FAMILIES OF RATIONALLY SIMPLY CONNECTED VARIETIES OVER SURFACES AND TORSORS FOR SEMISIMPLE GROUPS

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ABSTRACT

Under suitable hypotheses, we prove that a form of a projective homogeneous variety G/P defined over the function field of a surface over an algebraically closed field has a rational point. The method uses an algebro-geometric analogue of simple connectedness replacing the unit interval by the projective line. As a consequence, we complete the proof of Serre's Conjecture II in Galois cohomology for function fields over an algebraically closed field.

CONTENTS

1. Introduction

Let $f : X \to B$ be a morphism of smooth projective varieties over an algebraically closed field k of characteristic 0. In the paper [GHS03] it was shown that if B is a curve and the general fibre of f is a rationally connected variety then f has a section. In this paper we partially generalize this result to the case where B is a surface. In order to state it, we say that a *line* in a polarized variety $(X, \mathcal{L})$ is a morphism $\mathbf{P}^1 \to X$ such that $\mathcal{L}$ pulls back to $\mathcal{O}_{\mathbf{P}^1}(1)$.

Theorem **1.1** *(See Corollary 13.2). — Let $f : X \to S$ be a morphism of nonsingular projective varieties over an algebraically closed field k of characteristic zero with S a surface. If*

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1. *there exists a Zariski open subset* U *of* S *whose complement has codimension* 2 *such that* X_u *is irreducible for* $u \in U(k)$,
2. *there exists an invertible sheaf* $\mathcal{L}$ *on* $f^{-1}(U)$ *which is* f*-relatively ample,*
3. *the geometric generic fibre* $(X_{\overline{\eta}}, \mathcal{L}_{\overline{\eta}})$ *of* f *is rationally simply connected by chains of free lines and has a very twisting scroll,*

then there exists a rational section of f.

The assumption on the irreducibility of fibres is likely superfluous and can be removed in most instances, see Remark 11.2. The existence of the invertible sheaf $\mathcal{L}$ *globally* is necessary, even in the presence of conditions (1) and (3). Namely, there exist morphisms $f : X \to S$ as in the theorem all of whose fibres are isomorphic to $\mathbf{P}^n_k$, and which have no rational sections (e.g., conic bundles or more generally Brauer-Severi schemes over S). The third condition is what makes the proof work. In Section 2 we will discuss this condition informally in more detail. Morally speaking however the third condition is an algebraic geometric way of saying that $X_{\overline{\eta}}$ is "rationally simply connected".

In order to prove the theorem above we analyze carefully the space of sections of the restriction of the family to a general nonsingular projective curve $C \subset S$. It turns out that the result of the following theorem is enough to imply the theorem above.

Theorem **1.2** *(See Theorem 13.1).* — *Let* $X \to C$ *be a morphism of nonsingular projective varieties over an algebraically closed field* k *of characteristic* 0 *with* C *a curve. Let* $\mathcal{L}$ *be an invertible sheaf on* X *which is ample on each fibre of* $X \to C$. *Assume the geometric generic fibre of* $X \to C$ *is rationally simply connected by chains of free lines and contains a very twisting scroll. In this case, for* $e \gg 0$ *there exists a canonically defined irreducible component* $Z_e \subset \mathit{Sections}^e(X/C/k)$ *so that the restriction of*

$$\alpha_{\mathcal{L}} : \mathit{Sections}^e(X/C/k) \longrightarrow \underline{\mathit{Pic}}^e_{C/k}$$

to Z_e *has rationally connected fibres.*

In this theorem the space $\mathrm{Sections}^e(X/C/k)$ parametrizes sections $\sigma : C \to X$ of degree e (with respect to $\mathcal{L}$) and the map $\alpha_{\mathcal{L}}$ assigns to a section σ the point of $\underline{\mathrm{Pic}}^e_{C/k}$ corresponding to the invertible sheaf $\sigma^*\mathcal{L}$. In Section 2 we will informally discuss how this theorem is proved and how it implies Theorem 1.1.

Main application. — In an email dated Jul 8, 2005 Phillipe Gille sketched out how a theorem as above for families of homogeneous varieties (or Borel varieties) over surfaces might lead to a proof of Serre's Conjecture II for function fields of surfaces. Our method reduces Serre's Conjecture II to studying sections of families of Borel varieties over curves. In the case of curves over finite fields *the same method* was used by G. Harder to prove Serre's conjecture II for function fields of curves over finite fields, see for example [Har71], the references therein and [Har75]. Of course the actual details differ substantially.

Lots of work has been done by many authors on Serre's conjecture II, including Merkurjev and Suslin, Bayer and Parimala, Chernousov and Gille. A nice summary of these results, as well as additional results, can be found in the paper [CTGP04]. Our approach is tailored to function fields of surfaces, is geometric and does not involve any Galois cohomology groups.

Theorem **1.3** *(See Theorem 16.6). — Let k be an algebraically closed field of any characteristic. Let S be a quasi-projective surface over k. Let $X \to S$ be a projective morphism. Let $\overline{\eta}$ be the spectrum of the algebraic closure of the function field $k(S)$. If $X_{\overline{\eta}}$ is of the form G/P for some linear algebraic group G and parabolic subgroup P and $Pic(X) \to Pic(X_{\overline{\eta}})$ is surjective, then $X \to S$ has a rational section.*

Note that this theorem holds in characteristic $p > 0$ and the base surface does not need to be projective. The reason is that there are tricks one can do to reduce to a case where Theorem 1.1 applies, see Remark 16.4 and Remark 16.7. In any case, this theorem implies the following special case of Serre's Conjecture II over function fields.

Theorem **1.4** *(Serre's Conjecture II over function fields for groups defined over the ground field). — Let k be an algebraically closed field of any characteristic and let K/k be the function field of a surface. Let G be a connected, semisimple, simply connected algebraic group over k. Every G-torsor over K is trivial.*

Proof. — Suppose K is the function field of the quasi-projective surface S over k. Let T be a G-torsor over K (see [Mil80, page 120]). After possibly shrinking S we may assume that T extends to a G-torsor $\mathcal{T}$ over S. Let B be a Borel subgroup of G. Consider the family

$$X = \mathcal{T}/B \to S.$$

Since G is simply connected any invertible sheaf $\mathcal{L}_0$ on G/B has a G-linearization. Thus the map from the character group of B to $\mathrm{Pic}(G/B)$ is surjective. Note that the geometric generic fibre $X_{\overline{\eta}}$ is isomorphic to $(G/B)_{\overline{\eta}}$ by construction. The character group of B also maps to $\mathrm{Pic}(X)$. Comparing these maps over η and $\overline{\eta}$ we conclude that $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{\eta}})$ is surjective. In other words, the assumptions of Theorem 1.3 are satisfied and we see that X has a K-rational point. This means that T has a reduction of structure to B. But as B is a connected solvable group over k any B-torsor over K is trivial. We conclude that T is trivial as desired. □

To finish the proof of Serre's conjecture we use some standard results on linear algebraic groups (see e.g. [Spr98]) and the results of [CTGP04]. Suppose G_K is a simple algebraic group over K of type E_8. Note that G_K is simply connected. Let G be a form of G_K defined over k. Note that $\mathrm{Aut}(G) \cong \mathrm{Inn}(G) \cong G$. We see that G_K as a form of G over K corresponds to an element of $H^1(K, G)$ and hence by Theorem 1.4 it is itself split.

Combining this and Theorem 1.4 with [CTGP04, Theorem 1.2(v)] (which deals with all other types) this completes the proof of Serre's Conjecture II for function fields.

Theorem **1.5** *(Serre's Conjecture II for function fields). — Let k be an algebraically closed field. Let $k \subset K$ be a finitely generated field extension of transcendence degree* 2*. For every connected, simply connected, semisimple algebraic group G_K over K, every G_K-torsor over K is trivial.*

Minor Application. — We would like to point out that Theorem 1.1 and Theorem 1.2 can be used to reprove Tsen's theorem for families of hypersurfaces over surfaces. Whereas the proof of Tsen's theorem can fit on a napkin, this paper has 60+ pages. However, the aim of such an investigation is to stimulate research into different generalizations of Tsen's theorem. For example, one possible avenue for research is to study families of low degree hypersurfaces in homogeneous spaces. In any case, we briefly explain in Section 17 below how to reprove Tsen's theorem using the main results of this paper.

2. Leitfaden

In this section we try to introduce the reader to the techniques and methods used in this paper. We will assume the reader has read an exposition of the proof of the main result in the paper [GHS03]. All of the discussion in this section is informal; for the details the reader has to consult the body of this work.

The first important conceptual step is that Theorem 1.2 implies Theorem 1.1. Let $X \to S$ be a fibration as in Theorem 1.1. After possibly blowing up the surface S we may assume there exists a morphism $g : S \to B$ to a curve B whose general fibre is a smooth projective curve C. Let $b \in B$ be a general point. Here is a picture

$$\begin{array}{ccccccc} f^{-1}(C) & = & Y & \xrightarrow{f_b} & C & \longrightarrow & b \\ & & \downarrow & & \downarrow & & \downarrow \\ & & X & \xrightarrow{f} & S & \xrightarrow{g} & B \end{array}$$

How do we construct a rational section of f? To do this it is certainly necessary that we can find a rational section σ_b of f_b for a general b. Moreover, if we can find an algebraic family $\{\sigma_b\}_{b\in U}$ of sections for some $U \subset B$ nonempty open, then we have our rational section σ of f simply by setting $\sigma(s) = \sigma_{g(s)}(s)$. Let Sections$(X/S/B)$ denote the moduli space whose points correspond to pairs (b, σ_b) with $b \in B$ and σ_b a section of f_b as above. In particular, there is a canonical map Sections$(X/S/B) \to B$. The phrase "algebraic family" above means we want to find a (rational) section of this morphism Sections$(X/S/B) \to B$. Now, the main result of [GHS03] tells us that such a section exists, as soon as we can find a variety $Z \subset$ Sections$(X/S/B)$ such that the fibres of $Z \to B$ are rationally connected. For more details, please see the proof of Corollary 13.2.

Let k be an algebraically closed field, and let Y be a smooth projective variety over k. Let $\mathcal{L}$ be an ample invertible sheaf on Y. A *free line* on Y is a morphism $\varphi : \mathbf{P}^1 \to Y$ such that $\deg(\varphi^*\mathcal{L}) = 1$, and φ^*T_Y is globally generated. A *chain of free lines* is a morphism $C \to Y$, where C is a nodal curve of genus zero all of whose components are free lines and whose dual graph is a chain. We say Y is *rationally simply connected by chains of free lines* if

(1) the space of lines through a general point of Y is nonempty, irreducible, and rationally connected, and
(2) there exists an $n > 0$ such that for a general pair of points on Y the moduli space of length n chains of free lines connecting them is nonempty, irreducible, and birationally rationally connected.

A more precise formulation is that Y/k should satisfy Hypothesis 7.8. This explains the first of the two conditions of Theorem 1.1 (3).

Let $f : X \to C$ be a morphism of projective smooth varieties, where C is a curve. Let $\mathcal{L}$ be an ample invertible sheaf on X. In this situation we have the moduli stacks $\Sigma^e(X/C/k)$ whose points classify stable sections $h : C' \to X$ of degree e. This means that $C' = C \cup \bigcup_{i=1,\dots,s} C_i$, is a connected nodal curve, $h|_C : C \to X$ is a section of f, and the maps $h|_{C_i} : C_i \to X$ are 1-pointed stable maps from nodal rational curves into pairwise distinct fibres of f. Finally, the degree of the pullback of $\mathcal{L}$ to C has degree e. See Definition 6.2 for a more precise and more general definition. In Sections 7, 8, 9 we study these spaces of sections of $f : X \to C$ if the general fibre of f is rationally simply connected by chains of free lines. In some sense the main result is Corollary 9.8. It says that given two free sections σ, σ' of f, you can attach free lines in fibres to σ and σ' such that the resulting points p, p' of $\Sigma^e(X/C/k)$ are connected by a chain of rational curves in $\Sigma^e(X/C/k)$ whose nodes map to non-stacky, smooth points of $\Sigma^e(X/C/k)$.

This important intermediate result is the first hint that something like Theorem 1.2 holds. However, if $C = \mathbf{P}^1$ and $X \to C$ is a general pencil of cubic hypersurfaces in $\mathbf{P}^8$, then the moduli spaces of sections are not rationally connected, even though a general fibre is rationally simply connected by chains of free lines. The reason is (reversing arguments above) that a general net $X \to \mathbf{P}^2$ of cubics in $\mathbf{P}^8$ does not have a rational section! Hence a further condition on the general fibre is necessary.

Let k be an algebraically closed field, and let Y be a smooth projective variety over k. Let $\mathcal{L}$ be an ample invertible sheaf on Y. Loosely, a *very twisting scroll* on Y is given by a surface $R \subset Y$ which is ruled by free lines in $(Y, \mathcal{L})$ over $\mathbf{P}^1$, together with a section $\sigma : \mathbf{P}^1 \to R$ of the ruling with $\sigma^2 \geq 0$, such that the induced morphism $g : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$ is sufficiently free. (Recall here that $\overline{\mathcal{M}}_{g,n}(Y, e)$ is the Kontsevich moduli space of degree e stable maps from nodal n-pointed genus g curves into Y.) Namely, the pullback by g of T_{ev} twisted down by 2 should have vanishing H^1. Here T_{ev} is the relative tangent bundle of the map $ev : \overline{\mathcal{M}}_{0,1}(Y, 1) \to Y$. The formal definition is Definition 12.7 and the reformulation in terms of properties of g is Lemma 12.6. This explains the second of the two conditions of Theorem 1.1 (3).

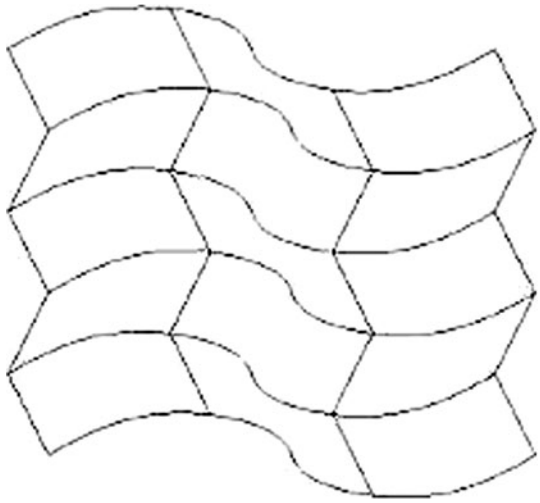

FIG. 1. — Connecting sections by scrolls

How are these assumptions used in the proof of Theorem 1.2? Let us return to the situation $f : X \to C$, $\mathcal{L}$ of three paragraphs ago, assuming now that a general fibre of f is rationally simply connected by chains of free lines and has a very twisting scroll. Let σ, σ' be two free sections of f. Let n be sufficiently large and consider the space Ch of length n chains of free lines in fibres X_c of f connecting $\sigma(c)$ to $\sigma(c')$. By the first assumption the fibres of the natural map $Ch \to C$ are rationally connected. Hence applying [GHS03] we obtain a rational section of $Ch \to C$. This means we have a sequence of surfaces $R_1, \ldots, R_n \subset X$ ruled by lines over C such that for each $c \in C$ we obtain a chain of lines connecting $\sigma(c)$ to $\sigma'(c)$. See Figure 1 and Lemma 9.6. What we want to do now is move the section σ in the surface R_1 so it becomes the common section of R_1 and R_2, etc, finally ending up with the section σ'. It turns out that this is possible on adding lines in fibres, see Proposition 9.7.

