is a partial LMMP$/Z$ on $K_X + B + \lambda_j C$. It is also a partial LMMP$/Z$ on $K_X + B + \lambda_j C + \epsilon A$ maybe after choosing a smaller $\epsilon > 0$. In particular, $(X_j/Z, B_j + \lambda_j C_j + \epsilon A_j)$ is $\mathbf{Q}$-factorial dlt. Now we can replace $(X/Z, B + C)$ with $(X_j/Z, B_j + \lambda_j C_j)$ hence assume that the LMMP consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log

flips/Z starting with $(X_1/Z, B_1) = (X/Z, B)$. Moreover, by replacing B with $B + \lambda C$ we may also assume that $\lambda = 0$.

We will reduce the problem to the situation in which not only $K_Y + B_Y$ is nef/Z but also $K_Y + B_Y + \lambda_i C_Y$ is nef/Z for some i. After that some simple calculations allow us to show that the LMMP terminates (see Step 5).

Step 2. We will show that, perhaps after replacing δ and ϵ with smaller positive numbers, for any number δ' and **R**-divisor A'_Y satisfying $0 \le \delta' \le \delta$ and $0 \le A'_Y \le \epsilon A_Y$, any LMMP/Z on $K_Y + B_Y + \delta' C_Y + A'_Y$ consists of only a sequence of log flips which are flops with respect to $(Y/Z, B_Y)$ (that is, $K_Y + B_Y$ is numerically trivial on each extremal ray that is contacted in the process). First we show that $K_Y + B_Y + \delta' C_Y + A'_Y$ is a limit of movable/Z **R**-divisors. Since δ is sufficiently small, we can assume that $\lambda_{i-1} \ge \delta' \ge \lambda_i$ for some i. By definition of LMMP with scaling, $K_{X_i} + B_i + \lambda_{i-1} C_i$ and $K_{X_i} + B_i + \lambda_i C_i$ are both nef/Z hence $K_{X_i} + B_i + \delta' C_i$ is also nef/Z. Thus, $K_{X_i} + B_i + \delta' C_i$ is (numerically) a limit of movable/Z **R**-divisors which in turn implies that $K_Y + B_Y + \delta' C_Y$ is also a limit of movable/Z **R**-divisors. On the other hand, each component of A_Y is movable/Z so A'_Y is a movable/Z **R**-divisor. Therefore, $K_Y + B_Y + \delta' C_Y + A'_Y$ is a limit of movable/Z **R**-divisors. This implies that any LMMP/Z on $K_Y + B_Y + \delta' C_Y + A'_Y$ consists of only a sequence of log flips. Finally, since $K_Y + B_Y$ is nef/Z, by [6, Proposition 3.2], perhaps after replacing δ and ϵ with smaller positive numbers, any step of such an LMMP would be a flop with respect to $(Y/Z, B_Y)$.

Step 3. Fix some $i \gg 0$ so that $\lambda_i < \delta$. Since $X \dashrightarrow X_i$ is a partial LMMP/Z on $K_X + B + \lambda_i C$, there is $0 < \tau < \epsilon$ such that $(X_i/Z, B_i + \lambda_i C_i + \tau A_i)$ is dlt. Run an LMMP/Z on $K_{X_i} + B_i + \lambda_i C_i + \tau A_i$ with scaling of some ample/Z divisor. For reasons as in Step 2, we can assume that this LMMP consists of only a sequence of log flips which are flops with respect to $(X_i/Z, B_i + \lambda_i C_i)$. On the other hand, since the components of A_i generate $N^1(X_i/Z)$, we can assume that there is an ample/Z **R**-divisor $H \ge 0$ such that $\tau A_i \equiv H + H'/Z$ where $H' \ge 0$ and

$$\left(X_i/Z, B_i + \lambda_i C_i + H + H' \right)$$

is dlt. The LMMP on $K_{X_i} + B_i + \lambda_i C_i + \tau A_i$ is also an LMMP on

$$K_{X_i} + B_i + \lambda_i C_i + H + H'$$

hence it terminates by Lemma 4.2 and we get a model T on which both

$$K_T + B_T + \lambda_i C_T \quad \text{and} \quad K_T + B_T + \lambda_i C_T + \tau A_T$$

are nef/Z. Again since the components of A_T generate $N^1(T/Z)$, there is $0 \le A'_T \le \tau A_T$ so that $K_T + B_T + \lambda_i C_T + A'_T$ is ample/Z and $\operatorname{Supp} A'_T = \operatorname{Supp} A_T$.

Step 4. Now run an LMMP/Z on $K_Y + B_Y + \lambda_i C_Y + A'_Y$ with scaling of some ample/Z divisor where A'_Y is the birational transform of A'_T. By Step 2, the LMMP consists of only a sequence of log flips which are flops with respect to $(Y/Z, B_Y)$, and it terminates for reasons similar to Step 3. Actually, the LMMP terminates on T because $K_T + B_T + \lambda_i C_T + A'_T$ is ample/Z. So, by replacing Y with T we could from now on assume that $K_Y + B_Y + \lambda_i C_Y$ is nef/Z. In particular, $K_Y + B_Y + \lambda_j C_Y$ is nef/Z for any $j \geq i$ since $\lambda_j \leq \lambda_i$ and since $K_Y + B_Y$ is nef/Z.

Step 5. Pick $j > i$ so that $\lambda_j < \lambda_{j-1}$ and let $r \colon U \to X_j$ and $s \colon U \to Y$ be a common resolution. Then,

$$r^*(K_{X_j} + B_j + \lambda_j C_j) = s^*(K_Y + B_Y + \lambda_j C_Y)$$
$$r^*(K_{X_j} + B_j) \gvertneqq s^*(K_Y + B_Y)$$
$$r^* C_j \lvertneqq s^* C_Y$$

where the first equality holds because both $K_{X_j} + B_j + \lambda_j C_j$ and $K_Y + B_Y + \lambda_j C_Y$ are nef/Z and X_j and Y are isomorphic in codimension one, the second inequality holds by Remark 2.6 because $K_Y + B_Y$ is nef/Z but $K_{X_j} + B_j$ is not nef/Z, and the third claim follows from the other two. Now

$$r^*(K_{X_j} + B_j + \lambda_{j-1} C_j) = r^*(K_{X_j} + B_j + \lambda_j C_j) + r^*(\lambda_{j-1} - \lambda_j)C_j$$
$$\lvertneqq s^*(K_Y + B_Y + \lambda_j C_Y) + s^*(\lambda_{j-1} - \lambda_j)C_Y$$
$$= s^*(K_Y + B_Y + \lambda_{j-1} C_Y)$$

However, since $K_{X_j} + B_j + \lambda_{j-1} C_j$ and $K_Y + B_Y + \lambda_{j-1} C_Y$ are both nef/Z, we have

$$r^*(K_{X_j} + B_j + \lambda_{j-1} C_j) = s^*(K_Y + B_Y + \lambda_{j-1} C_Y)$$

This is a contradiction and the sequence of log flips terminates as claimed.   □

The reader may want to have a look at Remark 2.9 before reading the rest of this section.

Proof of Theorem 4.1.

Step 1. In view of Lemma 4.2 it is enough to treat the case (iii). We can replace B with $B + \lambda C$ hence assume that $\lambda = 0$. Moreover, we may assume that the LMMP consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips starting with $(X_1/Z, B_1) = (X/Z, B)$. Pick i so that $\lambda_i > \lambda_{i+1}$. Then, $\operatorname{Supp} C_{i+1}$ does not contain any lc centre of $(X_{i+1}/Z, B_{i+1} + \lambda_{i+1} C_{i+1})$ because $(X_{i+1}/Z, B_{i+1} + \lambda_i C_{i+1})$ is lc. Thus, by replacing $(X/Z, B)$ with $(X_{i+1}/Z, B_{i+1})$ and C with $\lambda_{i+1} C_{i+1}$ we may assume that no lc centre of $(X/Z, B + C)$ is inside $\operatorname{Supp} C$. Moreover, since there are finitely many lc centres of $(X/Z, B)$, perhaps after truncating the sequence, we can assume that no lc centre is contracted in the sequence. We will reduce the problem to the situation of Lemma 4.3.

Step 2. By assumptions, there is a log minimal model $(Y/Z, B_Y)$ for $(X/Z, B)$. Let $\phi\colon X \dashrightarrow Y/Z$ be the corresponding birational map. Let $f\colon W \to X$ and $g\colon W \to Y$ be a common log resolution of $(X/Z, B + C)$ and $(Y/Z, B_Y + C_Y)$ where C_Y is the birational transform of C. By Remark 2.6,

$$E := f^*(K_X + B) - g^*(K_Y + B_Y)$$

is effective and exceptional/Y. Let B_W be the birational transform of B plus the reduced exceptional divisor of f, and let C_W be the birational transform of C. Pick a sufficiently small $\delta \geq 0$. Since $(X/Z, B)$ is lc,

$$E' := K_W + B_W - f^*(K_X + B)$$

is effective and exceptional/X. Actually, E' is also exceptional/Y because if D is a component of E' which is not exceptional/Y, then

$$0 = a(D, Y, B_Y) \geq a(D, X, B) \geq 0$$

which is a contradiction since $a(D, X, B) = 0$ means that D cannot be a component of E'.

On the other hand, since Y is $\mathbf{Q}$-factorial, there is an exceptional/Y $\mathbf{R}$-divisor F on W such that $C_W + F \equiv 0/Y$. So,

$$K_W + B_W + \delta C_W \equiv E + E' + \delta C_W \equiv E + E' - \delta F/Y$$

and since δ is sufficiently small, the support of $E + E'$ is contained in the support of the effective part of $E + E' - \delta F$. Now by Theorem 3.5, if we run an LMMP/Y on $K_W + B_W + \delta C_W$ with scaling of an ample divisor, then we arrive at a model V on which $E_V + E'_V = 0$, that is, $E + E'$ is contracted over V.

Step 3. We prove that $\phi\colon X \dashrightarrow Y$ does not contract any divisors. Assume otherwise and let D be a prime divisor on X contracted by ϕ. Then $D^\sim$ the birational transform of D on W is a component of E because by definition of log minimal models $a(D, X, B) < a(D, Y, B_Y)$. Now, in Step 2 take $\delta = 0$. The LMMP contracts $D^\sim$ since $D^\sim$ is a component of E and E is contracted. This is not possible as we can see as follows. Since $(X/Z, B + \lambda_i C)$ is lc,

$$K_W + B_W + \lambda_i C_W - f^*(K_X + B + \lambda_i C)$$

is effective and exceptional/X. On the other hand, $(X_i/Z, B_i + \lambda_i C_i)$ is a weak lc model of $(X/Z, B + \lambda_i C)$ hence $f^*(K_X + B + \lambda_i C) \geq M$ where M is the pullback of $K_{X_i} + B_i + \lambda_i C_i$ under $W \dashrightarrow X_i$, that is,

$$M = p_* q^*(K_{X_i} + B_i + \lambda_i C_i)$$

for some common resolution $p\colon W' \to W$ and $q\colon W' \to X_i$. Therefore,

$$K_W + B_W + \lambda_i C_W = M + G$$

where G is effective and exceptional/X. Since $K_{X_i} + B_i + \lambda_i C_i$ is nef/Z, it is (numerically) a limit of movable/Z **R**-divisors hence M is a limit of movable/Z **R**-divisors. If λ_i is sufficiently small then $W \dashrightarrow V$ (of Step 2) is a partial LMMP/Z on $K_W + B_W + \lambda_i C_W$. But since G is exceptional/X and since D is a divisor on X, $D^\sim$ is not a component of G hence $D^\sim$ cannot be contracted over V by the LMMP of Step 2, a contradiction. Thus ϕ does not contract divisors.

Step 4. We will construct a **Q**-factorial dlt blowup of $(Y/Z, B_Y)$ as follows. In Step 2, take $\delta > 0$ which is sufficiently small by assumptions. Let Y' be the model V obtained. Since $X \dashrightarrow Y$ does not contract divisors, by construction, each prime exceptional/Y divisor on Y' appears with coefficient one in $B_{Y'}$. So, in view of

$$K_{Y'} + B_{Y'} \equiv E_{Y'} + E'_{Y'} = 0/Y$$

we deduce that $K_{Y'} + B_{Y'}$ is the pullback of $K_Y + B_Y$ and that $(Y'/Z, B_{Y'})$ is a **Q**-factorial dlt blowup of $(Y/Z, B_Y)$. Moreover, $(Y'/Z, B_{Y'} + \delta C_{Y'})$ is dlt.

Step 5. We will construct a **Q**-factorial dlt blowup of $(X/Z, B)$ as follows. Since $(X/Z, B + C)$ is lc,

$$E'' := K_W + B_W + C_W - f^*(K_X + B + C)$$

is effective and exceptional/X. Run an LMMP/X on $K_W + B_W + C_W$ with scaling of an ample/X divisor. Since $K_W + B_W + C_W \equiv E''/X$, by Theorem 3.4, the LMMP terminates on a model X'. In particular, $(X'/Z, B' + C')$ is a **Q**-factorial dlt blowup of $(X/Z, B + C)$ where B' is the pushdown of B_W and C' is the pushdown of C_W. Moreover, the LMMP does not contract an exceptional prime divisor D of $W \to X$ iff $a(D, X, B + C) = 0$. Since $\operatorname{Supp} C$ does not contain any lc centre of $(X/Z, B + C)$ by Step 1, the LMMP does not contract an exceptional prime divisor D of $W \to X$ iff $a(D, X, B) = 0$. Therefore, $(X'/Z, B')$ is a **Q**-factorial dlt blowup of $(X/Z, B)$ and C' is the pullback of C. Note that the prime exceptional divisors of ϕ^{-1} are not contracted/X' since their log discrepancies with respect to $(X/Z, B)$ are all 0.

Step 6. By Remark 2.9(1), we can lift the sequence $X_i \dashrightarrow X_{i+1}/Z_i$ to an LMMP/Z on $K_{X'} + B'$ with scaling of C' if the property $(*)$ of the remark holds. If $(*)$ holds, then continue from the next paragraph. But if $(*)$ does not hold for some i, then we can replace our sequence $X_i \dashrightarrow X_{i+1}/Z_i$ with the LMMP in $(*)$ and repeat Steps 1–5 again. In particular, we could assume that each $(X_i/Z, B_i)$ is **Q**-factorial dlt. But in that case, by Remark 2.9 (2), $(*)$ holds so we will not need to deal with $(*)$ again.

From now on we assume that we have lifted $X_i \dashrightarrow X_{i+1}/Z_i$ to an LMMP/Z on $K_{X'} + B'$ with scaling of C'. We show that $Y' \dashrightarrow X'$ does not contract divisors. Suppose that D is a prime divisor on Y' which is exceptional/X'. If D is not exceptional/Y, then D is

an exceptional divisor of ϕ^{-1} and this contradicts the last sentence of Step 5. Thus, D is exceptional/Y. But since $(Y'/Z, B_{Y'})$ is a **Q**-factorial dlt blowup of $(Y/Z, B_Y)$, $a(D, Y, B_Y) = 0$ which in turn implies that $a(D, X, B) = 0$ and again Step 5 gives a contradiction.

We show that $X' \dashrightarrow Y'$ also does not contract divisors. By Step 3, it is enough to show that $(Y'/Z, B_{Y'})$ is a log minimal model of $(X'/Z, B')$. By construction, $B_{Y'}$ is the push-down of B'. So, it remains to compare log discrepancies. Assume that D is a prime divisor on X' which is exceptional/Y'. Since $X \dashrightarrow Y$ does not contract divisors by Step 3, D is exceptional/X. In particular, $a(D, X', B') = a(D, X, B) = 0$. If $a(D, Y', B_{Y'}) = 0$, then $a(D, Y, B_Y) = 0$ hence the birational transform of D cannot be a component of $E + E' + \delta C_W$ in Step 2 so it could not be contracted over Y' which is a contradiction. Therefore, $a(D, Y', B_{Y'}) > 0$. Thus, for every prime exceptional divisor of $X' \dashrightarrow Y'$ we have shown that $a(D, X', B') < a(D, Y', B_{Y'})$ which implies that $(Y'/Z, B_{Y'})$ is a log minimal model of $(X'/Z, B')$ by construction.

Step 7. Let $A \geq 0$ be a reduced divisor on W whose components are general ample/Z divisors such that they generate $N^1(W/Z)$. By Step 5, X' is obtained by running some LMMP on $K_W + B_W + C_W$. This LMMP is a partial LMMP on $K_W + B_W + C_W + \epsilon A$ for any sufficiently small $\epsilon > 0$. In particular, $(X'/Z, B' + C' + \epsilon A')$ is dlt where A' is the birational transform of A. For similar reasons, we can choose ϵ so that $(Y'/Z, B_{Y'} + \delta C_{Y'} + \epsilon A_{Y'})$ is also dlt. Now apply Lemma 4.3 to the LMMP/Z on $K_{X'} + B'$ with scaling of C' of Step 6. $\qquad\square$

5. Shokurov bss-ampleness, finite generation, and the klt case

The next theorem shows that more general versions of Theorem 1.1 and Theorem 1.4 hold when $(X/Z, B)$ has a finitely generated algebra $\mathcal{R}(X/Z, K_X + B)$. The result is an easy consequence of Lemmas 3.2 and 3.3 and it should be considered as a very special case of Shokurov's attempt in relating bss-ampleness and finite generation [31, Theorem 3.18] (see also [25], [13] for more recent adaptations). Though we are not using Shokurov's terminology of bss-ample divisors but the next theorem is saying that finite generation implies bss-ampleness in the specific situation we are concerned with. Shokurov proves that in general finite generation together with the so-called global almost generation property implies bss-ampleness.

Theorem **5.1** *(=Theorem 1.5).* — *Let* $(X/Z, B)$ *be a* **Q**-*factorial dlt pair where* B *is a* **Q**-*divisor and* $f : X \to Z$ *is surjective. Assume further that* $\mathcal{R}(X/Z, K_X + B)$ *is a finitely generated* $\mathcal{O}_Z$-*algebra, and that* $(K_X + B)|_{X_\eta} \sim_{\mathbf{Q}} 0$ *where* X_η *is the generic fibre of* f. *Then, any LMMP/Z on* $K_X + B$ *with scaling of an ample/Z divisor terminates with a good log minimal model.*

Proof. — We can assume that f is a contraction. Run an LMMP/Z on $K_X + B$ with scaling of some ample/Z divisor. Since termination and semi-ampleness/Z are both

local on Z, we can assume that Z is affine, say $\mathrm{Spec}\, R$. By Theorem 4.1, it is enough to show that $(X/Z, B)$ has a good log minimal model. Let I be a positive integer so that $I(K_X + B)$ is Cartier. Since $R(X/Z, K_X + B)$ is a finitely generated R-algebra, perhaps after replacing I with a multiple, there exist a log resolution $g\colon W \to X$, a divisor $E \geq 0$ and a free divisor F on W such that

$$\mathrm{Fix}\, g^* m I(K_X + B) = mE \quad \text{and} \quad \mathrm{Mov}\, g^* m I(K_X + B) = mF$$

for every $m > 0$ (cf. [31] or [7, Theorem 4.3]). Let $h\colon W \to T/Z$ be the contraction defined by $|F|$. Then, since $(K_X + B)|_{X_\eta} \sim_{\mathbf{Q}} 0$, the map $T \to Z$ is birational and E is vertical$/T$.

Choose a boundary B_W so that $(W/Z, B_W)$ is a log smooth model of $(X/Z, B)$ of type (1) as in Definition 2.3, that is, B_W is the birational transform of B plus the reduced exceptional divisor of g. We can write

$$I(K_W + B_W) = g^* I(K_X + B) + E'$$

where $E' \geq 0$ is exceptional$/X$. So,

$$\mathrm{Fix}\, m I(K_W + B_W) = mE + mE' \quad \text{and} \quad \mathrm{Mov}\, m I(K_W + B_W) = mF$$

Run the LMMP$/T$ on $K_W + B_W$ with scaling of some ample$/T$ divisor. Since $(K_X + B)|_{X_\eta} \sim_{\mathbf{Q}} 0$, there is a non-empty open subset $U \subseteq Z$ so that $K_X + B \sim_{\mathbf{Q}} 0$ over U and $T \to Z$ is an isomorphism over U. So, over U, $(X/Z, B)$ is a weak lc model of $(W/Z, B_W)$ hence by Corollary 3.7, $(W/Z, B_W)$ has a log minimal model over U (which is just a suitable $\mathbf{Q}$-factorial dlt blowup of $(X/Z, B)$). Thus, by Theorem 4.1, the LMMP terminates over U. By construction, over U we have

$$I(K_W + B_W) \sim E + E' + F \sim E + E'$$

so we reach a model Y' on which $E_{Y'} + E'_{Y'} \sim_{\mathbf{Q}} 0$ over U, in particular, $E_{Y'} + E'_{Y'}$ is vertical$/T$. On the other hand, since $W \dashrightarrow Y'$ is a partial LMMP$/T$ on $K_W + B_W$,

$$\mathrm{Fix}\, m I(K_{Y'} + B_{Y'}) = \mathrm{Fix}\big(mE_{Y'} + mE'_{Y'} + mF_{Y'}\big) = mE_{Y'} + mE'_{Y'}$$

hence by Lemma 3.2, $E_{Y'} + E'_{Y'}$ is very exceptional$/T$. Now, by Theorem 3.4, there is an LMMP$/T$ on $K_{Y'} + B_{Y'}$ which ends up with a log minimal model $(Y/T, B_Y)$ on which

$$I(K_Y + B_Y) \sim_{\mathbf{Q}} E_Y + E'_Y = 0/T$$

In particular, $I(K_Y + B_Y) \sim_{\mathbf{Q}} F_Y/Z$. Thus, $K_Y + B_Y$ is semi-ample$/Z$ and $(Y/Z, B_Y)$ is a good log minimal model of $(W/Z, B_W)$ hence of $(X/Z, B)$ by Remark 2.8.   □

Theorem **5.2.** — *We have the following for klt pairs:*

(1) *Let* $(X/Z, B)$ *be a* **Q**-*factorial klt pair where* B *is a* **Q**-*divisor and* $f : X \to Z$ *is surjective. Assume that* $(K_X + B)|_{X_\eta} \sim_{\mathbf{Q}} 0$ *where* X_η *is the generic fibre of* f. *Then, any LMMP/Z on* $K_X + B$ *with scaling of an ample/Z divisor terminates with a good log minimal model.*

(2) *Theorems 1.1 and 1.4 hold when* $(X/Z, B)$ *is klt.*

Proof. — (1) By [9], $\mathcal{R}(X/Z, K_X + B)$ is a finitely generated $\mathcal{O}_Z$-algebra so we can apply Theorem 5.1.

(2) Suppose that $(X/Z, B)$ is klt under the assumptions of Theorem 1.1. By taking a **Q**-factorial dlt blowup as in Corollary 3.6 we can assume that X is **Q**-factorial. If A is not vertical/Z, then $K_X + B$ is not pseudo-effective/Z hence $(X/Z, B)$ has a Mori fibre space by [9]. So, we can assume that A is vertical/Z. Then, $(K_X + B)|_{X_\eta} \sim_{\mathbf{Q}} 0$ where X_η is the generic fibre of f. Now use (1). We can treat Theorem 1.4 in a similar way. Note that here we do not need ACC for lc thresholds. $\square$

Though the last theorem settles the klt case of Theorems 1.1 and 1.4 but we prove further results in this direction as we will need them to deal with the lc case (e.g., proof of Lemma 5.5). The next two theorems unfortunately do not simply follow from Theorem 5.2 because the boundaries that appear are not necessarily rational.

Theorem **5.3.** — *Let* $(X/Z, B)$ *be a klt pair where* $f : X \to Z$ *is surjective. Assume further that there is a contraction* $g : X \to S/Z$ *such that* $K_X + B \sim_{\mathbf{R}} 0/S$ *and that* $S \to Z$ *is generically finite. Then,* $(X/Z, B)$ *has a good log minimal model.* (*Note that* B *is not necessarily a* **Q**-*divisor.*)

Proof. — As mentioned above we cannot apply Theorem 5.2 because B is not assumed to be rational. However, the proof given below is somewhat similar to the proof of Theorem 5.1.

We can assume that f is a contraction hence $S \to Z$ is birational. Moreover, by taking a **Q**-factorial dlt blowup as in Corollary 3.6 we can assume that X is **Q**-factorial. Since $K_X + B \sim_{\mathbf{R}} 0/S$, by Ambro [3] and Fujino-Gongyo [16, Theorem 3.1], there is a boundary B_S on S such that

$$K_X + B \sim_{\mathbf{R}} g^*(K_S + B_S)$$

and such that $(S/Z, B_S)$ is klt and of general type. So, by [9], there is a weak lc model $(T/Z, B_T)$ for $(S/Z, B_S)$ obtained by running an LMMP/Z on $K_S + B_S$ with scaling of some ample/Z divisor. Moreover, since $T \dashrightarrow S$ does not contract divisors, there are non-empty open subsets $U \subseteq S$ and $V \subseteq T$ such that the induced map $U \dashrightarrow V$ is an isomorphism and such that $\mathrm{codim}(T \setminus V) \geq 2$.

Take a log smooth model $(W/Z, B_W)$ of $(X/Z, B)$ of type (2) as in Definition 2.3 using a log resolution $e : W \to X$. Then, $(W/Z, B_W)$ is klt and

$$K_W + B_W = e^*(K_X + B) + E$$

where $E \geq 0$ is exceptional/X, and any log minimal model of $(W/Z, B_W)$ is also a log minimal model of $(X/Z, B)$ by Remark 2.8. Moreover, we can assume that the induced maps $\phi \colon W \dashrightarrow S$ and $\psi \colon W \dashrightarrow T$ are both morphisms. Run an LMMP/T on $K_W + B_W$ with scaling of some ample/T divisor. Since $K_X + B \sim_{\mathbf{R}} 0$ over U, $(X/Z, B)$ is a log minimal model of $(W/Z, B)$ over $U = V$ hence by Theorem 4.1 the LMMP terminates over V: we reach a model Y' such that $K_{Y'} + B_{Y'} \sim_{\mathbf{R}} 0$ over V, and the remaining steps of the LMMP contract extremal rays only over $T \setminus V$.