Now the problem with this result is that it only connects points in the boundary of $\Sigma^e(X/C/k)$ by chains of rational curves. Hence to prove Theorem 1.2 we need to find a way to connect points in the interior of $\Sigma^e(X/C/k)$ to points in the boundary. And this is where very twisting scrolls come in. Namely, we first show that if one has very twisting scrolls in fibres, then one has very twisting surfaces ruled by lines in X/C, see Lemma 12.9. Next, we show that this implies that "many" sections lie (as the defining section) on a very twisting surface, see Proposition 12.12. Finally, we show that if the section σ is a positive section of a very twisting surface, then it sits in a rational 1-parameter family which hits the boundary, see Lemma 12.5 and Corollary 12.13. These ideas form the backbone of the somewhat complicated proof of Theorem 1.2 (see Theorem 13.1).

3. Stacks of curves and maps from curves to a target

In this section we introduce the stack *Curves* classifying curves and the stack *CurveMaps*$(\mathcal{X}/\mathrm{S})$ classifying maps from curves to an algebraic stack $\mathcal{X}$. We recall that these are algebraic stacks. Given a smooth Z/S we construct a determinant morphism *CurveMaps*$_{\mathrm{LCI}}(\mathrm{Z} \times \mathrm{B}\mathbf{GL}_n/\mathrm{S}) \to \underline{\mathrm{Pic}}_{\mathrm{Z/S}}$.

The correct notion to use when doing moduli of algebraic varieties is to use families of varieties where the total space is an algebraic space – not necessarily a scheme. Even in the case of families of curves it can happen that the total space is not locally a scheme over the base. This leads to the following definitions.

Definition **3.1.** — *Let* S *be a scheme.*

1. *A* flat family of proper curves over S *is a morphism of algebraic spaces* $\pi : \mathrm{C} \to \mathrm{S}$ *which is proper, locally finitely presented, and flat of relative dimension* 1.[1]
2. *A* flat family of polarized, proper curves over S *is a pair* $(\pi : \mathrm{C} \to \mathrm{S}, \mathcal{L})$ *consisting of a flat family of proper curves over* S *and a* π*-ample invertible sheaf* $\mathcal{L}$ *on* C.
3. *For every morphism of schemes* $u : \mathrm{T} \to \mathrm{S}$ *and* $\pi : \mathrm{C} \to \mathrm{S}$ *as above the* pullback family *is the projection*
$$u^*\pi = \mathrm{pr}_{\mathrm{T}} : \mathrm{T} \times_{u,\mathrm{S},\pi} \mathrm{C} \to \mathrm{T}.$$
If the family is polarized by $\mathcal{L}$ *then the pullback family is polarized by the* $u^*\pi$*-ample invertible sheaf* $(u, \mathrm{Id}_{\mathrm{C}})^*\mathcal{L}$.
4. *For two flat families of proper curves* $\pi : \mathrm{C} \to \mathrm{S}$ *and* $\pi' : \mathrm{C}' \to \mathrm{S}'$, *a* morphism f *from the first family to the second is given by a Cartesian diagram*
$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{f} & \mathrm{C}' \\ \downarrow & & \downarrow \\ \mathrm{S} & \longrightarrow & \mathrm{S}'. \end{array}$$
If $\pi : \mathrm{C} \to \mathrm{S}$ *is polarized by* $\mathcal{L}$ *and* $\pi' : \mathrm{C}' \to \mathrm{S}'$ *is polarized by* $\mathcal{L}'$ *then a* morphism *between the polarized families is a pair* (f, ϕ) *with* f *as above and* $\phi : \mathcal{L} \to f^*\mathcal{L}'$ *is an isomorphism.*

Definition **3.2.** — *The* stack of proper curves *is the fibred category*
$$\mathit{Curves} \longrightarrow \mathit{Schemes}$$
whose objects are families of flat proper curves $\pi : \mathrm{C} \to \mathrm{S}$ *and morphisms are as above. The functor* *Curves* $\to$ *Schemes* *is the forgetful functor. Similarly the* stack of proper, polarized curves *is the*

[1] All fibres pure dimension 1.

fibred category

$$\mathit{Curves}_{\mathrm{pol}} \longrightarrow \mathit{Schemes}$$

whose objects are flat families of proper, polarized curves $(\pi : C \to S, \mathcal{L})$ *as in the definition above.*

Denote by (Aff) the category of affine schemes. A technical remark is that in [LMB00] stacks are defined as fibred categories over (Aff). So in the following propositions we state our results in a manner that is compatible with their notation. In particular when we speak of *Curves* over (Aff) we mean the restriction of the fibred category *Curves* to (Aff) $\subset$ *Schemes*.

There is a functor of fibred categories

$$F : \mathit{Curves}_{\mathrm{pol}} \to \mathit{Curves}$$

sending each object $(\pi : C \to S, \mathcal{L})$ to the object $(\pi : C \to S)$, i.e., forgetting about the invertible sheaves.

Proposition **3.3.** — *The categories* $\mathit{Curves}_{\mathrm{pol}}$ *and Curves are limit preserving algebraic stacks over (Aff) (with the fppf topology) with finitely presented, separated diagonals. Moreover* F *is a smooth, surjective morphism of algebraic stacks.*

Proof. — This is folklore; here is one proof. The category $\mathit{Curves}_{\mathrm{pol}}$ is a limit preserving algebraic stack with finitely presented, separated diagonal by [Sta06, Proposition 4.2]. Moreover, by [Sta06, Propositions 3.2 and 3.3], *Curves* is a limit preserving stack with representable, separated, locally finitely presented diagonal. To prove that *Curves* is an algebraic stack, it only remains to find a smooth cover of *Curves*, i.e., a smooth, essentially surjective 1-morphism from an algebraic space to *Curves*. Let $u : X \to \mathit{Curves}_{\mathrm{pol}}$ be such a smooth cover for the stack of polarized curves. We claim that $F \circ u : X \to \mathit{Curves}$ is a smooth cover.

Let $(S, \pi : C \to S)$ be an object of *Curves*. The 2-fibred product of F with the associated 1-morphism $S \to \mathit{Curves}$ is the stack $\mathcal{A}$ of π-ample invertible sheaves on C. Consider the diagram

$$\begin{array}{ccc} \mathcal{A} \times_{\mathit{Curves}_{\mathrm{pol}}} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathit{Curves}_{\mathrm{pol}} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathit{Curves}. \end{array}$$

Once we have shown $\mathcal{A}$ is an algebraic stack and $\mathcal{A} \to S$ is smooth surjective, then the claim above follows, as well as the smoothness and surjectivity of F in the proposition.

Before we continue we remark that $\mathcal{A}$ is an open substack of the Picard stack of $C \to S$ by [Gro63, Théorème 4.7.1]. Hence, by [Sta06, Proposition 4.1] (which is a variant of [LMB00, Théorème 4.6.2.1]) the stack $\mathcal{A}$ is a limit preserving algebraic stack with quasi-compact, separated diagonal. In particular, for every smooth cover $A \to \mathcal{A}$, the induced morphism $A \to S$ is locally finitely presented. Thus to prove $A \to S$ is smooth, and thus that $\mathcal{A}$ is smooth over S, it suffices to prove that $A \to S$ is formally smooth. Let s be a point of S and denote by C_s the fibre $\pi^{-1}(s)$. By [Ill71, Proposition 3.1.5], the obstructions to infinitesimal extensions of invertible sheaves on C_s live in $\mathrm{Ext}^2_{\mathcal{O}_{C_s}}(\mathcal{O}_{C_s}, \mathcal{O}_{C_s}) = H^2(C_s, \mathcal{O}_{C_s})$. Since C_s is a Noetherian, 1-dimensional scheme, $H^2(C_s, \mathcal{O}_{C_s})$ is zero. Thus there are no obstructions to infinitesimal extensions of π-ample invertible sheaves on C_s. Thus $\mathcal{A} \to S$ is smooth as desired.

Next, we prove that $\mathcal{A} \to S$ is surjective. Thus, let s be a point of S. It suffices to prove there is an ample invertible sheaf on C_s. This is essentially [Har77, Exercise III.5.8]. Therefore *Curves* is an algebraic stack.

It only remains to prove that the diagonal of *Curves* is quasi-compact. Let $(\pi_i : C_i \to S)$, $i = 1, 2$ be two families of proper flat curves over an affine scheme S. We have to show that the algebraic space

$$\mathrm{Isom}_S(C_1, C_2)$$

is quasi-compact. Using that F is smooth and surjective, after replacing S by a quasi-compact, smooth cover, we may assume there exist π_i-ample invertible sheaves $\mathcal{L}_i$ on C_i. We may assume the Hilbert polynomial of the fibres of $C_i \to S$ with respect to $\mathcal{L}_i$ are constant, say P_i. For every S-scheme T and every T-isomorphism $\phi : C_{1,T} \to C_{2,T}$, the graph $\Gamma_\phi : C_{1,T} \to (C_1 \times_S C_2)_T$ is a closed immersion. In the usual way this identifies $\mathrm{Isom}_S(C_1, C_2)$ as a locally closed subscheme of the Hilbert scheme of $C_1 \times_S C_2$ over S. We will show that whenever $T = \mathrm{Spec}(k)$ is the spectrum of a field, the Hilbert polynomial of Γ_ϕ with respect to the S-ample invertible sheaf $\mathrm{pr}_1^*\mathcal{L}_1 \otimes \mathrm{pr}_2^*\mathcal{L}_2$ is equal to $P_1 + P_2 - P_1(0)$. Quasi-compactness follows from the projectivity of the Hilbert scheme parametrizing closed subschemes with given Hilbert polynomial. By Snapper's theorem (see [Kle66]) the values of the Hilbert polynomial $\chi(C_{1,T}, \mathcal{L}_{1,T}^{\otimes n} \otimes \phi^*\mathcal{L}_{2,T}^{\otimes n})$ equal $\chi(C_{1,T}, \mathcal{L}_{1,T}^{\otimes n})) + \chi(C_{1,T}, \phi^*\mathcal{L}_{2,T}^{\otimes n}) - \chi(C_{1,T}, \mathcal{O})$. And of course $\chi(C_{1,T}, \phi^*\mathcal{L}_{2,T}^{\otimes n}) = \chi(C_{2,T}, \mathcal{L}_{2,T}^{\otimes n})$. The claim follows. ☐

At this point we need some definitions related to moduli of maps from curves into varieties. But since we are going to study the rational connectivity of moduli stacks of rational curves on varieties, we need to deal with moduli of morphisms from curves to stacks.

Definition **3.4.** — *Let $\mathcal{X}$ be an algebraic stack, let* S *be a scheme and let* $\mathcal{X} \to S$ *be a morphism. A* family of maps of proper curves to $\mathcal{X}$ over S *is a triple* $(T \to S, \pi : C \to T, \zeta : C \to \mathcal{X})$ *consisting of a flat family of proper curves over the* S*-scheme* T *and a* 1*-morphism* ζ *over* S.

We leave it to the reader to define morphisms between families of maps of proper curves to $\mathcal{X}$ over S.

Definition **3.5.** — *The* stack of maps of proper curves to $\mathcal{X}$ *is the fibred category*

$$\mathit{CurveMaps}(\mathcal{X}/\mathrm{S}) \longrightarrow \mathit{Schemes}/\mathrm{S}$$

whose objects are families of maps of proper curves to $\mathcal{X}$ *over* S*, and morphisms as above.*

We again have the technical remark that according to the conventions in [LMB00] we should really be working with the restriction of $\mathit{CurveMaps}(\mathcal{X}/\mathrm{S})$ to the category $(\mathrm{Aff}_{\mathrm{S}})$ of affine schemes over S. Denote by $\mathit{Curves}_{\mathrm{S}}$ the restriction of *Curves* over the category *Schemes*/S of S-schemes. There is an obvious functor

$$\mathrm{G} : \mathit{CurveMaps}(\mathcal{X}/\mathrm{S}) \longrightarrow \mathit{Curves}_{\mathrm{S}} \qquad (\mathbf{3.1})$$

forgetting the 1-morphisms to $\mathcal{X}$.

Proposition **3.6** *(See Lie06, Proposition 2.11). — Assume* S *excellent. Suppose that* $\mathcal{X} = [\mathrm{Z}/\mathrm{G}]$ *where* Z *is an algebraic space,* $\mathrm{Z} \to \mathrm{S}$ *is separated and of finite presentation, and where* G *is an* S*-flat linear algebraic group scheme. The stack* $\mathit{CurveMaps}(\mathcal{X}/\mathrm{S})$ *is a limit preserving algebraic stack over* $(\mathit{Aff}_{\mathrm{S}})$ *with locally finitely presented, separated diagonal. And the* 1*-morphism (3.1) is representable by limit preserving algebraic stacks.*

Proof. — The second assertion is precisely proved in [Lie06, Proposition 2.11] and implies the first assertion because of Proposition 3.3. ☐

Definition **3.7.** — *Let* S *be a scheme. A flat family of proper curves over* S*,* $\pi : \mathrm{C} \to \mathrm{S}$*, is* LCI *if* π *is a local complete intersection morphism in the sense of [Gro67, Définition 19.3.6].*

It is true that this is equivalent to [BGI71, Définition VIII.1.1]. See [BGI71, Proposition VIII.1.4].

Proposition **3.8.** — *The subcategory* $\mathit{Curves}_{\mathrm{LCI}}$ *of Curves of flat families which are* LCI *is an open substack of Curves.*

Proof. — Let $\pi : \mathrm{C} \to \mathrm{S}$ be a flat family of proper curves, and assume for the moment that C is a scheme. According to [Gro67, Corollaire 19.3.8] there exists an open subscheme $\mathrm{U} \subset \mathrm{S}$ such that C_s is LCI if and only if $s \in \mathrm{U}$. By definition [Gro67, Définition 19.3.6] the induced morphism $\mathrm{C}_{\mathrm{U}} \to \mathrm{U}$ is LCI. It follows from [Gro67, Proposition 19.3.9] that given a morphism of schemes $g : \mathrm{T} \to \mathrm{S}$ then the inverse image $g^{-1}(\mathrm{U}) \subset \mathrm{T}$ has the same property for the family over T.

In the general case (where we do not assume that C is a scheme) we use that there is a smooth cover $\mathrm{S}' \to \mathrm{S}$ such that the pullback family over S' has a total space which is a scheme (for example by using 3.3). Hence we obtain an open $\mathrm{U}' \subset \mathrm{S}'$ characterized as above. Clearly $U' \times_S S' = S' \times_S U'$ and we conclude that U' is the inverse image of an open U of S. Again by definition the restriction $C_U \to U$ is LCI, and all fibres of $C \to S$ at points of $S \setminus U$ are not LCI. Clearly this shows that any base change by $T \to S$ is LCI if and only if $T \to S$ factors through U. □

Definition **3.9.** — *Let $\mathcal{X}$ be an algebraic stack, let S be a scheme and let $\mathcal{X} \to S$ be a morphism. The* stack of maps of proper, LCI curves to $\mathcal{X}$ *is the open substack $CurveMaps_{LCI}(\mathcal{X}/S)$ of $CurveMaps(\mathcal{X}/S)$ obtained as the 2-fibred product of $CurveMaps(\mathcal{X}/S) \to Curves$ and $Curves_{LCI} \to Curves$.*

In order to define the determinant pushforward below we need the following lemma.