Let π be the induced map $W \dashrightarrow Y'$. Let $M \subseteq X$, $N \subseteq W$, and $O \subseteq Y'$ be the inverse images of $U = V$. Then, as mentioned $(M/V, B|_M)$ is a log minimal model of $(N/V, B_W|_N)$ hence M and O are isomorphic in codimension one. Thus, $E|_N$ is contracted over O. So, in particular, $E_{Y'} = \pi_* E$ is a divisor on Y' which is mapped into $T \setminus V$ hence it is very exceptional/T. On the other hand, since $(T/Z, B_T)$ is a weak lc model of $(S/Z, B_S)$,

$$\phi^*(K_S + B_S) = \psi^*(K_T + B_T) + G$$

where $G \geq 0$ is very exceptional/T because G is mapped into $T \setminus V$. Thus,

$$
\begin{aligned}
K_{Y'} + B_{Y'} = \pi_*(K_W + B_W) &= \pi_*\big(e^*(K_X + B) + E\big) \\
&= \pi_* e^*(K_X + B) + \pi_* E \\
&\sim_{\mathbf{R}} \pi_* \phi^*(K_S + B_S) + \pi_* E \\
&= \pi_* \psi^*(K_T + B_T) + \pi_* G + \pi_* E
\end{aligned}
$$

By construction, $\pi_* G + \pi_* E \geq 0$ is very exceptional/T. So, by Theorem 3.4, there is an LMMP/T on $K_{Y'} + B_{Y'}$ with scaling of some ample/T divisor which ends up with a model Y on which $K_Y + B_Y \sim_{\mathbf{R}} 0/T$ hence

$$K_Y + B_Y \sim_{\mathbf{R}} \nu^*(K_T + B_T)$$

where $\nu \colon Y \to T$ is the induced morphism. In particular, $K_Y + B_Y$ is semi-ample/Z as $K_T + B_T$ is semi-ample/Z. So, $(Y/Z, B_Y)$ is a good log minimal model of $(W/Z, B_W)$ hence of $(X/Z, B)$. $\qquad\square$

Theorem **5.4.** — *Let* $(X/Z, B)$ *be a klt pair where* B *is a* $\mathbf{Q}$*-divisor and* $f \colon X \to Z$ *is surjective. Assume further that* $(K_X + B)|_F \sim_{\mathbf{Q}} 0$ *for the generic fibre* F *of* f*. If* $K_X + B + H$ *is klt and nef/Z for some* $\mathbf{Q}$*-divisor* $H \geq 0$*, then the LMMP/Z on* $K_X + B$ *with scaling of* H *terminates if either*

(1) B *is big/Z or* H *is big/Z, or*

(2) H *is vertical/Z and* $\lambda \neq \lambda_j$ *for any* j *where* λ_j *are the numbers appearing in the LMMP with scaling (as in Definition 2.4) and* $\lambda = \lim_{i \to \infty} \lambda_i$*.*

Proof. — We can assume that f is a contraction. By Theorem 5.2, $(X/Z, B)$ has a good log minimal model. Assume that condition (1) holds. If B is big/Z, the termination follows from [9]. If H is big/Z, and if $\lambda > 0$, then again the termination follows from [9] as we can replace B with $B + \lambda H$. If $\lambda = 0$, use Theorem 4.1.

Now assume that condition (2) holds. Since B, H are **Q**-divisors, each λ_i is rational. However, at this stage we do not know whether λ is rational. So, we cannot apply Theorem 5.2 to get a log minimal model of $(X/Z, B + \lambda H)$ to allow us to apply Theorem 4.1.

We may assume that the LMMP consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips and that $X_1 = X$. Moreover, by Remark 2.9(1), (2), we can lift the sequence to the **Q**-factorial situation hence assume that X_i are **Q**-factorial. Now by Theorem 5.2, $(X/Z, B + \lambda_1 H)$ has a good log minimal model. But then every log minimal model of $(X/Z, B + \lambda_1 H)$ is good by Remark 2.7. Since $K_X + B + \lambda_1 H$ is nef/Z, $(X/Z, B + \lambda_1 H)$ is a log minimal model of itself hence $K_X + B + \lambda_1 H$ is semi-ample/Z. Let $g \colon X \to T$ be the contraction associated to it. Since H is vertical/Z and $(K_X + B)|_F \sim_{\mathbf{Q}} 0$, the map $T \to Z$ is birational so the generic fibre of g and f are the same. By Theorem 5.2, some LMMP/T on $K_X + B$ terminates with a good log minimal $(Y/T, B_Y)$ of $(X/T, B)$. Let $h \colon Y \to S/T$ be the contraction associated to $K_Y + B_Y$. Since

$$K_X + B + \lambda_1 H \sim_{\mathbf{Q}} 0/T$$

we have

$$K_Y + B_Y + \lambda_1 H_Y \sim_{\mathbf{Q}} 0/S$$

This combined with $K_Y + B_Y \sim_{\mathbf{Q}} 0/S$ implies that $H_Y \sim_{\mathbf{Q}} 0/S$. In particular, since $K_Y + B_Y$ is semi-ample/T, the map $S \to T$ is birational otherwise H_Y would be numerically negative on some curves not contained in H_Y which is not possible as $H_Y \geq 0$. Moreover,

$$K_Y + B_Y + \lambda H_Y \sim_{\mathbf{R}} 0/S$$

Therefore, by Theorem 5.3, $(Y/Z, B_Y + \lambda H_Y)$ has a log minimal model which is also a log minimal model of $(X/Z, B + \lambda H)$. Now the termination follows from Theorem 4.1. $\square$

An application of the above theorems concerns a certain LMMP with scaling which will be used in the proof of Lemmas 6.1 and 7.1 and Proposition 7.2.

Lemma 5.5 *(cf. [6, Lemma 4.1]).* — *Let* $(X/Z, B)$ *be a* **Q**-*factorial lc pair such that* $f \colon X \to Z$ *is surjective,* B *is a* **Q**-*divisor, and* $(K_X + B)|_F \sim_{\mathbf{Q}} 0$ *where* F *is the generic fibre of* f. *Moreover, assume that*

- *every lc centre of* $(X/Z, B)$ *is contained in* $\mathrm{Supp}\lfloor B \rfloor$,
- $K_X + B + cC$ *is lc and nef/Z for some* **Q**-*divisor* $C \geq 0$ *and some rational number* $c > 0$,
- $K_X + B \sim_{\mathbf{Q}} P + C/Z$ *for some* **Q**-*divisor* $P \geq 0$ *with* $\mathrm{Supp}\, P = \mathrm{Supp}\lfloor B \rfloor$.

Then, we can run an LMMP/Z on $K_X + B$ *with scaling of* cC *so that* $\lambda := \lim_{i \to \infty} \lambda_i = 0$ *where* λ_i *are the numbers that appear in the LMMP. (Note that we are not claiming that the LMMP terminates.)*

Proof.

Step 1. Since

$$(P + C)|_F \sim_{\mathbf{Q}} (K_X + B)|_F \sim_{\mathbf{Q}} 0$$

both P and C are vertical/Z, in particular, $\lfloor B \rfloor$ is also vertical/Z. On the other hand, for each rational number $t \in (0, 1]$ we can write

$$K_X + B + tcC \sim_{\mathbf{Q}} (1 + \epsilon)(K_X + B) - \epsilon P - \epsilon C + tcC$$

$$\sim_{\mathbf{Q}} (1 + \epsilon)\left(K_X + B - \frac{\epsilon}{1 + \epsilon}P + \frac{tc - \epsilon}{1 + \epsilon}C\right)\Big/Z$$

for some sufficiently small rational number $\epsilon > 0$. Under our assumptions,

$$\left(X, B - \frac{\epsilon}{1 + \epsilon}P + \frac{tc - \epsilon}{1 + \epsilon}C\right)$$

is klt and

$$\left(K_X + B - \frac{\epsilon}{1 + \epsilon}P + \frac{tc - \epsilon}{1 + \epsilon}C\right)\Big|_F = (K_X + B)|_F \sim_{\mathbf{Q}} 0$$

In particular, if $K_X + B + tcC$ is nef/Z, then

$$K_X + B - \frac{\epsilon}{1 + \epsilon}P + \frac{tc - \epsilon}{1 + \epsilon}C$$

is nef/Z hence semi-ample/Z by Theorem 5.2; this in turn implies that $K_X + B + tcC$ is semi-ample/Z. We will use this observation on X and on the birational models that will be constructed.

Step 2. Put $Y_1 := X$, $B_1 := B$, and $C_1 := C$. Let $\lambda_1 \geq 0$ be the smallest number such that $K_{Y_1} + B_1 + \lambda_1 cC_1$ is nef/Z. We may assume that $\lambda_1 > 0$. We will show that λ_1 is rational. Pick a rational number $\lambda' \in (0, \lambda_1)$. In view of Step 1, there is a boundary Δ and a rational number $\epsilon > 0$ such that $(Y_1/Z, \Delta)$ is klt and

$$K_{Y_1} + B_1 + \lambda' cC_1 \sim_{\mathbf{Q}} (1 + \epsilon)(K_{Y_1} + \Delta)/Z$$

Now $s = \frac{\lambda_1 - \lambda'}{1 + \epsilon}$ is the smallest number such that $K_{Y_1} + \Delta + scC_1$ is nef/Z. By [5, Lemma 3.1], s is rational which means that λ_1 is also rational (note that we did not apply [5, Lemma 3.1] directly to $(Y_1/Z, B_1)$ because this pair may not be dlt). By Step 1, $K_{Y_1} + B_1 + \lambda_1 cC_1$ is semi-ample/Z; let $Y_1 \to V_1/Z$ be the associated contraction. Run the LMMP/V_1 on $K_{Y_1} + B_1 + tcC_1$ with scaling of an ample/V_1 divisor,

for some rational number $t \in (0, \lambda_1)$. This terminates with a good log minimal model of $(Y_1/V_1, B_1 + tcC_1)$ by Step 1 and by Theorem 5.2. The LMMP is also an LMMP/V_1 on $K_{Y_1} + B_1$ with scaling of $\lambda_1 cC_1$ because

$$K_{Y_1} + B_1 + tcC_1 \sim_{\mathbf{Q}} -(\lambda_1 - t)cC_1 \sim_{\mathbf{Q}} \frac{(\lambda_1 - t)}{\lambda_1}(K_{Y_1} + B_1)/V_1$$

So, we get a model Y_2 such that $K_{Y_2} + B_2$ is semi-ample/V_1 where B_2 is the birational transform of B_1 (similar notation will be used for other divisors and models). Now since $K_{Y_2} + B_2 + \lambda_1 cC_2$ is the pullback of some ample/Z divisor on V_1,

$$K_{Y_2} + B_2 + \lambda_1 cC_2 + \delta(K_{Y_2} + B_2)$$

is semi-ample/Z for some sufficiently small $\delta > 0$. Put it in another way, $K_{Y_2} + B_2 + \tau cC_2$ is semi-ample/Z for some rational number $\tau < \lambda_1$. We can consider $Y_1 \dashrightarrow Y_2$ as a partial LMMP/Z on $K_{Y_1} + B_1$ with scaling of cC_1. We can continue as before. That is, let $\lambda_2 \geq 0$ be the smallest number such that $K_{Y_2} + B_2 + \lambda_2 cC_2$ is nef/Z, and so on (note that $\lambda_1 > \tau \geq \lambda_2$). This process is an LMMP/Z on $K_X + B$ with scaling of cC. The numbers λ_i that appear in the LMMP satisfy $\lambda := \lim_{i \to \infty} \lambda_i \neq \lambda_j$ for any j.

Step 3. In Step 2, we constructed an LMMP/Z on $K_X + B$ with scaling of cC. After modifying the notation, we may assume that the LMMP consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips, $X_1 = X$, and that $K_{X_i} + B_i + \lambda_i cC_i$ is nef/Z but numerically trivial/Z_i.

If $\lambda > 0$, then the LMMP is also an LMMP/Z on $K_X + B + \lambda' cC$ with scaling of $(1 - \lambda')cC$ for some rational number $\lambda' \in (0, \lambda)$. As in Step 1, there is a rational number $\epsilon > 0$ and a klt $(X/Z, \Delta)$ such that

$$K_X + B + \lambda' cC \sim_{\mathbf{Q}} (1 + \epsilon)(K_X + \Delta)/Z$$

So, we can consider the LMMP as an LMMP/Z on $K_X + \Delta$ with scaling of $\frac{1-\lambda'}{1+\epsilon}cC$. Since $(X/Z, \Delta + \frac{1-\lambda'}{1+\epsilon}cC)$ is klt and C is vertical/Z, Theorem 5.4(2) implies that the LMMP terminates. So, $\lambda = 0$. $\square$

6. Proof of Theorem 1.1 and Corollary 1.2

Corollary 1.2 follows from Theorem 1.1 easily (see the end of this section). We divide the proof of Theorem 1.1 into two cases. One case is when every lc centre of $(X/Z, B)$ is vertical/Z (the vertical case) and the other case is when some lc centre is horizontal/Z (the horizontal case). Each case needs a different kind of argument. Here we give a brief account of the main ideas (a very similar line of thought is used to prove Theorem 1.4).

Suppose that $(X/Z, B + A)$ is as in the statement of Theorem 1.1. By taking a $\mathbf{Q}$-factorial dlt blowup we can assume that $(X/Z, B)$ is $\mathbf{Q}$-factorial dlt. First note that if

A intersects the generic fibre of $X \to Z$, then $K_X + B$ is not pseudo-effective/Z and the result follows from [9]. So, we could assume that A is vertical/Z.

Now assume that every lc centre of $(X/Z, B)$ is vertical/Z which means that $\lfloor B \rfloor$ is vertical/Z. By Theorem 4.1, the termination statement in Theorem 1.1(3) holds for $(X/Z, B)$ if statement (1) holds. Thus, we only need to construct a good log minimal model of $(X/Z, B)$. Pick a sequence $t_1 > t_2 > \cdots$ of sufficiently small positive rational numbers such that $\lim_{i \to \infty} t_i = 0$. Then, $(X/Z, B - t_i \lfloor B \rfloor)$ is a klt pair for each i. So, by Theorem 5.2, for each i, we get a good log minimal model $(Y_i/Z, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor)$ of $(X/Z, B - t_i \lfloor B \rfloor)$ by running some LMMP/Z on $K_X + B - t_i \lfloor B \rfloor$.

Assume that $Y_i = Y_{i+1}$ for every $i \gg 0$ and let Y be this common model. Then, a simple calculation on log discrepancies show that $(Y/Z, B_Y)$ is actually a weak lc model of $(X/Z, B)$ from which we get a log minimal model as in Corollary 3.7. In general the Y_i may be different but at least we could assume that they are all isomorphic in codimension one. In particular, if $\lfloor B_{Y_1} \rfloor = 0$, then we could replace each Y_i with Y_1 and proceed as before. We may then assume that $\lfloor B_{Y_1} \rfloor \neq 0$.

Since $K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor$ is semi-ampleness/Z,

$$K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor \sim_{\mathbf{Q}} C_{Y_1}/Z$$

for some $C_{Y_1} \geq 0$ so that $K_{Y_1} + B_{Y_1} + C_{Y_1}$ is lc. If $t_i' = \frac{t_i}{t_1 - t_i}$ and $i \neq 1$, then

$$K_{Y_1} + B_{Y_1} + t_i' C_{Y_1} \sim_{\mathbf{Q}} \left(1 + t_i'\right)\left(K_{Y_1} + B_{Y_1} - t_i \lfloor B_{Y_1} \rfloor\right)/Z$$

We can make sure that $t_i' \leq 1$ for every $i > 1$. Next we run an LMMP/Z on $K_{Y_2} + B_{Y_2}$ with scaling of $t_2' C_{Y_2}$. We use special termination (see Lemma 6.1) to show that the LMMP terminates with a model Y . We can proceed as before by replacing each Y_i with Y, for $i \gg 0$ (see Proposition 6.2 for more details). Finally, we will use Theorem 1.7 to show that the log minimal model we have constructed is actually good (see Proposition 6.3).

Now the horizontal case: assume that some lc centre of $(X/Z, B)$ is horizontal/Z, that is, some component of $\lfloor B \rfloor$ is horizontal/Z. The above arguments do not work since $K_X + B - t_i \lfloor B \rfloor$ is not pseudo-effective/Z so we do not have a log minimal model of $(X/Z, B - t_i \lfloor B \rfloor)$. However, $(X/Z, B - t_i \lfloor B \rfloor)$ has a Mori fibre space. Assume that for some i we already have a Mori fibre structure on X, that is, we have a $(K_X + B - t_i \lfloor B \rfloor)$-negative extremal contraction $g \colon X \to T/Z$ which is of fibre type, that is, $\dim X > \dim T$. The condition $K_X + B + A \sim_{\mathbf{Q}} 0/Z$ and the pseudo-effectivity of $K_X + B$ ensures that $K_X + B \sim_{\mathbf{Q}} 0/T$, in particular, $K_X + B \sim_{\mathbf{Q}} g^* M/Z$ for some $\mathbf{Q}$-Cartier divisor M on T. Now there is a component of $\lfloor B \rfloor$, say S, which maps onto T since $\lfloor B \rfloor$ is ample/T. By applying induction to

$$K_S + B_S + A_S := (K_X + B + A)|_S$$

we deduce that $(S/Z, B_S)$ has a good log minimal model hence the algebra $\mathcal{R}(S/Z, K_S + B_S)$ is finitely generated over $\mathscr{O}_Z$. It turns out that this implies that $\mathcal{R}(T/Z, M)$

is finitely generated (see Lemma 6.4) hence $\mathcal{R}(X/Z, K_X + B)$ is also finitely generated. Next, we can apply Theorem 5.1 (see Lemma 6.5). In general, there may not be any Mori fibre structure on X but there is such a structure on some birational model of X which can be used in a somewhat similar way (see Proposition 6.6 and Lemma 6.7).

The vertical case. — In this subsection, we deal with Theorem 1.1 in the vertical case, i.e. when every lc centre of $(X/Z, B)$ is vertical/Z, in particular, when $X \to Z$ is birational. First we prove a kind of special termination which will enable us to do induction on dimension. It is helpful to recall Remarks 2.9 and 2.10 before reading the proof of the next result.

Lemma **6.1.** — *Let* $(X/Z, B + A)$ *be a lc pair of dimension d as in Theorem 1.1 such that* $(X/Z, B)$ *satisfies the assumptions of Lemma 5.5. Then, assuming Theorem 1.1 in dimension $d - 1$, there is an LMMP/Z on $K_X + B$ with scaling of cC which terminates.*

Proof. — By Lemma 5.5, there is an LMMP/Z on $K_X + B$ with scaling of cC such that $\lambda = \lim_{i \to \infty} \lambda_i = 0$. We can assume that the LMMP consists of only log flips. Let $(X'/Z, B')$ be a **Q**-factorial dlt blowup of $(X/Z, B)$ and let C' be the birational transform of C. By Remark 2.9(1), (2), we can lift the above LMMP/Z on $K_X + B$ to an LMMP/Z on $K_{X'} + B'$ with scaling of cC'. We could assume that the latter LMMP consists of only log flips.

Let S be a component of $\lfloor B' \rfloor$ and let T be the normalisation of the image of S in Z. Put $K_S + B_S := (K_{X'} + B')|_S$ and let $(S', B_{S'})$ be a **Q**-factorial dlt blowup of (S, B_S). By construction,

$$K_{S'} + B_{S'} + A_{S'} \sim_{\mathbf{Q}} 0/T$$

where $A_{S'}$ is the pullback of A. By Remark 2.10, we may assume that the LMMP on $K_{X'} + B'$ induces an LMMP/T on $K_{S'} + B_{S'}$ with scaling of $cC_{S'}$ where $C_{S'}$ is the pullback of C.

Since $\lambda = \lim_{i \to \infty} \lambda_i = 0$, $K_{S'} + B_{S'}$ is pseudo-effective/T which means that $A_{S'}$ is vertical/T. By induction, $(S'/T, B_{S'})$ has a log minimal model. Thus, by Theorem 4.1, the above LMMP/T on $K_{S'} + B_{S'}$ terminates. Thus, the LMMP on $K_{X'} + B'$ terminates near S. The same argument applied to each component of $\lfloor B' \rfloor$ shows that the LMMP on $K_{X'} + B'$ terminates near $\lfloor B' \rfloor$.

By assumptions, $K_X + B \sim_{\mathbf{Q}} P + C/Z$ where $P \geq 0$ and $\operatorname{Supp} P = \operatorname{Supp} \lfloor B \rfloor$. Moreover, $\operatorname{Supp} \lfloor B \rfloor$ contains all the lc centres of $(X/Z, B)$. Thus, there is a **Q**-divisor $P' \geq 0$ with $\operatorname{Supp} P' = \operatorname{Supp} \lfloor B' \rfloor$ such that $K_{X'} + B' \sim_{\mathbf{Q}} P' + C'/Z$. So, each extremal ray contracted by the LMMP on $K_{X'} + B'$ intersects P'. But the LMMP terminates near $\lfloor B' \rfloor$ which in turn implies that it terminates near P' therefore it terminates everywhere. $\square$

Next we use the above special termination to derive parts (1) and (3) of Theorem 1.1, in the vertical case, from Theorem 1.1 in lower dimensions. The semi-ampleness statement in part (2) will be proved afterwards using Theorem 1.7.

Proposition **6.2.** — *Theorem 1.1 in dimension $d - 1$ implies Theorem 1.1(1), (3) in dimension d in the vertical case, i.e. when every lc centre of $(X/Z, B)$ is vertical/Z.*

Proof.

Step 1. After taking a **Q**-factorial dlt blowup using Corollary 3.6 we may assume that $(X/Z, B)$ is **Q**-factorial dlt, in particular, $\lfloor B \rfloor$ is vertical/Z. Run an LMMP/Z on $K_X + B$ with scaling of an ample/Z divisor. If $K_X + B$ is not pseudo-effective/Z, then the LMMP ends up with a Mori fibre space by [9]. So, from now on we assume that $K_X + B$ is pseudo-effective/Z. By Theorem 4.1, the LMMP terminates if we show that $(X/Z, B)$ has a log minimal model. Since $K_X + B \sim_{\mathbf{Q}} -A/Z$, $A|_F = 0$ where F is the generic fibre of $f \colon X \to Z$. Thus, A is vertical/Z and $(K_X + B)|_F \sim_{\mathbf{Q}} 0$.

Step 2. Let $t_1 > t_2 > \cdots$ be a sequence of sufficiently small rational numbers with $\lim_{i \to \infty} t_i = 0$. Each $(X/Z, B - t_i \lfloor B \rfloor)$ is klt and $(K_X + B - t_i \lfloor B \rfloor)|_F \sim_{\mathbf{Q}} 0$ hence by Theorem 5.2 each $(X/Z, B - t_i \lfloor B \rfloor)$ has a good log minimal model $(Y_i/Z, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor)$ so that $Y_i \dashrightarrow X$ does not contract divisors. Moreover, $\mathrm{Supp} \lfloor B_{Y_i} \rfloor$ contains all the lc centres of $(Y_i/Z, B_{Y_i})$ because $(Y_i/Z, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor)$ is klt which means that $(Y_i/Z, B_{Y_i})$ is klt outside $\mathrm{Supp} \lfloor B_{Y_i} \rfloor$. Now, since A and $\lfloor B \rfloor$ are vertical/Z, there are vertical/Z **Q**-divisors $M, N \geq 0$ such that

$$K_X + B \sim_{\mathbf{Q}} -A \sim_{\mathbf{Q}} M/Z \quad \text{and} \quad -\lfloor B \rfloor \sim_{\mathbf{Q}} N/Z$$

and

$$K_X + B - t_i \lfloor B \rfloor \sim_{\mathbf{Q}} M + t_i N/Z$$

So, any prime divisor contracted by $X \dashrightarrow Y_i$ is a component of $M + N$ hence after replacing the sequence with a subsequence we can assume that the maps $X \dashrightarrow Y_i$ contract the same divisors, i.e. Y_i are isomorphic in codimension one.

Step 3. Assume that $Y_i = Y_{i+1}$ for $i \gg 0$, and let Y be this common model. Since $K_Y + B_Y - t_i \lfloor B_Y \rfloor$ is nef/Z for each $i \gg 0$, $K_Y + B_Y$ is also nef/Z. For any prime divisor D on X and each $i \gg 0$, we have

$$a(D, X, B) \leq a\big(D, X, B - t_i \lfloor B \rfloor\big) \leq a\big(D, Y, B_Y - t_i \lfloor B_Y \rfloor\big)$$

which implies that

$$a(D, X, B) \leq \lim_{i \to \infty} a\big(D, Y, B_Y - t_i \lfloor B_Y \rfloor\big) = a(D, Y, B_Y)$$

Therefore, $(Y/Z, B_Y)$ is a weak lc model of $(X/Z, B)$, and by Corollary 3.7, we can construct a log minimal model of $(X/Z, B)$ as required.

Step 4. Assume that $\lfloor B_{Y_1} \rfloor = 0$. Then, $K_{Y_1} + B_{Y_1} - t_i \lfloor B_{Y_1} \rfloor$ is nef/Z for each i, and since $Y_1 \dashrightarrow Y_i$ is an isomorphism in codimension one, we can replace each Y_i with Y_1 and then apply Step 3.