Lemma **3.10.** — *Suppose we are given a diagram of morphisms of algebraic spaces*

$$\begin{array}{ccc} C & \longrightarrow & Z \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $C \to T$ is an LCI, flat proper family of curves, and $Z \to S$ is a smooth morphism. Then the induced morphism $C \to T \times_S Z$ is a local complete intersection morphism in the sense of [BGI71, Définition VIII.1.1]. In particular, it is a perfect morphism *in the sense of [BGI71, Définition III.4.1].*

Proof. — Note that $T \times_S Z$ is smooth over T, and hence it suffices to prove the lemma in the case that $S = T$. By [BGI71, Proposition VIII.1.6] it suffices to prove the lemma after a faithfully flat base change. Hence it is also sufficient to prove the proposition in the case that $C \to S$ is projective, by 3.3. Choose a closed immersion $C \to \mathbf{P}^N_S$ over S. This is regular by [BGI71, Proposition VIII.1.2]. According to [BGI71, Corollaire VIII.1.3] this implies that $C \to \mathbf{P}^N_S \times_S Z$ is regular, which in turn by definition implies that $C \to Z$ is a local complete intersection morphism. □

We are going to use the stack $CurveMaps(\mathcal{X}/S)$ in the case where $\mathcal{X} = Z \times B\mathbf{GL}_n$ for some algebraic space Z separated and of finite presentation over S. Note that by Proposition 3.6 this is a limit preserving algebraic stack. An object of this stack is given by a datum $(T \to S, \pi : C \to T, \zeta : C \to Z, \mathcal{E})$ where $(T \to S, \pi : C \to T, \zeta : C \to Z)$ is a family of maps of proper curves to Z over S and where $\mathcal{E}$ is a locally free sheaf of rank n over C. It is straightforward to spell out what the morphisms are in this stack (left to the reader). The object $(T \to S, \pi : C \to T, \zeta : C \to Z, \mathcal{E})$ belongs to $CurveMaps_{LCI}(\mathcal{X}/S)$ if and only if π is LCI.

Assuming that S is excellent, Z is a scheme, $Z \to S$ is quasi-compact and smooth and $\mathcal{X} = Z \times B\mathbf{GL}_n$ we define a functor

$$\det(R(-)_*) : CurveMaps_{LCI}(\mathcal{X}/S) \to \mathrm{Hom}_S(Z, B\mathbf{G}_m) = Pic_{Z/S}$$

to the Picard stack of Z over S. First we note that, by our discussion above both sides are limit preserving algebraic stacks over S. Thus we need only define the functor on fibre categories over schemes of finite type over S. Consider an object $(T \to S, \pi : C \to T, \zeta : C \to Z, \mathcal{E})$ of the left hand side with T/S finite type. The morphism $\zeta : C \to T \times_S Z$ is a perfect morphism by Lemma 3.10. Since ζ is perfect $R\zeta_*\mathcal{E}$ is a perfect complex of bounded amplitude on $T \times_S Z$ by [BGI71, Corollaire III.4.8.1]. (We leave it to the reader to check that we have put in enough finiteness assumptions so the Corollary applies.) The "det" construction of [KM76] associates an invertible sheaf $\det(R\zeta_*\mathcal{E})$ to the perfect complex $R\zeta_*\mathcal{E}$ on the scheme $T \times_S Z$. Also, by [KM76, Definition 4(iii)] and [BGI71, Proposition IV.3.1.0], formation of $\det(R\zeta_*\mathcal{E})$ is compatible with morphisms of objects of $\mathit{CurveMaps}_{\mathrm{LCI}}(\mathcal{X}/S)$.

Definition **3.11.** — *Given* S, $Z \to S$ *and* n *as above. The* determinant pushforward 1-morphism *is the functor*

$$\mathit{det}(R(-)_*) : \mathit{CurveMaps}_{\mathrm{LCI}}(Z \times B\mathbf{GL}_n/S) \to \mathit{Pic}_{Z/S}$$

defined above. If the Picard stack $\mathit{Pic}_{Z/S}$ *has a coarse moduli space* $\underline{\mathit{Pic}}_{Z/S}$ *then the composite morphism*

$$\mathit{CurveMaps}_{\mathrm{LCI}}(Z \times B\mathbf{GL}_n/S) \to \underline{\mathit{Pic}}_{Z/S}$$

will also be called the determinant pushforward 1-morphism.

The following is an important special case for our paper. Namely, suppose that $S = \mathrm{Spec}(\kappa)$ is the spectrum of a field κ, and suppose that $Z = C$ is a smooth, projective, geometrically connected κ-curve of genus g. In this case we can "compute" the value of the determinant 1-morphism on some special points. Assume that we have a $\mathrm{Spec}(\kappa)$-valued point of $\mathit{CurveMaps}_{\mathrm{LCI}}(C \times B\mathbf{GL}_n/S)$ given by a datum $(\mathrm{Spec}(\kappa) \to \mathrm{Spec}(\kappa), \pi : C' \to \mathrm{Spec}(\kappa), \zeta : C' \to C, \mathcal{E})$. Assume furthermore that C' is proper, at worst nodal, geometrically connected of genus g. Finally, assume that ζ is an isomorphism over a dense open of C'. For every such map, there exists a unique section $s : C \to C'$ whose image is the unique irreducible component of C' mapping dominantly to C. The scheme $\overline{C' - s(C)}$ is a disjoint union of trees of rational curves $C_1, \dots, C_\delta$. Each C_i meets $s(C)$ in a single point t_i.

Lemma **3.12.** — *Assumptions and notation as above. There is an isomorphism*

$$\mathit{det}(R\zeta_*(\mathcal{E})) = \mathit{det}(s^*\mathcal{E})(d_1 \cdot t_1 + \cdots + d_\delta \cdot t_\delta)$$

where d_i is the degree of the base change of $\mathcal{E}$ to the connected nodal curve C_i over the field $\kappa(t_i)$.

Proof. — For a comb $C' = s(C) \cup C_1 \cup \cdots \cup C_\delta$ as above, there is a surjection of $\mathcal{O}_C$-modules $\mathcal{E} \to \mathcal{E}|_{s(C)}$ whose kernel we will write as $\oplus \mathcal{E}|_{C_i}(-t_i)$. We will use later on that $\chi(C_i, \mathcal{E}|_{C_i}(-t_i)) = d_i$ by Riemann-Roch on C_i over $\kappa(t_i)$. Pushing this forward by ζ gives a triangle of perfect complexes:

$$\longrightarrow R\zeta_*(\oplus\mathcal{E}|_{C_i}(-t_i)) \longrightarrow R\zeta_*(\mathcal{E}) \xrightarrow{\lambda} R\zeta_*(\mathcal{E}|_{s(C)}) \longrightarrow$$

Note that $R\zeta_*(\mathcal{E}|_{s(C)}) = s^*(\mathcal{E})[0]$, and that the term $R\zeta_*(\oplus\mathcal{E}|_{C_i}(-t_i))$ is supported in the points t_i. Hence λ is "good" and we may apply the "Div" construction of [KM76]. It follows that $\det(R\zeta_*(\mathcal{E})) \cong \det(s^*\mathcal{E})(-\mathrm{Div}(\lambda))$. Using [KM76, Theorem 3(iii) and (vi)], it follows that $\mathrm{Div}(\lambda) = -\sum d_i \cdot t_i$ (minus sign because the complex $\mathcal{H}^{\cdot}$ of locus citatus is our complex $\zeta_*(\oplus\mathcal{E}|_{C_i}(-t_i))$ shifted by 1). □

We end this section with a simple semi-continuity lemma.

Lemma **3.13.** — *Let* S *be an affine scheme. Let* $\pi : C \to S$ *be a flat family of proper curves over* S. *Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* C *which is locally finitely presented and flat over* S. *For every integer* $i \geq 0$ *there exists an open subscheme* U_i *of* S *such that for every* S*-scheme* T, *the derived pushforward to* T *of the pullback of* $\mathcal{F}$ *to* $T \times_S C$ *is concentrated in degrees* $\leq i$ *if and only if* T *factors through* U_i. *Moreover, after base change from* S *to* U_i, *the formation of* $R^i\pi_*(\mathcal{F})$ *is compatible with arbitrary base change.*

This also holds when S *is an arbitrary scheme, algebraic space, etc., instead of an affine scheme.*

Proof. — This is probably true in some vast generality for morphisms of algebraic stacks, but we do not know a reference. In our case we may deduce it from the schemes case as follows. By Proposition 3.3, there exists a faithfully flat morphism of affine schemes $S' \to S$ such that $S' \times_S C$ is projective over S'. By [LMB00, Proposition 13.1.9], the statement for the original family over S follows from the statement over S'. Also by limit arguments, it suffices to consider the case when S' is of finite type. Now the result follows from [Gro63, Section 7] or [Mum70, Section 5]. The general case follows from the affine case by [LMB00, Proposition 13.1.9]. □

4. Free sections

This section introduces the stack $\mathrm{Sections}(\mathcal{X}/\mathcal{C}/S)$ of sections of a morphism $f : \mathcal{X} \to \mathcal{C}$ of algebraic stacks over a base S. Recent work on Hom stacks allows us to show $\mathrm{Sections}(\mathcal{X}/\mathcal{C}/S)$ is algebraic in many cases. Especially important are free sections, which are also defined and discussed in this section. Proposition 4.15 gives a geometric criterion for when sections are free, and Lemma 4.17 proves free sections lift along a rationally connected fibration.

Let S be an algebraic space. Let $\mathcal{C} \to S$ be a proper, flat, finitely presented algebraic stack over S. Let $f : \mathcal{X} \to \mathcal{C}$ be a 1-morphism of algebraic S-stacks.

Definition **4.1.** — *With notation as above. Let* T *be an* S*-scheme, or an algebraic space over* S*.*

1. *A* family of sections of f over T *is a pair* (τ, θ) *consisting of a* 1*-morphism of* S*-stacks* $\tau : \mathrm{T} \times_{\mathrm{S}} \mathcal{C} \to \mathcal{X}$ *together with a* 2*-morphism* $\theta : f \circ \tau \Rightarrow \mathrm{pr}_{\mathcal{C}}$*, giving a* 2*-commutative diagram*

$$\begin{array}{ccc} & & \mathcal{X} \\ & \overset{\tau}{\nearrow} & \downarrow f \\ \mathrm{T} \times_{\mathrm{S}} \mathcal{C} & \overset{f \circ \tau}{\underset{\mathrm{pr}_{\mathcal{C}}}{\rightrightarrows}} \Downarrow \theta & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{T} & \longrightarrow & \mathrm{S} \end{array}$$

2. *For two families of sections of* f*, say* $(\mathrm{T}', \tau', \theta')$ *and* $(\mathrm{T}, \tau, \theta)$*, a* morphism (u, η) *from the first family to the second family is given by a morphism* $u : \mathrm{T}' \to \mathrm{T}$ *and a* 2*-morphism* $\eta : \tau \circ (u \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \tau'$ *such that*

$$\begin{array}{ccc} f \circ \tau' & \xrightarrow{\eta} & f \circ \tau \circ (u \times \mathrm{id}_{\mathcal{C}}) \\ \downarrow \theta' & & \downarrow \theta \\ \mathrm{pr}_{\mathcal{C}}^{\mathrm{T}' \times_{\mathrm{S}} \mathcal{C}} & \xrightarrow{=} & \mathrm{pr}_{\mathcal{C}}^{\mathrm{T} \times_{\mathrm{S}} \mathcal{C}} \circ (u \times \mathrm{id}_{\mathcal{C}}) \end{array}$$

commutes.

Definition **4.2.** — *The* stack of sections of f *is the fibred category*

$$\mathit{Sections}(\mathcal{X}/\mathcal{C}/\mathrm{S}) \longrightarrow \mathit{Schemes}/\mathrm{S}$$

whose objects are family of sections of f *and whose morphisms are morphisms of families of sections of* F*.*

We recall some results from the literature.

Theorem **4.3.** — *Let* $\mathcal{C} \to \mathrm{S}, f : \mathcal{X} \to \mathcal{C}$ *be as above.*

1. *See [Gro62, Part IV.4.c, p. 221–219]. Assume that* $\mathrm{C} = \mathcal{C}$ *is a projective scheme over* S *and that* $\mathrm{X} = \mathcal{X}$ *is a quasi-projective scheme over* C *either globally over* S*, resp. fppf locally over* S*. Then* $\mathit{Sections}(\mathrm{X}/\mathrm{C}/\mathrm{S})$ *is an algebraic space which is locally finitely presented and separated over* S*. Moreover, globally over* S*, resp. fppf locally over* S*, the connected components of* $\mathit{Sections}(\mathrm{X}/\mathrm{C}/\mathrm{S})$ *are quasi-projective over* S*.*
2. *See [Ols06, Theorem 1.1]. Assume that* $\mathcal{C}$ *and* $\mathcal{X}$ *are separated over* S *and have finite diagonal. Also assume that fppf locally over* S *there is a finite, finitely presented cover of* $\mathcal{C}$ *by an algebraic space. Then* $\mathit{Sections}(\mathcal{X}/\mathcal{C}/\mathrm{S})$ *is a limit preserving algebraic stack over* S *with separated, quasi-compact diagonal.*

3. *See [Lie06, Proposition 2.11]. Assume* S *excellent and* $\mathcal{C} = \mathrm{C}$ *is a proper flat family of curves. Assume that, fppf locally over* S*, we can write* $\mathcal{X} = [\mathrm{Z}/\mathrm{G}]$ *for some algebraic space* Z *separated and of finite type over* S *and some linear group scheme* G *flat over* S*. Then* *Sections*$(\mathcal{X}/\mathrm{C}/\mathrm{S})$ *is a limit preserving algebraic stack over* S.

Proof. — Besides the references, here are some comments. For the first case we may realize the space of sections as an open subscheme of Hilbert scheme of $\mathrm{C} \times_{\mathrm{S}} \mathrm{X}$ over S. In the second and third case we think of Sections$(\mathcal{X}/\mathcal{C}/\mathrm{S})$ as a substack of the Hom-stack $\mathrm{Hom}_{\mathrm{S}}(\mathcal{C}, \mathcal{X})$. The references guarantee the existence of the Hom-stack as an algebraic stack. Next we consider the 2-Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sections}(\mathcal{X}/\mathcal{C}/\mathrm{S}) & \longrightarrow & \mathrm{Hom}_{\mathrm{S}}(\mathcal{C}, \mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{S} & \longrightarrow & \mathrm{Hom}_{\mathrm{S}}(\mathcal{C}, \mathcal{C}) \end{array}$$

In cases 2 and 3 the lower horizontal arrow is representable by the results from [Ols06] and [Lie06] as the target is an algebraic stack. It follows that Sections$(\mathcal{X}/\mathcal{C}/\mathrm{S})$ is an algebraic stack. □

Of course we will most often use the space of sections for a family of varieties over a curve. We formulate a set of hypothesis that is convenient for the developments later on.

Hypothesis **4.4.** — *Let* κ *be a field. Let* C *be a projective, smooth, geometrically connected curve over* κ*. Let* $f : \mathrm{X} \to \mathrm{C}$ *be a quasi-projective morphism. Let* $\mathcal{L}$ *be an* f*-ample invertible sheaf on* X*. Denote the smooth locus of* X *by* $\mathrm{X}_{\mathrm{smooth}}$*. Denote the open subset of* $\mathrm{X}_{\mathrm{smooth}}$ *on which* f *is smooth by* $\mathrm{X}_{f,\mathrm{smooth}}$.

According to the theorem above the space of sections Sections$(\mathrm{X}/\mathrm{C}/\kappa)$ is a union of quasi-projective schemes over κ in this case. The locus Sections$(\mathrm{X}_{f,\mathrm{smooth}}/\mathrm{C}/\kappa)$ is an open subscheme of Sections$(\mathrm{X}/\mathrm{C}/\kappa)$.