From now on we assume that $\lfloor B_{Y_1} \rfloor \neq 0$. Since $(Y_1/Z, B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor)$ is klt and $K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor$ is nef/Z, $K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor$ is semi-ample/Z by Theorem 5.2. Thus,

$$K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor \sim_{\mathbf{Q}} C_{Y_1}/Z$$

for some $C_{Y_1} \geq 0$ such that $K_{Y_1} + B_{Y_1} + C_{Y_1}$ is lc, in particular, $\mathrm{Supp}\, C_{Y_1}$ does not contain any lc centre of $(Y_1/Z, B_{Y_1})$. For each rational number t we have

$$K_{Y_1} + B_{Y_1} + tC_{Y_1} \sim_{\mathbf{Q}} (1+t)\left(K_{Y_1} + B_{Y_1} - \frac{tt_1}{1+t}\lfloor B_{Y_1} \rfloor\right)\Big/Z$$

In particular, if $t_i' = \frac{t_i}{t_1 - t_i}$ and $i \neq 1$, then

$$K_{Y_1} + B_{Y_1} + t_i' C_{Y_1} \sim_{\mathbf{Q}} (1 + t_i')(K_{Y_1} + B_{Y_1} - t_i \lfloor B_{Y_1} \rfloor)/Z$$

We may assume that $2t_i \leq t_1$ for each $i > 1$ which implies that $t_i' \in [0, 1]$. So, $(Y_i/Z, B_{Y_i} + t_i' C_{Y_i})$ is a weak lc model of $(Y_1/Z, B_{Y_1} + t_i' C_{Y_1})$ for each $i > 1$. Note that $(Y_1/Z, B_{Y_1} + t_i' C_{Y_1})$ is lc because $t_i' \leq 1$.

Step 5. By construction,

$$K_{Y_2} + B_{Y_2} + A_{Y_2} \sim_{\mathbf{Q}} 0/Z$$

and every lc centre of $(Y_2/Z, B_{Y_2})$ is contained in $\lfloor B_{Y_2} \rfloor$ because $(Y_2/Z, B_{Y_2} - t_2 \lfloor B_{Y_2} \rfloor)$ is klt. On the other hand, since A is vertical/Z, A_{Y_2} is also vertical/Z, and if we put $P_{Y_2} = t_1 \lfloor B_{Y_2} \rfloor$, then by construction

$$K_{Y_2} + B_{Y_2} \sim_{\mathbf{Q}} P_{Y_2} + C_{Y_2}/Z$$

Now $K_{Y_2} + B_{Y_2} + t_2' C_{Y_2}$ is nef/Z, so by Lemma 6.1, we can run an LMMP/Z on $K_{Y_2} + B_{Y_2}$ with scaling of $t_2' C_{Y_2}$ which terminates on a model Y on which $K_Y + B_Y + \delta C_Y$ is nef/Z for any sufficiently small $\delta \geq 0$. Since $Y_2 \dashrightarrow Y_i$ is an isomorphism in codimension one, $K_{Y_i} + B_{Y_i} + t_i' C_{Y_i}$ is nef/Z, and $\lim_{i \to \infty} t_i' = 0$, we deduce that $K_{Y_2} + B_{Y_2}$ is (numerically) a limit of movable/Z **R**-divisors. So, the LMMP does not contract any divisors, i.e. $Y \dashrightarrow Y_i$ is an isomorphism in codimension one.

For any $i \gg 0$, $K_Y + B_Y + t_i' C_Y$ is nef/Z which in turn implies that $K_Y + B_Y - t_i \lfloor B_Y \rfloor$ is also nef/Z. Since $Y \dashrightarrow Y_i$ is an isomorphism in codimension one and since $(Y_i/Z, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor)$ is a log minimal model of $(X/Z, B - t_i \lfloor B \rfloor)$, $(Y/Z, B_Y - t_i \lfloor B_Y \rfloor)$ is a log minimal model of $(X/Z, B - t_i \lfloor B \rfloor)$, for every $i \gg 0$. By Step 3, we are done. $\quad\square$

Proposition **6.3.** — *Theorem 1.1 in dimension $d - 1$ implies Theorem 1.1 in dimension d in the vertical case, i.e. when every lc centre of $(X/Z, B)$ is vertical/Z.*

Proof. — After taking a **Q**-factorial dlt blowup using Corollary 3.6 we may assume that $(X/Z, B)$ is **Q**-factorial dlt, in particular, $\lfloor B \rfloor$ is vertical$/Z$. By Proposition 6.2, any LMMP$/Z$ on $K_X + B$ with scaling of an ample$/Z$ divisor ends up with a Mori fibre space or a log minimal model $(Y/Z, B_Y)$. Assume that $(Y/Z, B_Y)$ is a log minimal model. Let $g\colon W \to X$ and $h\colon W \to Y$ be a common resolution, and let $A_Y := h_* g^* A$. Then,

$$K_Y + B_Y + A_Y = h_* g^*(K_X + B + A) \sim_{\mathbf{Q}} 0/Z$$

where we use the fact that $h_* g^*(K_X + B) = K_Y + B_Y$ which in turn follows from the fact that $g^*(K_X + B) - h^*(K_Y + B_Y)$ is exceptional$/Y$, by Remark 2.6. Also, $(Y/Z, B_Y + A_Y)$ is lc because

$$h^*(K_Y + B_Y + A_Y) = g^*(K_X + B + A)$$

So, by replacing $(X/Z, B+A)$ with $(Y/Z, B_Y+A_Y)$ we may assume that $K_X + B$ is nef$/Z$. It remains to prove that $K_X + B$ is semi-ample$/Z$.

Run an LMMP$/Z$ on $K_X + B - \epsilon \lfloor B \rfloor$ with scaling of some ample$/Z$ divisor, for some sufficiently small rational number $\epsilon > 0$. Since $K_X + B$ is nef$/Z$ and $K_X + B + A \sim_{\mathbf{Q}} 0/Z$, $(K_X + B)|_F \sim_{\mathbf{Q}} 0$ where F is the generic fibre of $X \to Z$. Moreover, since $\lfloor B \rfloor$ is vertical$/Z$,

$$\left(K_X + B - \epsilon \lfloor B \rfloor \right)\big|_F \sim_{\mathbf{Q}} 0$$

So, by Theorem 5.2, the LMMP terminates on a model X' on which $K_{X'} + B_{X'} - \epsilon \lfloor B_{X'} \rfloor$ is semi-ample$/Z$. On the other hand, by [6, Proposition 3.2], $K_X + B$ is numerically trivial on each extremal ray contracted by the LMMP. Therefore, $K_{X'} + B_{X'}$ is also nef$/Z$. Another application of Theorem 5.2 shows that $K_{X'} + B_{X'} - \delta \lfloor B_{X'} \rfloor$ is semi-ample$/Z$ for any $\delta \in (0, \epsilon]$. In addition, $X \dashrightarrow X'$ is an isomorphism over the generic point of Z.

The pair $(X'/Z, B_{X'})$ is **Q**-factorial and lc and $(X'/Z, B_{X'} - \epsilon \lfloor B_{X'} \rfloor)$ is klt. If $\lfloor B_{X'} \rfloor = 0$ we are done so we can assume that $\lfloor B_{X'} \rfloor \neq 0$. Let $(Y/Z, B_Y)$ be a **Q**-factorial dlt blowup of $(X'/Z, B_{X'})$ and let S be a component of $T := \lfloor B_Y \rfloor$. Let A_Y be the pullback of $A_{X'}$. Then, from

$$K_{X'} + B_{X'} + A_{X'} \sim_{\mathbf{Q}} 0/Z$$

we get

$$K_Y + B_Y + A_Y \sim_{\mathbf{Q}} 0/Z$$

which implies that

$$K_S + B_S + A_S \sim_{\mathbf{Q}} 0/Z$$

where $K_S + B_S := (K_Y + B_Y)|_S$ and $A_S := A_Y|_S$. Now, by induction, $K_S + B_S$ is semi-ample/Z. On the other hand, if $P := e^*\lfloor B_{X'}\rfloor$ and if $\delta > 0$ is any sufficiently small rational number, where e is the morphism $Y \to X'$, then

$$K_Y + B_Y - \delta P = e^*\big(K_{X'} + B_{X'} - \delta\lfloor B_{X'}\rfloor\big)$$

is semi-ample/Z. Now, by Theorem 1.7, $K_Y + B_Y$ is semi-ample/Z hence $K_X + B$ is also semi-ample/Z. $\square$

The horizontal case. — In this subsection, we deal with the horizontal case of Theorem 1.1, that is, when some lc centre of $(X/Z, B)$ is horizontal/Z. First, we need the following result on finite generation of algebras.

Lemma **6.4.** *— Let $f: X \to Y/Z$ be a surjective morphism of normal varieties, projective over an affine variety $Z = \operatorname{Spec} R$, and L a Cartier divisor on Y. If $R(X/Z, f^*L)$ is a finitely generated R-algebra, then $R(Y/Z, L)$ is also a finitely generated R-algebra.*

The lemma can be easily derived from the deep fact that $\mathscr{O}_Y$ splits $f_*\mathscr{O}_X$. However, a simpler proof (suggested by the referee and independently by Kollár and Totaro) uses only basic commutative algebra: $R(X/Z, f^*L)$ is integral over $R(Y/Z, L)$ so the latter is finitely generated since the former is assumed to be finitely generated (cf. [4, Proposition 7.8]). Okawa [27, Theorem 4.1] proves a more general statement.

We will reduce the horizontal case to the next lemma by finding a suitable Mori fibre space.

Lemma **6.5.** *— Assume Theorem 1.1 in dimension $d - 1$. Let $(X/Z, B + A)$ be of dimension d as in Theorem 1.1 such that $(K_X + B)|_F \sim_{\mathbf{Q}} 0$ for the generic fibre F of $f: X \to Z$. Moreover, assume that there is a contraction $g: X \to T/Z$ such that*

(1) $K_X + B \sim_{\mathbf{Q}} 0/T$,
(2) *some lc centre of $(X/Z, B)$ is horizontal over T.*

Then, $(X/Z, B)$ has a good log minimal model.

Proof. — By replacing $(X/Z, B)$ with a **Q**-factorial dlt blowup, we can assume that $(X/Z, B)$ is **Q**-factorial dlt and that there is a component S of $\lfloor B\rfloor$ which is horizontal/T. Run an LMMP/Z on $K_X + B$ with scaling of some ample/Z divisor. Since termination and semi-ampleness/Z are local on Z, we can assume that Z is affine, say $\operatorname{Spec} R$. By Theorem 4.1, the LMMP terminates with a good log minimal model if we prove that $(X/Z, B)$ has a good log minimal model.

By adjunction define $K_S + B_S := (K_X + B)|_S$ and let $A_S := A|_S$. Then,

$$K_S + B_S + A_S = (K_X + B + A)|_S \sim_{\mathbf{Q}} 0/Z$$

and $K_S + B_S \sim_\mathbf{Q} 0/T$. Moreover, $(K_S + B_S)|_H \sim_\mathbf{Q} 0$ where H is the generic fibre of the induced morphism $h\colon S \to Z$.

Since we are assuming Theorem 1.1 in dimension $d - 1$, $(S/Z, B_S)$ has a good log minimal model. Therefore, if $I(K_S + B_S)$ is Cartier for some $I \in \mathbf{N}$, then $R(S/Z, I(K_S + B_S))$ is a finitely generated R-algebra (cf. [7]). We can choose I such that $I(K_X + B)$ is Cartier and such that $I(K_X + B) \sim g^*L$ for some Cartier divisor L on T. So, $I(K_S + B_S) \sim e^*L$ where $e\colon S \to T$ is the induced morphism. Now, by Lemma 6.4, $R(T/Z, L)$ is a finitely generated R-algebra which in turn implies that $R(X/Z, I(K_X + B))$ is a finitely generated R-algebra since $X \to T$ is a contraction. Therefore, $R(X/Z, K_X + B)$ is a finitely generated R-algebra and according to Theorem 5.1, $(X/Z, B)$ has a good log minimal model. $\square$

Proposition **6.6.** — *Assume Theorem 1.1 in dimension $d - 1$, and assume Theorem 1.1 in dimension d in the vertical case. Then, Theorem 1.1 holds in dimension d in the horizontal case, i.e. when some lc centre of $(X/Z, B)$ is horizontal over Z.*

Proof. — We can assume that $(X/Z, B)$ is $\mathbf{Q}$-factorial dlt and that $f\colon X \to Z$ is a contraction. By assumptions, some component of $\lfloor B \rfloor$ is horizontal/Z. Run an LMMP/Z on $K_X + B$ with scaling of some ample/Z divisor. If A is not vertical/Z, then $K_X + B$ is not pseudo-effective/Z hence the LMMP terminates with a Mori fibre space by [9]. So, we can assume that A is vertical/Z hence $(K_X + B)|_F \sim_\mathbf{Q} 0$ where F is the generic fibre of $f\colon X \to Z$. By Theorem 4.1, the LMMP terminates with a good log minimal model if we show that $(X/Z, B)$ has a good log minimal model.