Definition **4.5.** — *In Situation 4.4. For every integer e, the* scheme of degree e sections of f *is the open and closed subscheme* $\mathit{Sections}^e(\mathrm{X}/\mathrm{C}/\kappa)$ *of* $\mathit{Sections}(\mathrm{X}/\mathrm{C}/\kappa)$ *parametrizing sections* τ *of* f *which pullback* $\mathcal{L}$ *to a degree e invertible sheaf on* C*. The* universal degree e section of f *is denoted*

$$\sigma : \mathit{Sections}^e(\mathrm{X}/\mathrm{C}/\kappa) \times_\kappa \mathrm{C} \longrightarrow \mathrm{X}.$$

The Abel map *is the* κ*-morphism*

$$\alpha : \mathit{Sections}^e(\mathrm{X}/\mathrm{C}/\kappa) \longrightarrow \underline{\mathit{Pic}}^e_{\mathrm{C}/\kappa}$$

associated to the invertible sheaf $\sigma^*\mathcal{L}$ *by the universal property of the Picard scheme* $\underline{Pic}^e_{C/\kappa}$.

There are several variations of the notion of "free curve" as described in [Kol96, Definition II.3.1]. Of the following two notions, the weaker notion arises more often geometrically and often implies the stronger notion.

Definition **4.6.** — *In Situation 4.4. Let* T *be a* κ*-scheme. Let* $\tau : T \times_\kappa C \to X_{f,\mathrm{smooth}}$ *be a family of degree e sections of f with image in* $X_{f,\mathrm{smooth}}$.

1. *Let* D *be an effective Cartier divisor in* C. *The family* τ *is* weakly D-free *if the* $\mathcal{O}_T$*-module homomorphism*

$$(\mathrm{pr}_T)_*(\tau^* T_f) \longrightarrow (\mathrm{pr}_T)_*(\tau^* T_f \otimes_{\mathcal{O}_{T\times_\kappa C}} \mathrm{pr}_C^* \mathcal{O}_D)$$

   is surjective. Here $T_f = \mathcal{H}om(\Omega^1_{X/C}, \mathcal{O}_X)$.
2. *Let* D *be an effective Cartier divisor in* C. *The family* τ *is* D-free *if the sheaf of* $\mathcal{O}_T$*-modules*

$$R^1(\mathrm{pr}_T)_*(\tau^* T_f \otimes_{\mathcal{O}_{T\times_\kappa C}} \mathrm{pr}_C^* \mathcal{O}_C(-D))$$

   is zero.
3. *For an integer* $m \geq 0$, *the family is* weakly m-free *if the base change to* $\mathrm{Spec}(\overline{\kappa})$ *is weakly* D*-free for every effective, degree m, Cartier divisor* D *in* $C \times_\kappa \mathrm{Spec}(\overline{\kappa})$.
4. *For an integer* $m \geq 0$, *the family is* m-free *if the base change to* $\mathrm{Spec}(\overline{\kappa})$ *is* D*-free for every effective, degree m, Cartier divisor* D *in* $C \times_\kappa \mathrm{Spec}(\overline{\kappa})$.

Whenever we talk about D*-free, weakly* D*-free, weakly m-free or m-free sections we tacitly assume that the image of the section is contained in* $X_{f,\mathrm{smooth}}$.

Definition **4.7.** — *In Situation 4.4 a section* $s : C \to X$ *of* f *is called* unobstructed *if it is* 0*-free. It is called* free *if it is* 1*-free.*

It is true and easy to prove that these correspond to the usual notions. Namely, if a section is unobstructed then its formal deformation space is the spectrum of a power series ring. Also, a section s is free if and only if it is unobstructed and $s^*T_{X/C}$ is globally generated, which implies that its deformations sweep out an open set in X.

Lemma **4.8.** — *In Situation 4.4.*

1. *Let* D *be an effective Cartier divisor on* C. *There is an open subscheme* U *of* $\mathit{Sections}^e(X/C/\kappa)$ *parametrizing exactly those sections whose image lies in* $X_{f,\mathrm{smooth}}$ *and which are* D*-free.*
2. *Let* $m \geq 0$ *be an integer. There is an open subscheme* V *of* $\mathit{Sections}^e(X/C/\kappa)$ *parametrizing exactly those families whose image lies in* $X_{f,\mathrm{smooth}}$ *and which are m-free.*
3. *The set of* $\overline{\kappa}$*-points of* $\mathit{Sections}^e(X/C/\kappa) \times \mathit{Sym}^m(C)$ *corresponding to pairs* (τ, D) *such that* τ *is* D*-free is open.*

Proof. — The first part follows from Lemma 3.13. To see the second part consider a family $\tau : T \times_\kappa C \to X_{f,\mathrm{smooth}}$. Let $T' = T \times_\kappa \mathrm{Sym}^m(C)$. On $T' \times_\kappa C$ we have a "universal" relative Cartier divisor $\mathcal{D}$ (of degree m over T'). Hence we may consider the open $U \subset T' = T \times_\kappa \mathrm{Sym}^m(C)$ where the relevant higher direct image sheaf vanishes. Let $V \subset T$ be the largest open subset such that $V \times_\kappa \mathrm{Sym}^m(C)$ is contained in U. We leave it to the reader to verify that a base change by $T'' \to T$ of τ is m-free if and only if $T'' \to T$ factors through V. The third part is similar to the others. □

Lemma **4.9.** — *In Situation 4.4. Let* $\tau : T \times_\kappa C \to X_{f,\mathrm{smooth}}$ *be a family of sections, let* $u : T' \to T$ *be a morphism and let* $\tau' = \tau \circ (u, \mathrm{id}_C)$.

1. *Let* D *be an effective Cartier divisor in* C*. If* τ *is weakly* D*-free, then so is* τ'.
2. *Let* $m \geq 0$ *be an integer. If* τ *is weakly* m*-free then so is* τ'.
   The converse holds if u *is faithfully flat.*

Proof. — We first prove 1. The sheaf of $\mathcal{O}_T$-modules $\mathcal{G} = (\mathrm{pr}_T)_*(\tau^* T_f \otimes_{\mathcal{O}_{T\times C}} \mathrm{pr}_C^* \mathcal{O}_D)$ from Definition 4.6 is finite locally free (of rank the degree of D times the dimension of X over C) and its formation commutes with base change. The sheaf $\mathcal{F} = (\mathrm{pr}_T)_*(\tau^* T_f)$ is quasi-coherent but its formation only commutes with flat base change in general. Denote $\mathcal{F}'$ the corresponding sheaf for τ'. Assume the map $\mathcal{F} \to \mathcal{G}$ is surjective. Then $\mathcal{F}' \to \mathcal{G}' = u^*\mathcal{G}$ is surjective because $u^*\mathcal{F} \to u^*\mathcal{G}$ factors through $\mathcal{F}'$. The statement about the faithfully flat case follows from this discussion as well.

To prove 2 we use the arguments of 1 for the universal situation over $T \times_\kappa \mathrm{Sym}^m(C)$ as in the proof of Lemma 4.8 above (left to the reader). □

Lemma **4.10.** — *In Situation 4.4. Let* $\tau : T \times_\kappa C \to X_{f,\mathrm{smooth}}$ *be a family of sections.*

1. *Let* $D' \leq D$ *be effective Cartier divisors on* C*. If* τ *is* D*-free, resp. weakly* D*-free, then it is also* D'*-free, resp. weakly* D'*-free.*
2. *For integers* $m \geq m' \geq 0$*, if* τ *is* m*-free, resp. weakly* m*-free, then it is also* m'*-free, resp. weakly* m'*-free.*

Proof. — Consider the finite locally free sheaves of $\mathcal{O}_T$-modules defined by $\mathcal{G} = (\mathrm{pr}_T)_*(\tau^* T_f \otimes_{\mathcal{O}_{T\times C}} \mathrm{pr}_C^* \mathcal{O}_D)$ and $\mathcal{G}' = (\mathrm{pr}_T)_*(\tau^* T_f \otimes_{\mathcal{O}_{T\times C}} \mathrm{pr}_C^* \mathcal{O}_{D'})$ that occur in Definition 4.6. It is clear that there is a surjection $\mathcal{G} \to \mathcal{G}'$. On the other hand consider the sheaves $\mathcal{H} = R^1(\mathrm{pr}_T)_*(\tau^* T_f \otimes_{\mathcal{O}_{T\times C}} \mathrm{pr}_C^* \mathcal{O}_C(-D))$ and $\mathcal{H}' = R^1(\mathrm{pr}_T)_*(\tau^* T_f \otimes_{\mathcal{O}_{T\times C}} \mathrm{pr}_C^* \mathcal{O}_C(-D'))$. There is a surjection $\mathcal{H} \to \mathcal{H}'$ induced by the map $\mathcal{O}_C(-D) \to \mathcal{O}_C(-D')$ with finite cokernel (tensored with $\tau^* T_f$ this has a cokernel finite over T). In this way we deduce 1.

In order to prove 2 we work on $T' = T \times_\kappa \mathrm{Sym}^m(C) \times_\kappa \mathrm{Sym}^{m'-m}(C)$. There are "universal" relative effective Cartier divisors $\mathcal{D} \subset T' \times_\kappa C$ (of degree m) and $\mathcal{D}' \subset T' \times_\kappa C$ (of degree m') with $\mathcal{D} \subset \mathcal{D}'$. There are sheaves of $\mathcal{O}_{T'}$-modules $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{H}$, and $\mathcal{H}'$ which are variants of the sheaves above defined using the relative Cartier divisors $\mathcal{D}$ and $\mathcal{D}'$. If

τ is m-free, resp. weakly m-free, then $\mathcal{H} = 0$, resp. $(\mathrm{pr}_{T'})_*((\tau')^*T_f)$ surjects onto $\mathcal{G}$. The arguments just given to prove part 1 show that there are canonical surjections $\mathcal{H} \to \mathcal{H}'$ and $\mathcal{G} \to \mathcal{G}'$. Hence we deduce that $\mathcal{H}' = 0$, resp. that $(\mathrm{pr}_{T'})_*((\tau')^*T_f)$ surjects onto $\mathcal{G}'$. Then this in turn implies that τ is m-free, resp. weakly m-free. □

Lemma **4.11.** — *In Situation 4.4. Let $\tau : T \times_\kappa C \to X_{f,\mathrm{smooth}}$ be a family of sections.*

1. *Let D be an effective Cartier divisor on C. Then τ is D-free if and only if it is both 0-free and weakly D-free.*
2. *Let $m \geq 0$ be an integer. Then τ is m-free if and only if it is both 0-free and weakly m-free.*

Proof. — We first prove 1. Saying a family is 0-free is equivalent to saying it is D′-free where D′ is the empty Cartier divisor. Thus, if a family is D-free then it is 0-free by Lemma 4.10. The long exact sequence of higher direct images associated to the short exact sequence

$$0 \to \tau^*T_f \otimes_{\mathcal{O}_{T\times_\kappa C}} \mathrm{pr}_C^*\mathcal{O}_C(-D) \to \tau^*T_f \to \tau^*T_f \otimes_{\mathcal{O}_{S\times_\kappa C}} \mathrm{pr}_C^*\mathcal{O}_D \to 0$$

shows that

$$(\mathrm{pr}_T)_*(\tau^*T_f) \to (\mathrm{pr}_T)_*(\tau^*T_f \otimes_{\mathcal{O}_{T\times_\kappa C}} \mathrm{pr}_C^*\mathcal{O}_D)$$

is surjective so long as $R^1(\mathrm{pr}_T)_*(\tau^*T_f \otimes_{\mathcal{O}_{T\times_\kappa C}} \mathrm{pr}_C^*\mathcal{O}_C(-D))$ is the zero sheaf. Moreover, the converse holds so long as $R^1(\mathrm{pr}_S)_*(\tau^*T_f)$ is the zero sheaf. From this it follows that if a family of sections is D-free, then it is weakly D-free and the converse holds so long as the family is 0-free.

The proof of 2 is left to the reader. (It is similar to the proofs above.) □

Lemma **4.12.** — *In Situation 4.4. Let $\tau : T \times_\kappa C \to X_{f,\mathrm{smooth}}$ be a family of sections.*

1. *Let D be a nontrivial effective Cartier divisor of C. If $\deg(D) \geq 2g(C)$ and if τ is weakly D-free, then τ is $(\deg(D) - 2g(C))$-free (and thus D-free by the previous lemmas).*
2. *Let $m \geq 2g$ be an integer with $m \geq 1$ if $g(C) = 0$. Every weakly m-free family is m-free.*

Proof. — We prove part 1. By Lemma 4.8, there is a maximal open subscheme V of T over which τ is $(\deg(D) - 2g(C))$-free. The goal is to prove that V is all of T. For this, it suffices to prove that every geometric point of S factors through V. By Lemma 4.9, the base change of the family to any geometric point is weakly D-free. Thus, it suffices to prove the lemma when κ is algebraically closed and $T = \mathrm{Spec}(\kappa)$, i.e., τ is a section $\tau : C \to X_{f,\mathrm{smooth}}$.

Let m denote deg(D). Let us write $\mathcal{T} = \tau^*T_f$, and let r be the rank of $\mathcal{T}$. The assumption is that the restriction map

$$H^0(C, \mathcal{T}) \to H^0(C, \mathcal{T} \otimes_{\mathcal{O}_C} \mathcal{O}_D)$$

is surjective. Let $p \in D$ be a point. The restriction map

$$H^0(C, \mathcal{T} \otimes_{\mathcal{O}_C} \mathcal{O}_C(-D+p)) \to H^0(C, \mathcal{T} \otimes_{\mathcal{O}_C} \kappa(p))$$

is also surjective. In other words, there exist r global sections of $\mathcal{T}(-D+p)$ generating the fibre at p. These global sections give an injective $\mathcal{O}_C$-module homomorphism $\mathcal{O}_C^{\oplus r} \to \mathcal{T}(-D+p)$ which is surjective at p. Thus the cokernel is a torsion sheaf. Let $E \subset C$ be any effective divisor of degree $m - 2g(C)$. Twisting by $D - E - p$, this gives an injective $\mathcal{O}_C$-module homomorphism

$$\mathcal{O}_C(D - E - p)^{\oplus r} \to \mathcal{T}(-E)$$

whose cokernel is torsion. From the long exact sequence of cohomology, there is a surjection

$$H^1(C, \mathcal{O}_C(D - E - p))^{\oplus r} \to H^1(C, \mathcal{T}(-E)).$$

Since $\deg(D - E - p) > 2g(C) - 2$ we see that $h^1(C, \mathcal{O}_C(D - E - p))$ equals 0. Therefore $h^1(C, \mathcal{T}(-E))$ also equals 0.

Part 2 is proved in a similar fashion. □

Proposition **4.13.** — *Situation as in 4.4. Let* $D \subset C$ *be an effective divisor.*

1. *Let* $\tau : T \times_\kappa C \to X_{f,\text{smooth}}$ *be a family of sections. If the induced morphism* $(\tau|_{T\times_\kappa D}, \mathrm{Id}_D) : T \times_\kappa D \to X_{f,\text{smooth}} \times_C D$ *is smooth, then the family is weakly* D*-free.*
2. *Let* U *be the open subscheme of Sections*$^e(X_{f,\text{smooth}}/C/\kappa)$ *over which the universal section is* D*-free (see Lemma 4.8). The morphism*

$$(\sigma, \mathrm{Id}_D) : U \times_\kappa D \to X_{f,\text{smooth}} \times_C D$$

is smooth.