If $\epsilon > 0$ is a sufficiently small rational number, then $(X/Z, B - \epsilon\lfloor B \rfloor)$ is klt and $K_X + B - \epsilon\lfloor B \rfloor$ is not pseudo-effective/Z. Thus, by [9] there is a Mori fibre space $(Y/Z, B_Y - \epsilon\lfloor B_Y \rfloor)$ for $(X/Z, B - \epsilon\lfloor B \rfloor)$ obtained by running an LMMP/Z on $K_X + B - \epsilon\lfloor B \rfloor$. Let $g\colon Y \to T/Z$ be the $K_Y + B_Y - \epsilon\lfloor B_Y \rfloor$-negative extremal contraction which defines the Mori fibre space structure, and let R be the corresponding extremal ray. Since A is vertical/Z, A_Y is vertical over T. So, from

$$K_Y + B_Y + A_Y \sim_\mathbf{Q} 0/Z$$

we deduce that $(K_Y + B_Y) \cdot R = 0$ which in turn implies that $K_Y + B_Y \sim_\mathbf{Q} 0/T$. Since $K_Y + B_Y - \epsilon\lfloor B_Y \rfloor$ is numerically negative/T, $\lfloor B_Y \rfloor \neq 0$. We could apply Lemma 6.5 to get a good log minimal model of $(Y/Z, B_Y)$ but this does not give a good log minimal model of $(X/Z, B)$ because we cannot easily compare the singularities of $(Y/Z, B_Y)$ and $(X/Z, B)$. We need to find a model which is closely related to both $(Y/Z, B_Y)$ and $(X/Z, B)$.

Let $(W/Z, B_W)$ be a log smooth model of $(X/Z, B)$ of type (1) as in Definition 2.3 such that W dominates Y, say by a morphism $h\colon W \to Y$. We can write

$$K_W + B_W = h^*(K_Y + B_Y) + G$$

where G is exceptional/Y. Run the LMMP/Y on $K_W + B_W$ with scaling of some ample/Y divisor. By Theorem 3.5, at some step of this LMMP we reach a model X' on which $G_{X'} \leq 0$. Also,

$$K_{X'} + B_{X'} - G_{X'} \sim_{\mathbf{Q}} 0/T$$

as it is the pullback of $K_Y + B_Y$. Since $K_X + B$ is pseudo-effective/Z, $K_W + B_W$ and $K_{X'} + B_{X'}$ are pseudo-effective over both Z and T. Thus, $G_{X'}$ is vertical/T hence $K_{X'} + B_{X'} \sim_{\mathbf{Q}} 0$ over the generic point of T. By construction, $\lfloor B_Y \rfloor$ is ample/T hence in particular some of its components are horizontal/T. This implies that some component of $\lfloor B_{X'} \rfloor$ is horizontal/T because $\lfloor B_Y \rfloor$ is the pushdown of $\lfloor B_{X'} \rfloor$.

Now by Lemma 6.7 below, we can run an LMMP/T on $K_{X'} + B_{X'}$ which ends up with a model X'' on which $K_{X''} + B_{X''}$ is semi-ample/T (to be more precise, in place of X, Y, Z, B, A in the lemma use X', Y, T, $B_{X'}$, $-G_{X'}$ respectively). Since $K_{X'} + B_{X'} \sim_{\mathbf{Q}} 0$ over the generic point of T, $X' \dashrightarrow X''$ is an isomorphism over the generic point of T. So, if $X'' \rightarrow T''/T$ is the contraction associated to $K_{X''} + B_{X''}$, then $T'' \rightarrow T$ is birational and some component of $\lfloor B_{X''} \rfloor$ is horizontal/T''.

On the other hand, if we denote $X' \rightarrow Y$ by e, then we can write

$$K_{X'} + B_{X'} + A_{X'} = e^*(K_Y + B_Y + A_Y)$$

by taking $A_{X'} := -G_{X'} + e^* A_Y$. Thus,

$$K_{X'} + B_{X'} + A_{X'} \sim_{\mathbf{Q}} 0/Z$$

and

$$K_{X''} + B_{X''} + A_{X''} \sim_{\mathbf{Q}} 0/Z$$

where $A_{X''}$ is the birational transform of $A_{X'}$. Since $K_{X''} + B_{X''}$ is pseudo-effective/Z, $A_{X''}$ is vertical/Z and $K_{X''} + B_{X''} \sim_{\mathbf{Q}} 0$ over the generic point of Z. Now we can apply Lemma 6.5 (by taking X, Z, T, B, A in the lemma to be X'', Z, T'', $B_{X''}$, $A_{X''}$ respectively) to get a good log minimal model of $(X''/Z, B_{X''})$ which would give a good log minimal model of $(W/Z, B_W)$ hence of $(X/Z, B)$, by Remark 2.8. □

Lemma **6.7.** *— Assume Theorem 1.1 in dimension d in the vertical case. Let $(X/Z, B + A)$ be of dimension d as in Theorem 1.1 such that $(K_X + B)|_F \sim_{\mathbf{Q}} 0$ for the generic fibre F of $f : X \rightarrow Z$. Moreover, assume that*

- *$(X/Z, B)$ is **Q**-factorial dlt,*
- *there exist contractions $e : X \rightarrow Y$ and $g : Y \rightarrow Z$,*
- *e is birational, A is exceptional/Y, and Y is **Q**-factorial,*
- *$D := K_Y + B_Y - \epsilon \lfloor B_Y \rfloor$ is klt and g is a D-negative extremal contraction where $\epsilon > 0$ and $B_Y := e_* B$.*

Then, any LMMP/Z on $K_X + B$ *with scaling of an ample/Z divisor terminates with a good log minimal model of* $(X/Z, B)$.

Proof. — By Theorem 4.1, it is enough to prove that $(X/Z, B)$ has a good log minimal model. By adding a small multiple of A to B we can assume that $\operatorname{Supp} A \subseteq \operatorname{Supp} B$. Since e is birational, by Theorem 1.1 in dimension d in the vertical case, we can assume that $K_X + B$ is actually semi-ample/Y. By replacing X with the lc model of $(X/Y, B)$ we can assume that $K_X + B$ is ample/Y (we may loose the **Q**-factorial dlt property of $(X/Z, B)$ but we will recover it later). Since $-A$ is ample/Y, $\operatorname{Supp} A$ contains all the prime exceptional/Y divisors on X. In particular,

$$\operatorname{Supp} e^* \lfloor B_Y \rfloor \subseteq \big(\operatorname{Supp} \lfloor B \rfloor \cup \operatorname{Supp} A \big) \subseteq \operatorname{Supp} B$$

By replacing $(X/Z, B)$ with a **Q**-factorial dlt blowup, we can again assume that $(X/Z, B)$ is **Q**-factorial dlt; note that this preserves the property $\operatorname{Supp} e^* \lfloor B_Y \rfloor \subseteq \operatorname{Supp} B$.

Since A is exceptional/Y and $K_X + B + A \sim_{\mathbf{Q}} 0/Z$, $K_Y + B_Y \sim_{\mathbf{Q}} 0/Z$. So, $\lfloor B_Y \rfloor$ is ample/Z hence $e^* \lfloor B_Y \rfloor$ is semi-ample/Z. Thus, for a small rational number $\tau > 0$ we can write

$$K_X + B = K_X + B - \tau e^* \lfloor B_Y \rfloor + \tau e^* \lfloor B_Y \rfloor \sim_{\mathbf{Q}} K_X + \Delta/Z$$

where Δ is some rational boundary such that $(X/Z, \Delta)$ is klt: since $(Y/Z, B_Y - \epsilon \lfloor B_Y \rfloor)$ is klt, $\lfloor B_Y \rfloor$ contains all the lc centres of $(Y/Z, B_Y)$, in particular, the image of all the lc centres of $(X/Z, B)$; so, $\operatorname{Supp} e^* \lfloor B_Y \rfloor$ contains all the components of $\lfloor B \rfloor$ hence $(X/Z, B - \tau e^* \lfloor B_Y \rfloor)$ is klt and we can indeed find Δ with the required properties. By Theorem 5.2, we can run an LMMP/Z on $K_X + \Delta$ which ends up with a good log minimal model of $(X/Z, \Delta)$ hence a good log minimal model of $(X/Z, B)$. $\qquad \square$

Proof of Theorem 1.1. — We argue by induction so in particular we may assume that Theorem 1.1 holds in dimension $d - 1$. By Proposition 6.3, Theorem 1.1 holds in dimension d in the vertical case. On the other hand, by Proposition 6.6, Theorem 1.1 also holds in dimension d in the horizontal case. $\qquad \square$

Proof of Corollary 1.2. — First assume that B has rational coefficients. Since $-(K_X + B)$ is ample/Y, we can find

$$A \sim_{\mathbf{Q}} -(K_X + B)/Y$$

such that $A \geq 0$, $(X/Z, B + A)$ is lc, and

$$K_X + B + A \sim_{\mathbf{Q}} 0/Y$$

Now by Theorem 1.1, $(X/Y, B)$ has a good log minimal model and its lc model gives the $(K_X + B)$-flip. When B is not rational, we can find rational lc divisors $K_X + B_i$ and real numbers $r_i > 0$ such that

$$K_X + B = \sum r_i(K_X + B_i) \quad \text{and} \quad \sum r_i = 1$$

Moreover, we can assume that $-(K_X + B_i)$ is ample$/Y$ for every i. For each i, j there is a rational number $b_{i,j}$ such that

$$K_X + B_i \equiv b_{i,j}(K_X + B_j)/Y$$

By the cone theorem for lc pairs proved by Ambro [2] and Fujino [14],

$$K_X + B_i \sim_{\mathbf{Q}} b_{i,j}(K_X + B_j)/Y$$

Therefore, there is a morphism $X^+ \to Y$ which gives the flip of $K_X + B_i$ for every i. The morphism also gives the $(K_X + B)$-flip. $\square$

7. Proof of Theorem 1.4

In this section, we give the proof of Theorem 1.4 which is parallel to the arguments of section 6 with some small changes.

The vertical case. — First, we deal with Theorem 1.4 in the vertical case. It is helpful to recall Remarks 2.9 and 2.10 before reading the proof of the next special termination result.

Lemma **7.1.** — *Let* $(X/Z, B)$ *be a lc pair of dimension d as in Theorem 1.4 such that it satisfies the assumptions of Lemma 5.5. Then, assuming Conjecture 1.3 and Theorem 1.4 in dimension $d - 1$, there is an LMMP$/Z$ on $K_X + B$ with scaling of cC which terminates.*

Proof. — By Lemma 5.5, there is an LMMP$/Z$ on $K_X + B$ with scaling of cC such that $\lambda = \lim_{i \to \infty} \lambda_i = 0$. We can assume that the LMMP consists of only log flips. Let $(X'/Z, B')$ be a $\mathbf{Q}$-factorial dlt blowup of $(X/Z, B)$ and let C' be the birational transform of C. By Remark 2.9(1), (2), we can lift the above LMMP$/Z$ on $K_X + B$ to an LMMP$/Z$ on $K_{X'} + B'$ with scaling of cC'. We could assume that the latter LMMP consists of only log flips.

Let S be a component of $\lfloor B' \rfloor$ and let T be the normalisation of the image of S in Z. Put $K_S + B_S := (K_{X'} + B')|_S$ and let $(S', B_{S'})$ be a $\mathbf{Q}$-factorial dlt blowup of (S, B_S). By Remark 2.10, we may assume that the LMMP on $K_{X'} + B'$ induces an LMMP$/T$ on $K_{S'} + B_{S'}$ with scaling of $cC_{S'}$ where $C_{S'}$ is the pullback of C. On the other hand, by assumptions, there is a non-empty open subset $U \subseteq Z$ such that $K_X + B \sim_{\mathbf{Q}} 0$ over U and

such that the generic point of each lc centre of $(X/Z, B)$ is mapped into U. Let $V \subseteq T$ be the inverse image of U under $T \to Z$. Then, $K_{S'} + B_{S'} \sim_{\mathbf{Q}} 0$ over V and the generic point of each lc centre of $(S'/T, B_{S'})$ is mapped into V.

By induction, $(S'/T, B_{S'})$ has a log minimal model. Since $\lambda = \lim_{i \to \infty} \lambda_i = 0$, by Theorem 4.1, the above LMMP/T on $K_{S'} + B_{S'}$ terminates. Thus, the LMMP/Z on $K_{X'} + B'$ terminates near S. The same argument applied to each component of $\lfloor B' \rfloor$ shows that the LMMP/Z on $K_{X'} + B'$ terminates near $\lfloor B' \rfloor$. By assumptions, $K_X + B \sim_{\mathbf{Q}} P + C/Z$ where $P \geq 0$ and $\mathrm{Supp}\, P = \mathrm{Supp}\lfloor B \rfloor$. Moreover, $\mathrm{Supp}\lfloor B \rfloor$ contains all the lc centres of $(X/Z, B)$. Thus, there is a $\mathbf{Q}$-divisor $P' \geq 0$ with $\mathrm{Supp}\, P' = \mathrm{Supp}\lfloor B' \rfloor$ such that $K_{X'} + B' \sim_{\mathbf{Q}} P' + C'/Z$. So, each extremal ray contracted by the LMMP/Z on $K_{X'} + B'$ intersects P'. But the LMMP terminates near $\lfloor B' \rfloor$ which in turn implies that it terminates near P' therefore it terminates everywhere. $\square$

Proposition **7.2.** — *Theorem 1.4 in dimension $d - 1$ implies Theorem 1.4(1), (3) in dimension d in the vertical case, i.e. when every lc centre of $(X/Z, B)$ is vertical/Z.*

Proof.