3. *The scheme Sections*$(X/C/\kappa)$ *is smooth at every point which corresponds to a section that is* D*-free.*

Proof. — When D is the empty divisor, both 1 and 2 are vacuous, and 3 follows from the vanishing of the obstruction space for the deformation theory. When D is not empty, one argues as in [Kol96, Proposition II.3.5]. □

In order to state the following proposition we need some notation. In Situation 4.4, consider the Hilbert scheme $\mathrm{Hilb}^m_{X/\kappa}$ of finite length m closed subschemes of X over κ. There is an open subscheme $H^m_{X,f} \subset \mathrm{Hilb}^m_{X/\kappa}$ parametrizing those $Z \subset X$ such that (a) $Z \subset X_{f,\text{smooth}}$ and (b) $f|_Z : Z \to C$ is a closed immersion. By construction there is a natural quasi-projective morphism

$$H^m_{X,f} \longrightarrow \mathrm{Sym}^m(C). \tag{4.1}$$

We claim this morphism is smooth. Namely, a point of $H^m_{X,f}$ corresponds to a pair (Z', σ), where $Z' \subset C$ is a closed subscheme of length m, and $\sigma : Z' \to X_{f,\mathrm{smooth}}$ which is a section of f. There are no obstructions to deforming a map from a given zero dimensional scheme to a smooth scheme, whence the claim.

Let D be a degree m effective Cartier divisor on C. The fibre $H^m_{X,f,D}$ of (4.1) over $[D] \in \mathrm{Sym}^m(C)$ is the space of sections of $D \times_C X_{f,\mathrm{smooth}} \to D$. Note that $H^m_{X,f,D}$ equals the Weil restriction of scalars $\mathrm{Res}_{D/\kappa}(D \times_C X_{f,\mathrm{smooth}})$. In fact $H^m_{X,f}$ is a Weil restriction of scalars as well, which gives an alternative construction of $H^m_{X,f}$. Namely, $H^m_{X,f} = \mathrm{Res}_{\mathcal{D}/\mathrm{Sym}^m(C)}(\mathcal{D} \times_C X_{f,\mathrm{smooth}})$, where $\mathcal{D} \subset \mathrm{Sym}^m(C) \times_\kappa C$ is the universal degree m effective Cartier divisor on C.

Consider any family of sections $\tau : T \times_\kappa C \to X_{f,\mathrm{smooth}}$. Given a divisor D of C of degree m we may restrict τ to $T \times_\kappa D$ and obtain a T-valued point of $H^m_{X,f,D}$. This construction can be done more generally with D replaced by the universal Cartier divisor $\mathcal{D}$ to obtain a morphism

$$T \times_\kappa \mathrm{Sym}^m(C) \longrightarrow H^m_{X,f} \tag{4.2}$$

over $\mathrm{Sym}^m(C)$. Since this is functorial in T we obtain a morphism of algebraic stacks

$$res : \mathrm{Sections}(X_{f,\mathrm{smooth}}/C/\kappa) \times_\kappa \mathrm{Sym}^m(C) \longrightarrow H^m_{X,f} \tag{4.3}$$

over $\mathrm{Sym}^m(C)$.

Lemma **4.14.** — *In Situation 4.4, the morphism res is smooth at any point* (τ, D) *such that* τ *is* D*-free.*

Proof. — This is akin to Proposition 4.13 and proved in the same way. It also follows from part 2 of that Proposition by the fibrewise criterion of smoothness since both $\mathrm{Sections}(X_{f,\mathrm{smooth}}/C/\kappa) \times_\kappa \mathrm{Sym}^m(C)$ and $H^m_{X,f}$ are smooth over $\mathrm{Sym}^m(C)$ at the corresponding points. □

Here are two technical results that will be used later.

Proposition **4.15.** — *See [KMM92a, 1.1]. In Situation 4.4, with char*$(\kappa) = 0$. *Let e and* $m \geq 0$ *be integers. There exists a constructible subset* W^m_e *of* $H^m_{X,f}$ *whose intersection* $W^m_e \cap H^m_{X,f,D}$ *with the fibre over every point* $[D]$ *of* $\mathrm{Sym}^m(C)$ *contains a dense open subset of* $H^m_{X,f,D}$ *with the following property: For every* κ*-scheme* T *and for every family of degree e sections*

$$\tau : T \times_\kappa C \longrightarrow X_{f,\mathrm{smooth}},$$

if the restriction of τ *to* $T \times_\kappa D$ *is in* $W^m_e \cap H^m_{X,f,D}$, *then* τ *is weakly* D*-free.*

Proof. — It suffices to construct W^m_e over $\overline{\kappa}$, and hence we may and do assume κ algebraically closed. Stratify $\mathrm{Sections}^e(X_{f,\mathrm{smooth}}/C/\kappa)$ by a finite set of irreducible locally closed strata T_i, each *smooth* over κ. For each i we have the restriction morphism $res_i : T_i \times \mathrm{Sym}^m(C) \to H^m_{X,f}$. This is a morphism of smooth schemes of finite type over $\mathrm{Sym}^m(C)$. Let $S_i \subset T_i \times \mathrm{Sym}^d(C)$ be the closed set of points where the derivative of res_i

$$\mathrm{d}res_i : T_{T_i \times \mathrm{Sym}^m(C)} \longrightarrow res_i^*(T_{H^m_{X,f}})$$

is not surjective. By Sard's theorem, or generic smoothness the image $E_i = res_i(S_i) \subset H^m_{X,f}$ is not dense in any fibre $H^m_{X,f,D}$. It is constructible as well by Chevalley's theorem. Take W^m_e to be the complement of the finite union $\bigcup E_i$. Let us prove that W^m_e has the desired properties. Suppose that $\tau : C \to X_{f,\mathrm{smooth}}$ is a section such that $\tau|_D$ is in W^m_e. Let i be such that τ corresponds to a point $[\tau] \in T_i$. By construction of W^m_e the morphism $T_i \to H^m_{X,f,D}$ is smooth with surjective derivative at τ. Note that $T_\tau(T_i) \subset H^0(C, \tau^* T_f)$, and that $T_{res(\tau \times [D])}(H^m_{X,f,D}) = H^0(C, \tau^* T_f|_D)$. Thus the surjectivity of the derivative in terms of Zariski tangent spaces implies that τ is weakly D-free. ☐

Situation **4.16.** — *Here k is an algebraically closed field and we have a commutative diagram*

$$\begin{array}{ccccc} \tau : C \times T & \longrightarrow & Y & \hookleftarrow & U \\ & \searrow & \downarrow & \swarrow & \\ & & C & & \end{array}$$

of varieties over k where C is a smooth projective curve and Y is proper over C. Moreover, we assume given a dense open $U \subset Y$ such that $U \to C$ is a smooth morphism whose nonempty fibres are geometrically irreducible. Finally, we assume that $\tau(C \times T) \cap U \neq \emptyset$.

In this situation, for $t \in T(k)$ we denote $\tau_t : C \to Y$ the restriction of τ to $C \times \{t\}$. In other words, we think of τ as a family of maps from C to Y. Let $m > 0$ be an integer. Let $H^m_U = H^m_{U,U\to C} \to \mathrm{Sym}^m(C)$ be as in (4.1). In the following we will use a canonical rational map

(**4.4**) $$\tau_m : T \times \mathrm{Sym}^m(C) \dashrightarrow H^m_U.$$

over $\mathrm{Sym}^m(C)$. Namely, it is defined on the open subscheme $W \subset \mathrm{Sym}^m(C) \times T$ consisting of the set of pairs (D, t) such that $\tau_t(D) \subset U$ which is a dense open by assumption. And for $(D, t) \in W$ we simply set $\tau_m(D, t) = \tau_t|_D : D \to U$.

Lemma **4.17.** — *In Situation 4.16 assume k is an uncountable algebraically closed field of characteristic 0, and assume given $m > 0$ such that τ_m is dominant. Finally, let V be a variety and let*

$\Phi : \mathrm{V} \to \mathrm{U}$ be a smooth, surjective morphism whose nonempty geometric fibres are birationally rationally connected. Then we can find a commutative diagram

$$\begin{array}{ccccc} \mathrm{C} \times \mathrm{T}' & \xrightarrow{\tau'} & \overline{\mathrm{V}} & \hookleftarrow & \mathrm{V} \\ \downarrow & & \downarrow{\scriptstyle \overline{\Phi}} & & \downarrow{\scriptstyle \Phi} \\ \mathrm{C} \times \mathrm{T} & \xrightarrow{\tau} & \mathrm{Y} & \hookleftarrow & \mathrm{U} \end{array}$$

such that $\mathrm{T}' \to \mathrm{T}$ is dominant, the top row is as in Situation 4.16, and $\tau'_m : \mathrm{T}' \times Sym^m(\mathrm{C}) \dashrightarrow \mathrm{H}^m_{\mathrm{V}}$ is dominant. Moreover, given an arbitray compactification $\mathrm{V} \subset \mathrm{W}$ by a proper variety W we can choose our $\overline{\mathrm{V}}$ to dominate W.

Remark **4.18.** — This technical lemma just says that a general m-free section of $\mathrm{Y} \to \mathrm{C}$ lifts to an m-free section of V.

Proof. — Note that V is a nonsingular variety. We choose a smooth proper compactification $\mathrm{V} \subset \overline{\mathrm{V}}$ such that Φ extends to a morphism $\overline{\Phi} : \overline{\mathrm{V}} \to \mathrm{Y}$. (First choose any proper compactification to which Φ extends, and then resolve singularities, see [Hir64, Hir64a]. To see the last statement of the lemma let $\overline{\mathrm{V}}$ be a resolution of W.) This gives us the right square of the diagram in the lemma. By generic smoothness we may shrink U and assume that the fibres of $\overline{\Phi}$ are proper smooth varieties over all points of U. In this case these fibres are rationally connected varieties by our assumption on Φ. We may shrink T and assume that $\tau_t(\mathrm{C})$ meets U for all $t \in \mathrm{T}(k)$.

The assumption on the fibres of Φ implies that for every $t \in \mathrm{T}(k)$ we may find a section $\sigma : \mathrm{C} \to \overline{\mathrm{V}}$ such that $\overline{\Phi} \circ \sigma = \tau_t$. See [GHS03]. It was already known before the results of [GHS03], see for example [KMM92b], that one may, given any m sufficiently general points $c_1, \ldots, c_m$ of the curve C, assume the m-tuple $(\sigma(c_i) \in \overline{\mathrm{V}}_{\tau_t(c_i)})_{i=1\ldots m}$ is a general point of $\mathrm{V}_{\tau_t(c_1)} \times \cdots \times \mathrm{V}_{\tau_t(c_m)}$. This is done by the "smoothing combs" technique of [KMM92b]. Namely, after attaching sufficiently many free teeth (in fibres of $\overline{\mathrm{V}} \to \mathrm{Y}$) to σ, the comb can be deformed to a section σ which passes through m very general points of m very general fibres. (This also follows from the main result of the paper [HT06]. Note that since the fibres of $\overline{\mathrm{V}} \to \mathrm{U}$ are smooth and rationally connected we are allowed to use that result. Of course [HT06] proves something much stronger!) Combined with the assumption that τ_m is dominant this implies that in fact the m-points $\sigma(c_i)$ hit a general point of $\mathrm{V}_{c_1} \times \cdots \times \mathrm{V}_{c_m}$ which is the fibre of $\mathrm{H}^m_{\mathrm{V}} \to \mathrm{Sym}^m(\mathrm{C})$ over the point $\sum c_i$.

The paragraph above solves the problem "pointwise", but since the field k is uncountable this implies the result of the lemma by a standard technique. Namely, the above produces lots (uncountably many) of points on the scheme $\mathrm{Sections}(\overline{\mathrm{V}} \times_{\mathrm{Y}} (\mathrm{C} \times \mathrm{T})/(\mathrm{C} \times \mathrm{T})/\mathrm{T})$ which implies that it must have a suitable irreducible component T' dominating T (details left to the reader). □

5. Kontsevich stable maps

In this section we review the stack $\overline{\mathcal{M}}_{g,\mathrm{I}}(\mathrm{Y}/\mathrm{B}, \beta)$ of Kontsevich stable maps. Standard references are [Kon95, FP97, BM96]. We will use these spaces in a relative setting and where β is a *partial curve class*.

Let k be an algebraically closed field and let Y be a quasi-projective k-scheme.

Definition **5.1.** — *A* stable map *to* Y *over* k *is given by*

1. *a proper, connected, at-worst-nodal* k*-curve* C*,*
2. *a finite collection* $(p_i)_{i\in\mathrm{I}}$ *of distinct, smooth, closed points of* C*, and*
3. *a* k*-morphism* $h : \mathrm{C} \to \mathrm{Y}$.

These data have to satisfy the stability *condition that the logarithmic dualizing sheaf* $\omega_{\mathrm{C}/k}(\sum_{i\in\mathrm{I}} p_i)$ *is* h*-ample.*

The stability of the triple $(\mathrm{C}, (p_i), h)$ can also be expressed in terms of the numbers of special points on h-contracted components of C of arithmetic genus 0 and 1, or by saying that the automorphism group scheme of the triple is finite. See [Kon95].

There are many invariants of a stable map of a numerical or topological nature. One invariant is the number $n = \#\mathrm{I}$ of marked points. Occasionally it is convenient to mark the points by some unspecified finite set I, but usually I is simply $\{1, 2, \ldots, n\}$. Another invariant is the arithmetic genus g of C. Yet another invariant is the "curve class" of $h_*([\mathrm{C}])$, which we define below. Since often we do not need the full curve class, the following definition makes precise the notion of a "partial curve class". We will use NS(Y) to denote the Néron-Severi group of Y, i.e., the group of Cartier divisors up to numerical equivalence on Y.

Definition **5.2.** — *Let* k *be an algebraically closed field and let* Y *be a quasi-projective* k*-scheme. A* partial curve class *on* Y *is a triple* (A, i, β) *consisting of an Abelian group* A *and homomorphisms of Abelian groups*

$$i : \mathrm{A} \to NS(\mathrm{Y}) \quad \text{and} \quad \beta : \mathrm{A} \to \mathbf{Z}.$$

The triple will usually be denoted by β*, with* A *being denoted* A_β *and* i *being denoted* i_β*. We say a stable map* $(\mathrm{C}, (p_i)_{i\in\mathrm{I}}, h)$ belongs to the partial curve class β *or* has partial curve class β *if for every element* a *in* A *and for every Cartier divisor* D *on* Y *with class* $i(a)$*, the degree of* $h^*\mathrm{D}$ *on* C *equals* $\beta(a)$*.*

Remark **5.3.** — If there exists a stable map belonging to β then $\mathrm{Ker}(\beta)$ contains $\mathrm{Ker}(i)$, i.e., β factors through $i(\mathrm{A}) \subset \mathrm{NS}(\mathrm{Y})$. From this point of view it is reasonable to restrict to cases where A is a subgroup of NS(Y). But it is sometimes useful to allow i to be noninjective.

We will also use a relative version of stable maps and of partial curve classes.

Definition **5.4.** — *Let* $Y \to S$ *be a quasi-projective morphism.*

1. *A* family of stable maps to Y/S *is given by*
   (1) *an* S*-scheme* T*,*
   (2) *a proper, flat family of curves* $\pi : C \to T$ *(see Definition 3.1),*
   (3) *a finite collection of* T*-morphisms* $(\sigma_i : T \to C)_{i\in I}$*, and*
   (4) *a* S*-morphism* $h : C \to Y$*.*
   These data have to satisfy the stability *condition that for every algebraically closed field* k *and morphism* $t : Spec(k) \to T$*, the base change* $(C_t, (\sigma_i(t))_{i\in I}, h_t)$ *is a stable map to* $Y_t = Spec(k) \times_{t,S} Y$ *as in Definition 5.1.*
2. *Given two families of stable maps* $(T' \to S, \pi' : C' \to T', (\sigma'_i)_{i\in I}, h')$ *and* $(T \to S, \pi : C \to T, (\sigma_i)_{i\in I}, h)$ *a* morphism *from the first family to the second family is given by an* S*-morphism* $u : T' \to T$*, and a morphism* $\phi : C' \to C$ *such that (a)* (ϕ, u) *forms a morphism of flat families of curves (see 3.1), (b)* $\sigma_i \circ u = \phi \circ \sigma'_i$*, and (c)* $h' = h \circ \phi$*.*

Let $Y \to S$ be a quasi-projective morphism. Denote $\mathrm{Pic}_{Y/S}$ the relative Picard functor, see [BLR90, Section 8.1]. It is an fppf sheaf on the category of schemes over S. We define the relative Néron-Severi functor $NS_{Y/S}$ of Y/S to be the quotient of $\mathrm{Pic}_{Y/S}$ by the subsheaf of sections numerically equivalent to zero on all geometric fibres. Finally, we define the relative Néron-Severi group $NS(Y/S) = NS_{Y/S}(S)$ to be the sections of this sheaf over S.

Definition **5.5.** — *A* partial curve class *on* Y/S *is a triple* (A, i, β) *consisting of an Abelian group* A*, and homomorphisms of Abelian groups*

$$i : A \to NS(Y/S) \text{ and } \beta : A \to \mathbf{Z}.$$

For every morphism of schemes $T \to S$ *there is a* pullback partial curve class *on* Y_T/T *given by* (A, i_T, β) *where* i_T *is the composition*

$$A \xrightarrow{i} NS(Y/S) \to NS(Y_T/T).$$

A family of stable maps $(T \to S, \pi : C \to T, (\sigma_i)_{i\in I}, h)$ *to* Y/S belongs to the partial curve class β *or* has partial curve class β *if for every algebraically closed field* k *and every morphism* $t : Spec(k) \to T$*, the pullback* $(C_t, (\sigma_i(t))_{i\in I}, h_t)$ *belongs to the pullback partial curve class* (A, i_t, β)*.*

Suppose that $(T \to S, \pi : C \to T, (\sigma_i)_{i\in I}, h)$ is a family of stable maps to Y/S as in Definition 5.4. For every $g \geq 0$ there is an open and closed subscheme of T where the geometric fibres of π have genus g. In addition, given a relative partial curve class β, there is an open and closed subscheme of T where the geometric fibres $h_t : C_t \to Y_t$ belong to the partial curve class β_t. This is because the degree of an invertible sheaf on a proper flat family of curves is locally constant on the base. Thus the following definition makes sense.

Definition **5.6.** — *Let* $Y \to S$ *be quasi-projective. For every finite set* I *the* Kontsevich stack of stable maps to Y/S *is the stack*

$$\overline{\mathcal{M}}_{*,I}(Y/S) \longrightarrow \mathit{Schemes}/S$$

whose objects and morphisms are as in Definition 5.4 above. When I *is just* $\{1,\dots,n\}$, *this stack is denoted* $\overline{\mathcal{M}}_{*,n}(Y/B)$. *For every integer* $g \geq 0$, *there is an open and closed substack* $\overline{\mathcal{M}}_{g,I}(Y/B)$ *parametrizing stable maps whose geometric fibres all have arithmetic genus* g. *And for every partial curve class* β *on* Y/S, *there is an open and closed substack* $\overline{\mathcal{M}}_{g,I}(Y/B,\beta)$ *parametrizing stable maps belonging to the partial curve class* β.

The following theorem is well known to the experts. We state it here for convenience, and because we do not know an exact reference.

Theorem **5.7.** — *Compare [Kon95]. Let* $Y \to S$ *be quasi-projective, and* S *excellent.*

1. *The stack* $\overline{\mathcal{M}}_{*,I}(Y/S)$ *is a locally finitely presented, algebraic stack over* S *with finite diagonal, satisfying the valuative criterion of properness over* S *when* Y *is proper over* S.
2. *If* S *lives in characteristic* 0 *then* $\overline{\mathcal{M}}_{*,I}(Y/S)$ *is a Deligne-Mumford stack over* S.
3. *If* Y *is proper over* S *and if* (A,i,β) *is a partial curve class such that* $i(A)$ *contains the class of a* S*-relatively ample invertible sheaf on* Y, *then* $\overline{\mathcal{M}}_{g,I}(Y/S,\beta)$ *is proper over* S *with projective coarse moduli space.*

Proof. — (Sketch.) Forgetting the sections gives a 1-morphism from $\overline{\mathcal{M}}_{*,I}(Y/S)$ to *CurveMaps*(Y/S). It is easy to see that this morphism is representable and smooth. Hence, by Proposition 3.6 the stack $\overline{\mathcal{M}}_{*,I}(Y/S)$ is algebraic locally of finite presentation over S, with finitely presented diagonal. For the valuative criterion of properness, see [Kon95, bottom page 338]. Finiteness of the diagonal is, in view of quasi-compactness (see above), a consequence of the valuative criterion of properness for Isom between stable maps (basically trivial), combined with the fact that by definition stable maps have finite automorphism groups (gives quasi-finiteness). This proves 1. In characteristic zero a stack with finite diagonal is Deligne-Mumford, because automorphism group schemes are reduced. Finally, to see 3 note that $\overline{\mathcal{M}}_{g,I}(Y/S,\beta)$ is quasi-compact in this case by the argument in [Kon95, page 338]. Thus it is proper over S, see [Ols05] and [Fal03, Appendix]. By [KM97] the coarse moduli space exists, and by the above it is proper over S. The projectivity of this coarse moduli space will not be used in the sequel; a reference is [AV02, Section 8.3]. □

Notation **5.8.** — *A simple case of a partial curve class is as follows. Let* $Y \to S$ *be quasi-projective and let* $\mathcal{L}$ *be an* S*-ample invertible sheaf on* Y. *For every integer* e *there is a partial curve* *class* $\beta = (\mathbf{Z}, 1 \mapsto [\mathcal{L}], 1 \mapsto e)$. *When* $\mathcal{L}$ *is understood, the corresponding Kontsevich moduli space is denoted* $\overline{\mathcal{M}}_{g,n}(\mathrm{Y}/\mathrm{S}, e)$.

6. Stable sections and Abel maps

In this section we introduce the stack $\Sigma^e(\mathrm{X}/\mathrm{C}/\mathrm{S})$ of stable sections as a special case of Kontsevich moduli spaces. This is a compactification of the space of sections. We explain the connection to combs, a ubiquitous notion in rational curves on algebraic varieties. Finally, we introduce the Abel map in this setting, see Lemma 6.7.

Situation **6.1.** — *In this section* $\mathrm{C} \to \mathrm{S}$ *is a proper, flat family of curves (see 3.1) whose geometric fibres are connected smooth projective curves of some fixed genus* $g(\mathrm{C}/\mathrm{S})$. *Moreover,* $f : \mathrm{X} \to \mathrm{C}$ *is a proper morphism, and* $\mathcal{L}$ *is an* f*-ample invertible sheaf on* X.

Here the group $\mathrm{NS}(\mathrm{C}/\mathrm{S})$ (see discussion preceding 5.5) is a free Abelian group of rank 1, simply because any two invertible sheaves on the family are numerically equivalent on geometric fibres if and only if they have the same degree on the fibres. Denote [point] the element of $\mathrm{NS}(\mathrm{C}/\mathrm{S})$ that fppf locally on S corresponds to an invertible sheaf on C having degree 1 on the fibres.

Consider the relative partial curve class $\beta_{\mathrm{C}/\mathrm{S}} = (\mathbf{Z}, 1 \mapsto [\mathrm{point}], 1 \mapsto 1)$ on C/S (compare 5.8). For all nonnegative integers g and n, we denote

$$\overline{\mathcal{M}}_{g,n}(\mathrm{C}/\mathrm{S}, 1) := \overline{\mathcal{M}}_{g,n}(\mathrm{C}/\mathrm{S}, \beta_{\mathrm{C}/\mathrm{S}}).$$

Loosely speaking the stable maps h arising from points of this algebraic stack have one component that maps birationally to a fibre of $\mathrm{C} \to \mathrm{S}$, and all other components are contracted. If $g = g(\mathrm{C}/\mathrm{S})$, then the contracted components all have genus zero.

Denote by [fibre] the pullback $f^*[\mathrm{point}]$ in $\mathrm{NS}(\mathrm{X}/\mathrm{S})$. For every integer e, denote by β_e the relative partial curve class $(\mathbf{Z} \oplus \mathbf{Z}, i, \beta_e)$ with $i(1,0) = [\mathrm{fibre}]$, $i(0,1) = \mathcal{L}$ and $\beta_e(1,0) = 1$, $\beta_e(0,1) = e$. Using this notation, for all nonnegative integers g and n, we may consider the algebraic stacks

$$\overline{\mathcal{M}}_{g,n}(\mathrm{X}/\mathrm{S}, \beta_e)$$

introduced in the previous section. When $g = g(\mathrm{C}/\mathrm{S})$ a stable map h arising from a point of this algebraic stack gives rise to a generalized section of $\mathrm{X}_s/\mathrm{C}_s$ having total degree e with respect to $\mathcal{L}$ for some geometric point s of S.

There is an obvious way in which β_e and $\beta_{\mathrm{C}/\mathrm{S}}$ are related. Thus by [BM96, Theorem 3.6], there are 1-morphisms of algebraic S-stacks

$$\overline{\mathcal{M}}_{g,n}(f) : \overline{\mathcal{M}}_{g,n}(\mathrm{X}/\mathrm{S}, \beta_e) \longrightarrow \overline{\mathcal{M}}_{g,n}(\mathrm{C}/\mathrm{S}, 1).$$

Definition **6.2.** — *In Situation 6.1. For every nonnegative integer n and every integer e,* the space of n-pointed, degree e, stable sections of f *is the algebraic stack*

$$\Sigma_n^e(X/C/S) := \overline{\mathcal{M}}_{g(C/S),n}(X/S, \beta_e).$$

We will also use the variant $\Sigma_I^e(X/C/S)$ *where the sections are labeled by a given finite set* I. *Finally,* $\Sigma^e(X/C/S) = \Sigma_0^e(X/C/S)$.

We also have the analogue of Definitions 4.2 and 4.5.

Definition **6.3.** — *In Situation 6.1. A* family of n-pointed, degree e sections *of* X/C/S *over* $T \to S$ *is given by a family* (τ, θ) *of sections of* f *(see Definition 4.1) together with pairwise disjoint sections* $\sigma_i : T \to T \times_S C$, $i = 1, \ldots, n$, *such that* $\tau^*\mathcal{L}$ *has degree e on the fibres of* $T \times_S C \to T$. *We leave it to the reader to define* morphisms *of families of n-pointed, degree e sections of* X/C/S.

We will denote such a family as $(T \to S, \sigma_i : T \to T \times_S C, h : T \times_S C \to X)$. In other words τ is replaced by h and we drop the notation θ since in this case it just signifies that $f \circ h$ equals the projection map $T \times_S C \to S$.

The functor which associates to T/S the set of isomorphism classes of families of n-pointed, degree e sections of X/C/S is an algebraic space $\text{Sections}_n^e(X/C/S)$ locally of finite presentation over S. There are two ways to see this. First, we can use that the natural 1-morphism

$$\text{Sections}_n^e(X/C/S) \longrightarrow \text{Sections}(X/C/S)$$

is representable and smooth as $C \to S$ is smooth (proof omitted). Theorem 4.3 part (2) guarantees Sections(X/C/S) is an algebraic stack locally of finite presentation over S. (If $C \to S$ is projective then it follows from part (1) of Theorem 4.3.) Combined we conclude that $\text{Sections}_n^e(X/C/S)$ is an algebraic stack locally of finite presentation over S. Second, there is another obvious 1-morphism, namely

$$\text{Sections}_n^e(X/C/S) \longrightarrow \Sigma_n^e(X/C/S).$$

This 1-morphism is representable by open immersions of schemes. In this precise sense, the proper, algebraic S-stack $\Sigma_n^e(X/C/S)$ is a "compactification" of $\text{Sections}_n^e(X/C/S)$. And of course this also implies that $\text{Sections}_n^e(X/C/S)$ is an algebraic stack locally of finite presentation over S.

We will also use the variant $\text{Sections}_I^e(X/C/S)$ where the sections are labeled by a given finite set I.

Notation **6.4.** — *Let n be a nonnegative integer. Let* I *be a set of cardinality n. Let e be an integer. Let* δ *be a nonnegative integer. Let* J *be a set of cardinality* δ. *Let* $\underline{e} = (e_0, (e_j)_{j\in J})$ *be a collection of integers such that* $e = e_0 + \sum_{j\in J} e_j$. *Let* $\underline{I} = (I_0, (I_j)_{j\in J})$ *be a collection of subsets of* I *such that*

$\mathrm{I} = \mathrm{I}_0 \sqcup \sqcup_{j\in\mathrm{J}} \mathrm{I}_j$ *is a partition of* I. *The triple* $(\mathrm{J}, \underline{e}, \underline{\mathrm{I}})$ *is called an* indexing triple *for* (e, I). *In the special case that* $\mathrm{I} = \emptyset$, *so that every subset* I_0 *and* I_j *is also* $\emptyset$, *this is called an* indexing pair *for* e *denoted* $(\mathrm{J}, \underline{e})$.

We are going to define an algebraic stack $\mathrm{Comb}^{\underline{e}}_{\underline{\mathrm{I}}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$ of $\underline{\mathrm{I}}$-pointed, degree $\underline{e}$ combs. A point will correspond to a degree e stable section with marked points indexed by $\mathrm{I}_0 \amalg \mathrm{J}$ and for every j a vertical genus zero degree e_j curve with marked points indexed by $\mathrm{I}_j \amalg \{j\}$. These will be "attached" by requiring the points marked j (on the handle and tooth) to map to the same point of X.

Definition **6.5.** — *In Situation 6.1. A* family of $\underline{\mathrm{I}}$-pointed, degree $\underline{e}$ combs *in* $(\mathrm{X}/\mathrm{C}/\mathrm{S}, \mathcal{L})$ *as above is given by the following data*

(1) *an* S*-scheme* T
(2) *an object* $\zeta_0 = (\pi_0 : \mathcal{C}_0 \to \mathrm{T}, (p_i : \mathrm{T} \to \mathcal{C}_0)_{i\in\mathrm{I}_0} \cup (q_j : \mathrm{T} \to \mathcal{C}_0)_{j\in\mathrm{J}}, h_0 : \mathcal{C}_0 \to \mathrm{X})$ *of* $\Sigma^{e_0}_{\mathrm{I}_0\sqcup\mathrm{J}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$ *over* T, *and*
(3) *for* $j \in \mathrm{J}$, *an object* $\zeta_j = (\pi_j : \mathcal{C}_j \to \mathrm{T}, (p_i : \mathrm{T} \to \mathcal{C}_j)_{i\in\mathrm{I}_j} \cup (r_j : \mathrm{T} \to \mathcal{C}_j), h_j : \mathcal{C}_j \to \mathrm{X})$ *of* $\overline{\mathcal{M}}_{0,\mathrm{I}_j\sqcup\{j\}}(\mathrm{X}/\mathrm{C}, e_j)$ *as in Notation 5.8 over* T.