Step 1. After taking a $\mathbf{Q}$-factorial dlt blowup using Corollary 3.6 we may assume that $(X/Z, B)$ is $\mathbf{Q}$-factorial dlt, in particular, $\lfloor B \rfloor$ is vertical/Z. Run an LMMP/Z on $K_X + B$ with scaling of an ample/Z divisor. By Theorem 4.1, the LMMP terminates if we prove that $(X/Z, B)$ has a log minimal model. By assumptions, $K_X + B \sim_{\mathbf{Q}} 0$ over some non-empty open subset $U \subseteq Z$ and if η is the generic point of any lc centre of $(X/Z, B)$, then $f(\eta) \in U$.

Step 2. Let $t_1 > t_2 > \cdots$ be a sequence of sufficiently small rational numbers with $\lim_{i \to \infty} t_i = 0$. Each $(X/Z, B - t_i \lfloor B \rfloor)$ is klt and $(K_X + B - t_i \lfloor B \rfloor)|_F \sim_{\mathbf{Q}} 0$ where F is the generic fibre of f. Hence by Theorem 5.2 each $(X/Z, B - t_i \lfloor B \rfloor)$ has a good log minimal model $(Y_i/Z, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor)$ so that $Y_i \dashrightarrow X$ does not contract divisors. Since we are assuming Conjecture 1.3 in dimension d, we can assume that each $(Y_i/Z, B_{Y_i})$ is lc. Moreover, $\mathrm{Supp}\lfloor B_{Y_i} \rfloor$ contains all the lc centres of $(Y_i/Z, B_{Y_i})$ because $(Y_i/Z, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor)$ is klt which means that $(Y_i/Z, B_{Y_i})$ is klt outside $\mathrm{Supp}\lfloor B_{Y_i} \rfloor$. Since $K_X + B \sim_{\mathbf{Q}} 0$ over U and since $\lfloor B \rfloor$ is vertical/Z, there are vertical/Z $\mathbf{Q}$-divisors $M, N \geq 0$ such that

$$K_X + B \sim_{\mathbf{Q}} M/Z \quad \text{and} \quad -\lfloor B \rfloor \sim_{\mathbf{Q}} N/Z$$

and

$$K_X + B - t_i \lfloor B \rfloor \sim_{\mathbf{Q}} M + t_i N/Z$$

So, any prime divisor contracted by $X \dashrightarrow Y_i$ is a component of $M + N$ hence after replacing the sequence with a subsequence we can assume that the maps $X \dashrightarrow Y_i$ contract the same divisors, i.e. Y_i are isomorphic in codimension one.

Step 3. Assume that $Y_i = Y_{i+1}$ for $i \gg 0$, and let Y be this common model. Since $K_Y + B_Y - t_i \lfloor B_Y \rfloor$ is nef/Z for each $i \gg 0$, $K_Y + B_Y$ is also nef/Z. For any prime divisor D on X and each $i \gg 0$, we have

$$a(D, X, B) \le a(D, X, B - t_i \lfloor B \rfloor) \le a(D, Y, B_Y - t_i \lfloor B_Y \rfloor)$$

which implies that

$$a(D, X, B) \le \lim_{i \to \infty} a(D, Y, B_Y - t_i \lfloor B_Y \rfloor) = a(D, Y, B_Y)$$

Therefore, $(Y/Z, B_Y)$ is a weak lc model of $(X/Z, B)$, and by Corollary 3.7, we can construct a log minimal model of $(X/Z, B)$ as required.

Step 4. Assume that $\lfloor B_{Y_1} \rfloor = 0$. Then, $K_{Y_1} + B_{Y_1} - t_i \lfloor B_{Y_1} \rfloor$ is nef/Z for each i, and since $Y_1 \dashrightarrow Y_i$ is an isomorphism in codimension one, we can replace each Y_i with Y_1 and then apply Step 3.

From now on we assume that $\lfloor B_{Y_1} \rfloor \ne 0$. Then, since $(Y_1/Z, B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor)$ is klt and $K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor$ is nef/Z, $K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor$ is semi-ample/Z by Theorem 5.2. Thus,

$$K_{Y_1} + B_{Y_1} - t_1 \lfloor B_{Y_1} \rfloor \sim_{\mathbf{Q}} C_{Y_1}/Z$$

for some $C_{Y_1} \ge 0$ such that $K_{Y_1} + B_{Y_1} + C_{Y_1}$ is lc, in particular, $\operatorname{Supp} C_{Y_1}$ does not contain any lc centre of $(Y_1/Z, B_{Y_1})$. For each rational number t we have

$$K_{Y_1} + B_{Y_1} + t C_{Y_1} \sim_{\mathbf{Q}} (1 + t)\left(K_{Y_1} + B_{Y_1} - \frac{t t_1}{1 + t} \lfloor B_{Y_1} \rfloor\right)\Big/Z$$

In particular, if $t_i' = \frac{t_i}{t_1 - t_i}$ and $i \ne 1$, then

$$K_{Y_1} + B_{Y_1} + t_i' C_{Y_1} \sim_{\mathbf{Q}} (1 + t_i')(K_{Y_1} + B_{Y_1} - t_i \lfloor B_{Y_1} \rfloor)/Z$$

We may assume that $2t_i \le t_1$ for each $i > 1$ which implies that $t_i' \in [0, 1]$. So, $(Y_i/Z, B_{Y_i} + t_i' C_{Y_i})$ is a weak lc model of $(Y_1/Z, B_{Y_1} + t_i' C_{Y_1})$ for each $i > 1$. Note that $(Y_1/Z, B_{Y_1} + t_i' C_{Y_1})$ is lc because $t_i' \le 1$.

Step 5. We will modify the situation so that if η is the generic point of any lc centre of $(Y_2/Z, B_{Y_2})$, then η is mapped into U. Put $P_{Y_2} = t_1 \lfloor B_{Y_2} \rfloor$. Then,

$$K_{Y_2} + B_{Y_2} \sim_{\mathbf{Q}} P_{Y_2} + C_{Y_2}/Z$$

By Lemma 5.5, we can run an LMMP/Z on $K_{Y_2} + B_{Y_2}$ with scaling of $t_2' C_{Y_2}$ so that if λ_k are the numbers appearing in the LMMP, then $\lambda = \lim_{k \to \infty} \lambda_k = 0$. Since $Y_2 \dashrightarrow Y_i$ is an isomorphism in codimension one, $K_{Y_i} + B_{Y_i} + t_i' C_{Y_i}$ is nef/Z, and $\lim_{i \to \infty} t_i' = 0$, we deduce that $K_{Y_2} + B_{Y_2}$ is (numerically) a limit of movable/Z $\mathbf{R}$-divisors. So, the LMMP does not contract any divisors. Now, by replacing the t_i with a subsequence, we

can assume that for each $i > 2$ there is k such that $\lambda_k \geq t'_i \geq \lambda_{k+1}$. If T is the variety corresponding to λ_{k+1}, then $K_T + B_T + t'_i C_T$ is nef/Z. By replacing Y_i with T, we could assume that Y_i occurs in some step of the LMMP. Thus, we can assume that every Y_i occurs in some step of the LMMP if $i > 1$.

After finitely many steps, the LMMP does not contract any lc centres. So, perhaps after replacing $(Y_2/Z, B_{Y_2})$ with some $(Y_j/Z, B_{Y_j})$, with $j \gg 0$, we may assume that the LMMP does not contract any lc centre of $(Y_2/Z, B_{Y_2})$. Assume that η is the generic point of some lc centre of $(Y_2/Z, B_{Y_2})$ which is mapped outside U. If E is a prime divisor over η such that $a(E, Y_2, B_{Y_2}) = 0$, then $a(E, X, B) > 0$ and for any $i \geq 2$ we have

$$a\big(E, Y_2, B_{Y_2} - t_i \lfloor B_{Y_2} \rfloor\big) = a\big(E, Y_i, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor\big)$$
$$\geq a\big(E, X, B - t_i \lfloor B \rfloor\big)$$
$$\geq a(E, X, B)$$

where the equality follows from the fact that $Y_2 \dashrightarrow Y_i$ is an isomorphism near η. Then,

$$0 = a(E, Y_2, B_{Y_2}) = \lim_{i \to \infty} a\big(E, Y_2, B_{Y_2} - t_i \lfloor B_{Y_2} \rfloor\big) \geq a(E, X, B) > 0$$

which is a contradiction. So, from now on we can assume that if η is the generic point of any lc centre of $(Y_2/Z, B_{Y_2})$, then η is mapped into U. Moreover, by construction, $K_{Y_2} + B_{Y_2} \sim_{\mathbf{Q}} 0$ over U.

Step 6. By Lemma 7.1, we can run an LMMP/Z on $K_{Y_2} + B_{Y_2}$ with scaling of $t'_2 C_{Y_2}$ which terminates on a model Y on which $K_Y + B_Y + \delta C_Y$ is nef/Z for any sufficiently small $\delta \geq 0$. Since $Y_2 \dashrightarrow Y_i$ is an isomorphism in codimension one, $K_{Y_i} + B_{Y_i} + t'_i C_{Y_i}$ is nef/Z, and $\lim_{i \to \infty} t'_i = 0$, we deduce that $K_{Y_2} + B_{Y_2}$ is (numerically) a limit of movable/Z **R**-divisors. So, the LMMP does not contract any divisors, i.e. $Y \dashrightarrow Y_i$ is an isomorphism in codimension one.

For any $i \gg 0$, $K_Y + B_Y + t'_i C_Y$ is nef/Z which in turn implies that $K_Y + B_Y - t_i \lfloor B_Y \rfloor$ is also nef/Z. Since $Y \dashrightarrow Y_i$ is an isomorphism in codimension one and since $(Y_i/Z, B_{Y_i} - t_i \lfloor B_{Y_i} \rfloor)$ is a log minimal model of $(X/Z, B - t_i \lfloor B \rfloor)$, $(Y/Z, B_Y - t_i \lfloor B_Y \rfloor)$ is a log minimal model of $(X/Z, B - t_i \lfloor B \rfloor)$, for every $i \gg 0$. By Step 3, we are done. $\quad\square$

Proposition **7.3.** — *Theorem 1.4 in dimension $d - 1$ implies Theorem 1.4 in dimension d in the vertical case, i.e. when every lc centre of* $(X/Z, B)$ *is vertical/Z.*

Proof. — By Proposition 7.2, statements (1) and (3) of Theorem 1.4 hold for $(X/Z, B)$, in particular, $(X/Z, B)$ has a log minimal model $(Y/Z, B_Y)$. By assumptions, $K_X + B \sim_{\mathbf{Q}} 0$ over some non-empty open subset $U \subseteq Z$ and if η is the generic point of any lc centre of $(X/Z, B)$ then $f(\eta) \in U$. This implies that $K_Y + B_Y \sim_{\mathbf{Q}} 0$ over U and that if η is the generic point of any lc centre of $(Y/Z, B_Y)$ then η is mapped into U. By replacing $(X/Z, B)$ with $(Y/Z, B_Y)$ we may assume that $(X/Z, B)$ is **Q**-factorial dlt and that $K_X + B$ is nef/Z. It remains to prove that $K_X + B$ is semi-ample/Z.

Run an LMMP/Z on $K_X + B - \epsilon \lfloor B \rfloor$ with scaling of some ample/Z divisor, for some sufficiently small rational number $\epsilon > 0$. By [6, Proposition 3.2], $K_X + B$ is numerically trivial on each step of this LMMP. Since $\lfloor B \rfloor$ is vertical/Z, $(K_X + B - \epsilon \lfloor B \rfloor)|_F \sim_{\mathbf{Q}} 0$ where F is the generic fibre of f. So, by Theorem 5.2, the LMMP terminates on a model X' on which $K_{X'} + B_{X'}$ and $K_{X'} + B_{X'} - \epsilon \lfloor B_{X'} \rfloor$ are both nef/Z. Another application of Theorem 5.2 shows that $K_{X'} + B_{X'} - \delta \lfloor B_{X'} \rfloor$ is semi-ample/Z for any $\delta \in (0, \epsilon]$. The pair $(X'/Z, B_{X'})$ is $\mathbf{Q}$-factorial and lc and $(X'/Z, B_{X'} - \epsilon \lfloor B_{X'} \rfloor)$ is klt. If $\lfloor B_{X'} \rfloor = 0$ we are done so we can assume that $\lfloor B_{X'} \rfloor \neq 0$. Let $(Y/Z, B_Y)$ be a $\mathbf{Q}$-factorial dlt blowup of $(X'/Z, B_{X'})$. By construction, $K_Y + B_Y \sim_{\mathbf{Q}} 0$ over U and if η is the generic point of any lc centre of $(Y/Z, B_Y)$ then η is mapped into U.

Let S be a component of $T := \lfloor B_Y \rfloor$ and put $K_S + B_S := (K_Y + B_Y)|_S$. If Q is the normalisation of the image of S in Z and if $V \subseteq Q$ is the inverse image of U, then $K_S + B_S \sim_{\mathbf{Q}} 0$ over V and if η is the generic point of any lc centre of $(S/Q, B_S)$ then η is mapped into V. So, by induction, $K_S + B_S$ is semi-ample/Q hence semi-ample/Z. On the other hand, if $P := e^* \lfloor B_{X'} \rfloor$ and if $\delta > 0$ is any sufficiently small rational number, where e is the morphism $Y \to X'$, then

$$K_Y + B_Y - \delta P = e^* \big(K_{X'} + B_{X'} - \delta \lfloor B_{X'} \rfloor \big)$$

is semi-ample/Z. Now, by Theorem 1.7, $K_Y + B_Y$ is semi-ample/Z hence $K_X + B$ is also semi-ample/Z. $\qquad \square$

In the proof of Proposition 7.3, it is also possible to compactify X and Z first before running the LMMP on $K_X + B - \epsilon \lfloor B \rfloor$; in this way we would need Theorem 1.7 only when X, Z are projective.

The horizontal case. — In this subsection, we deal with the horizontal case of Theorem 1.4.

Lemma 7.4. — *Assume Conjecture 1.3 and Theorem 1.4 in dimension $d - 1$. Let $(X/Z, B)$ be of dimension d as in Theorem 1.4. Moreover, assume that there is a contraction $g \colon X \to T/Z$ such that*

(1) $K_X + B \sim_{\mathbf{Q}} 0/T$,
(2) *some lc centre of $(X/Z, B)$ is horizontal over* T.

Then, $(X/Z, B)$ has a good log minimal model.

Proof. — By replacing $(X/Z, B)$ with a $\mathbf{Q}$-factorial dlt blowup, we can assume that $(X/Z, B)$ is $\mathbf{Q}$-factorial dlt and that there is a component S of $\lfloor B \rfloor$ which is horizontal/T. Run an LMMP/Z on $K_X + B$ with scaling of some ample/Z divisor. Since termination and semi-ampleness/Z are local on Z, we can assume that Z is affine, say Spec R. By

Theorem 4.1, the LMMP terminates with a good log minimal model if we prove that $(X/Z, B)$ has a good log minimal model.

By assumptions, $K_X + B \sim_{\mathbf{Q}} 0$ over some non-empty open subset $U \subseteq Z$ and if η is the generic point of any lc centre of $(X/Z, B)$ then $f(\eta) \in U$. By adjunction define $K_S + B_S := (K_X + B)|_S$. Then, $K_S + B_S \sim_{\mathbf{Q}} 0$ over U and if η is the generic point of any lc centre of $(S/Z, B_S)$ then $f(\eta) \in U$. Moreover, $K_S + B_S \sim_{\mathbf{Q}} 0/T$.

Since we are assuming Conjecture 1.3 and Theorem 1.4 in dimension $d - 1$, $(S/Z, B_S)$ has a good log minimal model. Therefore, if $I(K_S + B_S)$ is Cartier for some $I \in \mathbf{N}$, then $R(S/Z, I(K_S + B_S))$ is a finitely generated R-algebra (cf. [7]). We can choose I such that $I(K_X + B)$ is Cartier and such that $I(K_X + B) \sim g^*L$ for some Cartier divisor L on T. So, $I(K_S + B_S) \sim e^*L$ where $e\colon S \to T$ is the induced morphism. Now, by Lemma 6.4, $R(T/Z, L)$ is a finitely generated R-algebra which in turn implies that $R(X/Z, I(K_X + B))$ is a finitely generated R-algebra since $X \to T$ is a contraction. Therefore, $R(X/Z, K_X + B)$ is a finitely generated R-algebra and according to Theorem 5.1, $(X/Z, B)$ has a good log minimal model.   □

Proposition **7.5.** — *Assume Theorem 1.4 in dimension $d - 1$. Then, Theorem 1.4 holds in dimension d in the horizontal case, i.e. when some lc centre of $(X/Z, B)$ is horizontal over Z.*

Proof. — We can assume that $(X/Z, B)$ is **Q**-factorial dlt and that $f\colon X \to Z$ is a contraction. By assumptions, some component of $\lfloor B \rfloor$ is horizontal/Z. Moreover, $K_X + B \sim_{\mathbf{Q}} 0$ over some non-empty open subset $U \subseteq Z$ and if η is the generic point of any lc centre of $(X/Z, B)$ then η is mapped into U. Run an LMMP/Z on $K_X + B$ with scaling of some ample/Z divisor. By Theorem 4.1, the LMMP terminates with a good log minimal model if we show that $(X/Z, B)$ has a good log minimal model.

If $\epsilon > 0$ is a sufficiently small rational number, then $(X/Z, B - \epsilon\lfloor B \rfloor)$ is klt and $K_X + B - \epsilon\lfloor B \rfloor$ is not pseudo-effective/Z. Thus, by [9] there is a Mori fibre space $(Y/Z, B_Y - \epsilon\lfloor B_Y \rfloor)$ for $(X/Z, B - \epsilon\lfloor B \rfloor)$ obtained by running an LMMP/Z on $K_X + B - \epsilon\lfloor B \rfloor$. Let $g\colon Y \to T/Z$ be the $K_Y + B_Y - \epsilon\lfloor B_Y \rfloor$-negative extremal contraction which defines the Mori fibre space structure, and let R be the corresponding extremal ray. By construction, $K_Y + B_Y \sim_{\mathbf{Q}} 0$ over U hence $(K_Y + B_Y) \cdot R = 0$ which in turn implies that $K_Y + B_Y \sim_{\mathbf{Q}} 0/T$. Since $K_Y + B_Y - \epsilon\lfloor B_Y \rfloor$ is numerically negative/T, $\lfloor B_Y \rfloor \neq 0$. As we are assuming Conjecture 1.3 in dimension d, we can assume that $(Y/Z, B_Y)$ is lc.

Let $(W/Z, B_W)$ be a log smooth model of $(X/Z, B)$ of type (2) as in Definition 2.3 such that W dominates Y, say by a morphism $h\colon W \to Y$. In particular, we may assume that if η is the generic point of any lc centre of $(W/Z, B_W)$, then η is mapped into U. We can write

$$K_W + B_W = h^*(K_Y + B_Y) + G$$

where G is exceptional/Y. Run the LMMP/Y on $K_W + B_W$ with scaling of some ample/Y divisor. By Theorem 3.5, we reach a model X' on which $G_{X'} \le 0$. Moreover,

$$K_{X'} + B_{X'} - G_{X'} \sim_{\mathbf{Q}} 0/T$$

as it is the pullback of $K_Y + B_Y$.

Let π be the given morphism $W \to X$. By definition of log smooth models, we can write

$$K_W + B_W = \pi^*(K_X + B) + E$$

where E is effective and exceptional/X. On the other hand, since $K_X + B \sim_{\mathbf{Q}} 0$ and $K_Y + B_Y \sim_{\mathbf{Q}} 0$ over U, we have $\pi^*(K_X + B) = h^*(K_Y + B_Y)$ over U which means that $E = G$ over U. Therefore, every component of G with negative coefficient is mapped into $Z \setminus U$ hence $G_{X'}$ is also mapped into $Z \setminus U$. In particular, this means that $K_{X'} + B_{X'} \sim_{\mathbf{Q}} 0$ over U. Moreover, if η is the generic point of any lc centre of $(X'/Z, B_{X'})$ then η is mapped into U because the same holds for $(W/Z, B_W)$. Let $U_T \subseteq T$ be the inverse image of U. Then, $K_{X'} + B_{X'} \sim_{\mathbf{Q}} 0$ over U_T and if η is the generic point of any lc centre of $(X'/Z, B_{X'})$ then η is mapped into U_T.

By construction, $\lfloor B_Y \rfloor$ is ample/T hence in particular some of its components are horizontal/T. This implies that some component of $\lfloor B_{X'} \rfloor$ is horizontal/T because $\lfloor B_Y \rfloor$ is the pushdown of $\lfloor B_{X'} \rfloor$. By Lemma 7.6 below, we can run an LMMP/T on $K_{X'} + B_{X'}$ which ends up with a model X'' on which $K_{X''} + B_{X''}$ is semi-ample/T (more precisely, we should take X, Y, Z, B in the lemma to be our $X', Y, T, B_{X'}$ respectively). Over U_T, $X' \dashrightarrow X''$ is an isomorphism. So, if $X'' \to T''/T$ is the contraction associated to $K_{X''} + B_{X''}$, then $T'' \to T$ is birational and some component of $\lfloor B_{X''} \rfloor$ is horizontal/T''.

By construction, $K_{X''} + B_{X''} \sim_{\mathbf{Q}} 0/T''$, and $K_{X''} + B_{X''} \sim_{\mathbf{Q}} 0$ over U. Moreover, if η is the generic point of any lc centre of $(X''/Z, B_{X''})$ then η is mapped into U. Now we can apply Lemma 7.4 (by taking X, Z, T, B in the lemma to be our $X'', Z, T'', B_{X''}$ respectively) to get a good log minimal model of $(X''/Z, B_{X''})$ which would give a good log minimal model of $(W/Z, B_W)$ hence of $(X/Z, B)$, by Remark 2.8. $\square$

Lemma **7.6.** — *Assume Conjecture 1.3 in dimension d and Theorem 1.4 in dimension $d - 1$. Let $(X/Z, B)$ be of dimension d as in Theorem 1.4. Moreover, assume that*

- *$(X/Z, B)$ is $\mathbf{Q}$-factorial dlt,*
- *there exist contractions $e\colon X \to Y$ and $g\colon Y \to Z$,*
- *e is birational and Y is $\mathbf{Q}$-factorial,*
- *$K_Y + B_Y$ is lc, $D := K_Y + B_Y - \epsilon \lfloor B_Y \rfloor$ is klt, and g is a D-negative extremal contraction for some $\epsilon > 0$ where $B_Y := e_*B$.*

Then, any LMMP/Z on $K_X + B$ with scaling of an ample/Z divisor terminates with a good log minimal model of $(X/Z, B)$.

Proof. — By Theorem 4.1, it is enough to prove that $(X/Z, B)$ has a good log minimal model. By assumptions, $K_X + B \sim_{\mathbf{Q}} 0$ over some non-empty open subset $U \subseteq Z$

and if η is the generic point of any lc centre of $(X/Z, B)$ then $f(\eta) \in U$. Let $U_Y \subseteq Y$ be the inverse image of U. Then, $K_X + B \sim_{\mathbf{Q}} 0$ over U_Y and if η is the generic point of any lc centre of $(X/Z, B)$ then $e(\eta) \in U_Y$. Since e is birational, by Proposition 7.3 we can assume that $K_X + B$ is actually semi-ample/Y. By replacing X with the lc model of $(X/Y, B)$ we can assume that $K_X + B$ is ample/Y (we may loose the $\mathbf{Q}$-factorial dlt property of $(X/Z, B)$ but we will recover it later).

Since g is an extremal contraction and since $K_Y + B_Y \sim_{\mathbf{Q}} 0$ over U, $K_Y + B_Y \sim_{\mathbf{Q}} 0/Z$. Moreover, since $K_X + B$ is ample/Y, there is an e-exceptional $\mathbf{Q}$-divisor $A \geq 0$ such that

$$K_X + B + A = e^*(K_Y + B_Y)$$

and $K_X + B + A \sim_{\mathbf{Q}} 0/Z$, and since $K_Y + B_Y$ is lc, $K_X + B + A$ is lc too. Also, since $-A$ is ample/Y, $\operatorname{Supp} A$ contains all the prime exceptional/Y divisors on X. Now we can add a small multiple of A to B and assume that $\operatorname{Supp} B$ contains all the prime exceptional/Y divisors on X. Note that $A \sim_{\mathbf{Q}} 0$ over U_Y hence $A = 0$ over U_Y, so we still have $K_X + B \sim_{\mathbf{Q}} 0$ over U after adding a small multiple of A to B. Now,

$$\operatorname{Supp} e^* \lfloor B_Y \rfloor \subseteq \big(\operatorname{Supp} \lfloor B \rfloor \cup \operatorname{Supp} A \big) \subseteq \operatorname{Supp} B$$

Replacing $(X/Z, B)$ with a $\mathbf{Q}$-factorial dlt blowup, we can again assume that $(X/Z, B)$ is $\mathbf{Q}$-factorial dlt; the property $\operatorname{Supp} e^* \lfloor B_Y \rfloor \subseteq \operatorname{Supp} B$ is preserved.

By assumptions, $\lfloor B_Y \rfloor$ is ample/Z hence $e^* \lfloor B_Y \rfloor$ is semi-ample/Z. Thus, for a small rational number $\tau > 0$ we can write

$$K_X + B = K_X + B - \tau e^* \lfloor B_Y \rfloor + \tau e^* \lfloor B_Y \rfloor \sim_{\mathbf{Q}} K_X + \Delta/Z$$

where Δ is some rational boundary such that $(X/Z, \Delta)$ is klt: since $(Y/Z, B_Y - \epsilon \lfloor B_Y \rfloor)$ is klt, $\lfloor B_Y \rfloor$ contains all the lc centres of $(Y/Z, B_Y)$, in particular, the image of all the lc centres of $(X/Z, B)$; so, $\operatorname{Supp} e^* \lfloor B_Y \rfloor$ contains all the components of $\lfloor B \rfloor$ hence $(X/Z, B - \tau e^* \lfloor B_Y \rfloor)$ is klt and we can indeed find Δ with the required properties. By Theorem 5.2, we can run an LMMP/Z on $K_X + \Delta$ which ends up with a good log minimal model of $(X/Z, \Delta)$ hence a good log minimal model of $(X/Z, B)$.   □

Proof of Theorem 1.4. — This follows from Propositions 7.3 and 7.5.   □

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