These data should satisfy the requirement that $h_0 \circ q_j$ *equals* $h_j \circ r_j$ *as morphisms* $\mathrm{T} \to \mathrm{X}$ *for every* $j \in \mathrm{J}$.

Morphisms of $\underline{\mathrm{I}}$-pointed, degree $\underline{e}$ combs in X/C/S are defined in the obvious way. The category of families of $\underline{\mathrm{I}}$-pointed, degree $\underline{e}$ combs in X/C/S with pullback diagrams as morphisms is an S-stack denoted $\mathrm{Comb}^{\underline{e}}_{\underline{\mathrm{I}}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$.

Given a family $(\zeta_0, (\zeta_j)_{j\in\mathrm{J}})$ of combs, the family ζ_0 is the *handle* and the families $(\zeta_j)_{j\in\mathrm{J}}$ are the *teeth*. There is a "forgetful" 1-morphism

$$\Phi_{\mathrm{handle}} : \mathrm{Comb}^{\underline{e}}_{\underline{\mathrm{I}}}(\mathrm{X}/\mathrm{C}/\mathrm{S}) \longrightarrow \Sigma^{e_0}_{\mathrm{I}_0\sqcup\mathrm{J}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$$

called the *handle* 1*-morphism*. Similarly, for every $j \in \mathrm{J}$ there is a 1-morphism

$$\Phi_{\mathrm{tooth},j} : \mathrm{Comb}^{\underline{e}}_{\underline{n}}(\mathrm{X}/\mathrm{C}/\mathrm{S}) \longrightarrow \overline{\mathcal{M}}_{0,\mathrm{I}_j\sqcup\{j\}}(\mathrm{X}/\mathrm{C}, e_j)$$

called the j*th tooth* 1*-morphism*. Moreover as constructed in [BM96, Theorem 3.6], specifically Case II on p. 18, there is also a *total curve* 1*-morphism*

$$\Phi_{\mathrm{total}} : \mathrm{Comb}^{\underline{e}}_{\underline{\mathrm{I}}}(\mathrm{X}/\mathrm{C}/\mathrm{S}) \longrightarrow \Sigma^{e}_{\mathrm{I}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$$

obtained by attaching $r_j(\mathrm{T})$ in $\mathcal{C}_j$ to $q_j(\mathrm{T})$ in $\mathcal{C}_0$ for every $j \in \mathrm{J}$.

By definition, $\mathrm{Comb}^{\underline{e}}_{\underline{\mathrm{I}}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$ together with the handle and tooth 1-morphisms is the equalizer of the collection of marked point 1-morphisms associated to $(q_j, r_j)_{j\in\mathrm{J}}$. Since each of $\Sigma^{e_0}_{\mathrm{I}_0\sqcup\mathrm{J}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$ and $\overline{\mathcal{M}}_{0,\mathrm{I}_j\sqcup\{j\}}(\mathrm{X}/\mathrm{C}, e_j)$ is a proper, algebraic S-stack with finite diagonal, the same is true for the equalizer $\mathrm{Comb}^{\underline{e}}_{\underline{\mathrm{I}}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$.

The *canonical open substack*, $\mathrm{Comb}^{\underline{e}}_{\underline{\mathrm{I}},\mathrm{open}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$, is defined to be the inverse image under Φ_{handle} of the open substack $\mathrm{Sections}^{e_0}_{\mathrm{I}_0\sqcup\mathrm{J}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$ of $\Sigma^{e_0}_{\mathrm{I}_0\sqcup\mathrm{J}}(\mathrm{X}/\mathrm{C}/\mathrm{S})$.

Proposition **6.6.** — *In Situation 6.1. Given indexing triples* $(J', \underline{e}', \underline{I}')$ *and* $(J'', \underline{e}'', \underline{I}'')$, *the images of*

$$\Phi_{\mathrm{total}} : Comb^{\underline{e}'}_{\underline{I}',\mathrm{open}}(X/C/S) \to \Sigma^{e}_{I}(X/C/S)$$

and

$$\Phi_{\mathrm{total}} : Comb^{\underline{e}''}_{\underline{I}'',\mathrm{open}}(X/C/S) \to \Sigma^{e}_{I}(X/C/S)$$

intersect only if there exists a bijection $\phi : J' \to J''$ *such that* $e''_{\phi(j)} = e'_j$ *and* $I''_{\phi(j)} = I'_j$ *for every* $j \in J'$. *If such a bijection exists, the images are equal. Finally, for every algebraically closed field* k *over* S, *every object of* $\Sigma^{e}_{I}(X/C/S)$ *over Spec* k *is in the image of*

$$\Phi_{\mathrm{total}} : Comb^{\underline{e}}_{\underline{I},\mathrm{open}}(X/C/S) \to \Sigma^{e}_{I}(X/C/S)$$

for some collection $(J, \underline{e}, \underline{I})$.

Proof. — This is left to the reader. □

Lemma **6.7.** — *In Situation 6.1 assume* S *is excellent. For every integer* e *there exists a* 1*-morphism over* S

$$\alpha_{\mathcal{L}} : \Sigma^{e}(X/C/S) \longrightarrow Pic^{e}_{C/S}$$

with the following property: For every indexing pair $(J, \underline{e})$ *for* e *the composition*

$$Comb^{\underline{e}}_{\mathrm{open}}(X/C/S) \to \Sigma^{e}(X/C/S) \to Pic^{e}_{C/S}$$

on the level of objects over a scheme T *of finite type over* S *maps the object* (ζ_0, ζ_j) *as in Definition 6.5 to the class of the invertible sheaf*

$$h_0^*(\mathcal{L})\left(\sum\nolimits_{j\in J} e_j \underline{q_j(T)}\right)$$

on $T \times_S C$.

Proof. — We use the material preceding Definition 3.11. Any flat proper family of nodal curves is LCI in the sense of Definition 3.7. Thus there is a morphism of stacks

$$\Sigma^{e}(X/C/S) \longrightarrow CurveMaps_{\mathrm{LCI}}(C \times B\mathbf{G}_m)$$

which associates to $(T \to S, \pi : C' \to T, h : C' \to X)$ the object $(T \to S, \pi : C' \to T, f \circ h : C' \to C, h^*\mathcal{L})$. Hence we may use the construction of Definition 3.11 which gives a 1-morphism

$$CurveMaps_{\mathrm{LCI}}(C \times B\mathbf{G}_m) \longrightarrow Pic_{C/S}.$$

To prove this map satisfies the property expressed in the lemma it suffices to apply Lemma 3.12. □

7. Peaceful chains

In this section we study free lines and free chains of lines on a family of projective varieties. Their key feature is that they are unobstructed. An important notion is that of a peaceful point, see Definition 7.4. The key feature is that these are connected to any general point by a chain of free lines. Hypothesis 7.7 is our notion of being rationally simply connected by chains of free lines.

Let S be an excellent scheme. Let $f : X \to S$ be a proper, flat morphism with geometrically irreducible fibres. Let $\mathcal{L}$ be an f-ample invertible sheaf on X. Recall (5.8) that $\overline{\mathcal{M}}_{0,n}(X/S, 1)$ parametrizes stable n-pointed genus 0 degree 1 maps from connected nodal curves to fibres of $X \to S$. Over a geometric point $s : \mathrm{Spec}(k) \to S$ such a stable map $(C/k, p_1, \dots, p_n \in C(k), h : C \to X_s)$ is nonconstant on exactly one irreducible component L of C because $\mathcal{L}$ is ample. Then, $L \cong \mathbf{P}^1_k$, the map $L \to X_s$ is birational onto its image and corresponds to a point of $\overline{\mathcal{M}}_{0,0}(X/S, 1)$. We will call $L \to X_s$ a *line*, and $(C/k, p_1, \dots, p_n \in C(k), h : C \to X_s)$ an *n-pointed line*. If $n = 0$ or $n = 1$ then $L = C$. If $n = 2$ it may happen that $C = L \cup C'$, with $p_1, p_2 \in C'(k)$ and where C' is contracted. However, one gets the same algebraic stack (in the case $n = 2$) by considering moduli of triples $(L, p_1, p_2 \in L(k), h : L \to X_s)$ where p_1 and p_2 are allowed to be the same point of L. We will use this below without further mention. For any $n \geq 0$ the objects of the stack $\overline{\mathcal{M}}_{0,n}(X/S, 1)$ have no nontrivial automorphisms, as is clear from their description above. Hence $\overline{\mathcal{M}}_{0,n}(X/S, 1)$ is actually an algebraic space.

Definition **7.1.** — *Notation as above. An n-pointed line is* free *if the morphism* $L \to X_s \to X$ *factors through the smooth locus of* X/S *and the pullback of* T_{X_s} *via the morphism* $L \to X_s$ *is globally generated.*

We leave it to the reader to show that the locus of free lines defines an open subspace of $\overline{\mathcal{M}}_{0,n}(X/S, 1)$. For every integer n there is an algebraic space $\mathrm{FreeChain}_2(X/S, n)$ over S parametrizing 2-pointed chains of n free lines in fibres of f. To be precise, $\mathrm{FreeChain}_2(X/S, n)$ is an open subset of the n-fold fibre product[2]

$$\overline{\mathcal{M}}_{0,2}(X/S, 1) \times_{ev_2, X, ev_1} \overline{\mathcal{M}}_{0,2}(X/S, 1) \times_{ev_2, X, ev_1} \cdots \times_{ev_2, X, ev_1} \overline{\mathcal{M}}_{0,2}(X/S, 1).$$

The points of $\mathrm{FreeChain}_2(X/S)$ over a geometric point $s : \mathrm{Spec}(k) \to S$ correspond to n-tuples of 2-pointed lines $(C_i/k, p_{i,1}, p_{i,2} \in C_i, h_i : C_i \to X_s)$ such that $h_i(p_{i,2}) = h_{i+1}(p_{i+1,1})$ for $i = 1, \dots, n-1$ and such that each of the 2-pointed lines is free. We will sometime denote this by $(C/k, p, q \in C(k), h : C \to X_s)$. This means that C is the union of the curves C_i with $p_{i,2}$ identified with $p_{i+1,1}$, with h equal to h_i on C_i and $p = p_{1,1}$ and $q = p_{n+1,2}$. Note that there are $n+1$ natural evaluation morphisms

$$\mathrm{ev}_i : \mathrm{FreeChain}_2(X/S) \longrightarrow X, \quad i = 1, \dots, n+1$$

[2] This fibre product will be denoted $\mathrm{Chain}_2(X/C, n)$ later on.

over S, namely $\mathrm{ev}_i([(C_i/k, p_{i,1}, p_{i,2} \in C_i, h_i : C_i \to X_s)]_{i=1,\dots,n}) = h_i(p_{i,1})$ for $i = 1, \dots, n$ and ev_{n+1} on the same object evaluates to give $h_n(p_{n,2})$. In the variant notation described above we have $\mathrm{ev}_1((C/k, p, q \in C(k), h : C \to X_s)) = h(p)$ and $\mathrm{ev}_{n+1}((C/k, p, q \in C(k), h : C \to X_s)) = h(q)$. Note that these morphisms map into the *smooth* locus of the morphism $X \to S$.

Lemma **7.2.** — *With notation as above. For every positive integer n and for every integer* $i = 1, \dots, n+1$*, the evaluation morphism* $\mathrm{ev}_i : \mathit{FreeChain}_2(X/S, n) \longrightarrow X$ *is a smooth morphism. In particular* $\mathit{FreeChain}_2(X/S, n)$ *is smooth over* S.

Proof. — We claim the morphism $ev : \overline{\mathcal{M}}_{0,1}(X/S, 1) \to X$ is smooth on the locus of free lines. This is true because, given a free 1-pointed line $h : L \to X_s$, $p \in L(k)$ the corresponding deformation functor has obstruction space $H^1(L, h^*T_{X_s}(-p))$. This is zero as $h^*T_{X_s}$ is globally generated by assumption. A reference for the case where S is a point is [Kol96, Proposition II.3.5]. Combined with a straightforward induction argument, this proves that the morphisms ev_i are smooth. □

Lemma **7.3.** — *Notation as above. Suppose that* $s : \mathit{Spec}(k) \to S$ *is a geometric point. Suppose that* $(C/k, p, q \in C(k), h : C \to X_s)$ *is a free* 2*-pointed chain of lines of length n. If* $\mathrm{ev}_{1,n+1} : \mathit{FreeChain}_2(X/S, n) \to X \times_S X$ *is smooth at the corresponding point then:*

1. $H^1(C, h^*T_{X_s}(-p-q)) = 0$*, and*
2. *for any free line* $h_L : L \to X_s$ *passing through* $h(q)$ *and any point* $r \in L$ *the morphism* $\mathrm{ev}_{1,n+2} : \mathit{FreeChain}_2(X/S, n+1) \to X \times_S X$ *is smooth at the point corresponding to the chain of free lines of length* $n+1$ *given by* $(C \cup L, p, r \in (C \cup L)(k), h \cup h_L)$.

Proof. — If the morphism $\mathrm{ev}_{1,n+1}$ is smooth then in particular the map $h^*T_{X_s} \to h^*T_{X_s}|_p \oplus h^*T_{X_s}|_q$ is surjective on global sections. Using freeness of the chain this implies statement (1). Let $C' = C \cup L$, and $h' = h \cup h_L$. Part (1) and the freeness of L imply that $H^1(C', (h')^*T_{X_s}(-p-r))$ is zero. Part (2) follows by deformation theory from this vanishing. □

Definition **7.4.** — *In the situation above. A geometric point* $x \in X_s(k)$ *lying over the geometric point* $s : \mathit{Spec}(k) \to S$ *is* peaceful *if both*

1. *the evaluation morphism* $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/S, 1) \to X$ *is smooth at every point in* $\mathrm{ev}^{-1}(x)$*, and*
2. *for all* $n \gg 0$*, the evaluation morphism* $\mathrm{ev}_{1,n+1} : \mathit{FreeChain}_2(X/S, n) \to X \times_S X$ *is smooth at some point* $(C/k, p, q \in C(k), h : C \to X_s)$ *with* $h(p) = x$.

The second condition implies that there is at least one free line through every peaceful point, and hence that a peaceful point lies in the smooth locus of $X \to S$. The first implies, upon considering the derivative of ev that all lines through a peaceful point are free. Note that there may be situations where there are no peaceful points whatsoever.

Lemma **7.5.** — *In the situation above. The set* $X_{f,\mathrm{pax}}$ *of peaceful points of* X *is an open subset of* X*. Furthermore, there exists an* n_0 *such that the second condition of Definition 7.4 holds for all peaceful points and all* $n \geq n_0$.

Proof. — Denote by Z the singular set of the morphism $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/S, 1) \to X$. Since ev is proper, ev(Z) is a closed subset of X. Thus the complement $U' = X \setminus \mathrm{ev}(Z)$ is the maximal open subset of X such that $\mathrm{ev} : \mathrm{ev}^{-1}(U') \to U'$ is smooth. For every n, denote by V_n the open subset of $\mathrm{FreeChain}_2(X/S, n)$ on which

$$\mathrm{ev}_{1,n+1} : \mathrm{FreeChain}_2(X/S, n) \to X \times_S X$$

is smooth. The image $V'_n := \mathrm{ev}_1(V_n)$ is open by Lemma 7.2. By Lemma 7.3 we have $V'_1 \subset V'_2 \subset \ldots$. Since S and hence X is Noetherian we find n_0 such that $V_{n_0} = V_{n_0+1} = \ldots$. It is clear that the set $V' := U' \cap \bigcup_{n\geq 1} V'_n$ is open and equals $X_{f,\mathrm{pax}}$. □

Remark **7.6.** — In fact, consider the open subset $W_n \subset X \times_S X$ above which the morphism $\mathrm{ev}_{1,n+1} : \mathrm{FreeChain}_2(X/S, n) \to X \times_S X$ has some smooth point. This is an increasing sequence by Lemma 7.3 as well and hence stabilizes to W_{n_1} for some n_1.

Definition **7.7.** — *In the situation above. Let* U *be an open subset of* $X_{f,\mathrm{pax}}$*. Let* $n > 0$*. A* U-adapted free chain of n lines *is a point* $[h] \in \mathit{FreeChain}_2(X/S, n)$ *such that* $\mathrm{ev}_i([h]) \in U$ *for all* $i = 1, \ldots, n + 1$*. When* U *is all of* $X_{f,\mathrm{pax}}$*, this is sometimes called a* peaceful chain*. The subset of* $\mathit{FreeChain}_2(X/S, n)$ *parametrizing* U*-adapted free chains is denoted* $\mathit{FreeChain}_2^U(X/S, n)$*. When* U *is all of* $X_{f,\mathrm{pax}}$ *we sometimes use the notation* $\mathit{PeaceChain}_2(X/S, n)$.

Without more hypotheses on our morphism $f : X \to S$ and relatively ample invertible sheaf $\mathcal{L}$ we cannot say much more (since after all the set of lines in fibres might be empty). A variety V over an algebraically closed field k is rationally connected if there exists a variety T and a morphism $T \times \mathbf{P}^1 \to V$ such that the induced morphism $T \times \mathbf{P}^1 \times \mathbf{P}^1 \to V \times V$ is dominant. A variety is birationally rationally connected if it is birational to a rationally connected variety. Note that $\mathbf{A}^n_k$ is birationally rationally connected but not rationally connected. On the other hand, a proper birationally rationally connected variety is rationally connected.

Hypothesis **7.8.** — *Let* S *be an excellent scheme of characteristic* 0*. Let* $f : X \to S$ *be a proper flat morphism with geometrically irreducible fibres. Let* $\mathcal{L}$ *be* f*-ample. In addition we assume the following.*

(1) *There exists an open subset* U *of* X *surjecting to* S *and such that the geometric fibres of the evaluation morphism*

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/S, 1) \to X$$

over U *are nonempty, irreducible and rationally connected.*

(2) *There exists a positive integer* m_0 *and an open subset* V *of* $X \times_S X$ *surjecting to* S *and such that the geometric fibres of the evaluation morphism*

$$\mathrm{ev}_{1,n+1} : \mathit{FreeChain}_2(X/S, m_0) \longrightarrow X \times_S X$$

over V *are nonempty, irreducible and birationally rationally connected.*

Note that condition (2) in particular implies that the smooth locus of f intersects every fibre of f. Namely, the open V is contained in $\mathrm{Smooth}(X/S) \times_S \mathrm{Smooth}(X/S)$ by the definition of free lines. In addition condition (2) shows that the smooth locus of each geometric fibre of $X \to S$ is rationally chain connected by free rational curves, and hence rationally connected.

Lemma **7.9.** — *In Situation 7.8 the set* $X_{f,\mathrm{pax}}$ *of peaceful points is dense in every smooth fibre of* $X \to S$. *(Warning: There may not be any smooth fibres.)*

Proof. — First replace S by the open subscheme over which the morphism is smooth. In characteristic 0 lines passing through a general point of a smooth variety are free, see [KMM92a, Proposition 1.1]. Hence the open U' of the proof of Lemma 7.5 has a nonempty intersection with every fibre of f. On the other hand, condition 2 of 7.8 implies that for every point s of S the morphism $\mathrm{FreeChain}_2(X/S, m_0)_s \to (X \times_S X)_s$ is a dominant morphism of smooth varieties. Hence it is smooth at some point $\xi \in \mathrm{FreeChain}_2(X/S, m_0)_s$ by generic smoothness. The "critère de platitude par fibre" [Gro67, Theorem 11.3.10] implies that $\mathrm{ev}_{1,n+1}$ is flat at ξ and hence smooth as desired. Thus the open V'_{m_0} of the proof of lemma 7.5 has nonempty fibre $V'_{m_0,s}$. We win because $X_{f,\mathrm{pax}} \supset U' \cap V'_{m_0}$. ☐

Remark **7.10.** — In Situation 7.8 if X_s is a geometric fibre such that for a general point $x \in X_s$ the lines through x avoid the singular locus of X_s, then the set of peaceful points is dense in X_s also. Hypothesis 7.8 holds if X is a cubic hypersurface with a single ordinary double point in $\mathbf{P}^9_k$ over $S = \mathrm{Spec}\, k$. In this case lines through a general point do not pass through the double point and we conclude that the peaceful points are dense. We will not use this remark in the sequel.

Lemma **7.11.** — *In Situation 7.8.*

(1) *The geometric fibres of* $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/S, 1) \to X$ *over* $X_{f,\mathrm{pax}}$ *are proper, smooth and rationally connected.*
(2) *If* U *is a nonempty open subset of* $X_{f,\mathrm{pax}}$, *then* $\mathit{FreeChain}_2^U(X/S, n)$ *is an open dense subset of* $\mathit{FreeChain}_2(X/S, n)$. *In fact, it is dense in the fibre of* $\mathit{FreeChain}_2(X/S, n) \to S$ *over any point* s *for which* $U_s \neq \emptyset$.
(3) *For every integer* $i = 1, \ldots, n+1$, *each morphism*

$$\mathrm{ev}_i : \mathit{FreeChain}_2^U(X/S, n) \longrightarrow U,$$

is a smooth morphism and every geometric fibre is nonempty, irreducible and birationally rationally connected.

Proof. — Since $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/S, 1) \to X$ is a proper morphism and since the geometric generic fibre of ev is rationally connected, every fibre in the smooth locus of ev is rationally connected by [KMM92b, 2.4] and [Kol96, Theorem IV.3.11]. In particular, the fibre over every peaceful point is in the smooth locus and thus is rationally connected. This proves (1).

By Lemma 7.2, the morphisms ev_i of (3) are smooth. Thus the preimages $\mathrm{ev}_i^{-1}(U)$ are dense open subsets. It follows that the finite intersection $\mathrm{FreeChain}_2^U(X/S, n) = \bigcap_{i=1}^{n+1}(\mathrm{ev}_i^{-1}(U))$ is also a dense open subset. The same argument works for fibres. This proves (2).

Next let x be a geometric point of U. The fibre of

$$\mathrm{ev}_1 : \mathrm{FreeChain}_2^U(X/S, n) \longrightarrow U$$

over x is the total space of a tower of birationally rationally connected fibrations. Indeed, the variety parametrizing choices for the first line L_1 of the chain containing $x = p_{1,1}$ is the fibre of $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/S, 1) \to X$ over x. Next, given L_1, the variety parametrizing choices for the attachment point $p_{1,2}$ is the intersection of L_1 with U, which is birationally rationally connected since it is open in $L_1 \cong \mathbf{P}^1$. Next, the variety parametrizing lines L_2 containing $p_{1,2} = p_{2,1}$ is rationally connected for the same reason that the variety parametrizing lines L_1 is rationally connected, etc. By [GHS03], the total space of a tower of birationally rationally connected fibrations is itself birationally rationally connected. Thus the geometric fibres of ev_1 are birationally rationally connected. The proof of (3) for the other morphisms ev_i is similar. □

Proposition **7.12.** — *In Situation 7.8. There exists a positive integer m_1 such that for every integer $n \geq m_1$ and for every nonempty open subset U of $X_{f,\mathrm{pax}}$ there exists an open subset U_n of $\mathit{FreeChain}_2^U(X/S, n)$ whose intersection with every geometric fibre of*

$$\mathrm{ev}_{1,n+1} : \mathit{FreeChain}_2^U(X/S, n) \longrightarrow U \times_S U$$

is nonempty, smooth, irreducible and birationally rationally connected.

Proof. — Let n_0 be the integer of Lemma 7.5. Let m_0 be the integer of Hypothesis 7.8. We claim that $m_1 = 2n_0 + m_0$ works.

Namely, suppose that $n \geq m_1$. Write $n = a + m_0 + b$, with $a \geq n_0$ and $b \geq n_0$. Consider the open sets

$$A = \mathrm{ev}_{a+1}(\mathrm{FreeChain}_2^U(X/S, a)) \subset X, \quad \text{and}$$
$$B = \mathrm{ev}_1(\mathrm{FreeChain}_2^U(X/S, b)) \subset X.$$

These have nonempty fibre over every point of S over which the fibre of $U \to S$ is nonempty by Lemma 7.11. For B use that there is an automorphism on the space of chains which switches the start with the end of a chain. As a first approximation we let

$$U_n = ev_{a+1}^{-1}(A) \cap ev_{a+1+m_0}^{-1}(B) \cap ev_{a+1,a+1+m_0}^{-1}(V) \subset \mathrm{FreeChain}_2^U(X/S, n).$$

Here $V \subset X \times_S X$ is the open set of Hypothesis 7.8. Let us describe the geometric points of U_n. Let $s : \mathrm{Spec}(k) \to S$ be a geometric point of S such that $U_s \neq \emptyset$. A point $\xi \in U_n(k)$ is given by the following data:

(1) a point $x \in A_s(k)$, a point $y \in B_s(k)$, such that $(x, y) \in V_s(k)$,
(2) a U-adapted free chain of a lines ξ_a with end point at x,
(3) a U-adapted free chain of b lines ξ_b with start point at y, and
(4) a U-adapted 2-pointed free chain ξ_m of m_0 lines connecting x to y.

The morphism $U_n \to U \times_S U$ maps ξ as above to the pair $(x', x'') \in U_s(k)^2$ consisting of the start point x' of ξ_a and the end point x'' of ξ_b. According to Lemma 7.11 the set of choices of ξ_a and ξ_b, given x' and x'', is a nonempty smooth irreducible, birationally rationally connected variety. For every pair (ξ_a, ξ_b) we get a pair of points (x, y) as above and since $V_s \neq \emptyset$ it is a nonempty open condition to have $(x, y) \in V_s(k)$. By the Hypothesis 7.8 the set of choices of ξ_m connecting x with y forms an irreducible, birationally rationally connected variety. Using [GHS03], it follows that the fibre of U_n over (x', x'') is smooth and birationally rationally connected.

We are not yet assured that $ev_{1,n+1} : U_n \to U \times_S U$ is smooth. Let $U'_n \subset U_n$ be the smooth locus of this morphism. We claim that $U'_n \subset U_n$ is dense in every geometric fibre $ev_{1,n+1}^{-1}((x', x''))$ we studied above, which will finish the proof of the proposition. By irreducibility of this fibre it suffices to show U'_n intersects the fibre. Since $a \geq n_0$ we know by Lemma 7.5 that there is a chain ξ_a as above which represents a point where $ev_{1,a+1}$ is smooth. Similarly for ξ_b. By openness of smoothness, we may assume the associated pair (x, y) is an element of $V_s(k)$. Pick any free chain ξ_m connecting x and y as above. We claim that the resulting chain of length $a + m_0 + b$ represents a smooth point for the morphism $ev_{1,n+1}$. The last part of the proof is similar to the proof of Lemma 7.3 and uses the results of that lemma. Namely, ξ_m corresponds to a chain of lines $L_1, \ldots, L_n$ and points $p_{i,j}$, $i \in \{1, \ldots, n\}$, $j \in \{1, 2\}$, such that (where we simply write T_{X_s} to denote the pullback) $H^1(L_1 \cup \ldots \cup L_a, T_{X_s}(-p_{1,1} - p_{a,2})) = 0$, $H^1(L_{a+m_0+1} \cup \ldots \cup L_n, T_{X_s}(-p_{a+m_0+1,1} - p_{n,2})) = 0$, and the lines $L_{a+1}, \ldots, L_{a+m_0}$ are free. It is easy to see that this implies that $H^1(L_1 \cup \ldots \cup L_n, T_{X_s}(-p_{1,1} - p_{n,2})) = 0$. This finishes the proof as in Lemma 7.3. □

8. Porcupines

In this section we define the notion of a porcupine for a family X/C of varieties over a curve. Using porcupines and an irreducible component Z of $\Sigma(X/C/k)$

parametrizing a free section Lemma 8.4 produces a sequence of irreducible components Z_e of $\Sigma(X/C/k)$. Namely Z_e is the component containing porcupines of degree e whose body is in Z. It will turn out later that this sequence is asymptotically canonical, i.e., independent of the choice of Z, see Corollary 9.8. An interesting fact is that the Abel map restricted to Z_e has irreducible geometric generic fibre when $e \gg 0$, see Lemma 8.9 and Remark 8.10.

Hypothesis **8.1.** — *Here k is an uncountable algebraically closed field of characteristic* 0. *Let* C *be a smooth, irreducible, proper curve over k. Let* X *be smooth and proper over k. Let $f : X \to C$ be proper and flat with geometrically irreducible fibres. The invertible $\mathcal{O}_X$-module $\mathcal{L}$ is f-ample.*

We are going to apply the material of the previous section to this situation. In particular, let $X_{f,\mathrm{pax}}$ be the set of peaceful points relative to $f : X \to C$, see Lemma 7.5. Here is one more type of stable map that arises often in what follows.

Definition **8.2.** — *Notation as above. Let e be an integer and let n be a nonnegative integer. A degree e* porcupine with n quills *over k is given by the following data*

1. *a section $s : C \to X$ of degree e with respect to $\mathcal{L}$,*
2. *n distinct points $q_1, \ldots, q_n \in C(k)$, and*
3. *n 1-pointed lines $(r_i \in L_i(k), L_i \to X_{q_i})$.*

These data have to satisfy the requirements that (a) the set $s^{-1}(X_{f,\mathrm{pax}}) \neq \emptyset$ and each q_i is in this open set, (b) the points r_i and q_i get mapped to the same k point of X*, and (c) the section s is free (see Definition 4.7).*

It may be useful to discuss this a bit more. The first remark is that by our definitions the image of s lies in the smooth locus of X/C and all the lines L_i are free and lie in the smooth locus as well. In other words, the union $C \cup (\bigcup L_i) \to X$ is going to be a comb (see Definition 6.5) in the smooth locus. We will call the handle $s : C \to X$ the *body* of the porcupine, and we will call the teeth $L_i \to X$ the *quills*.

Families of degree e porcupines with n quills are defined in the obvious manner; for example we may observe that the space of porcupines of degree e and n quills is an open substack of the moduli stack $\mathrm{Comb}^{\underline{e}}_{\underline{I}}(X/C/k)$ with $\underline{I} = (\emptyset, \emptyset, \ldots, \emptyset)$, and $\underline{e} = (e, 1, \ldots, 1)$ of combs studied in Section 6. The parameter space for porcupines is denoted $\mathrm{Porcupine}^{e,n}(X/C/k)$.

Denote by

$$\begin{array}{llll}
\Phi_{\mathrm{total}} : & \mathrm{Porcupine}^{e,n}(X/C/\kappa) & \longrightarrow & \Sigma^{e+n}(X/C/\kappa), \\
\Phi_{\mathrm{body}} : & \mathrm{Porcupine}^{e,n}(X/C/\kappa) & \longrightarrow & \mathrm{Sections}^{e}_{n}(X/C/k), \\
\Phi_{\mathrm{quill},i} : & \mathrm{Porcupine}^{e,n}(X/C/\kappa) & \longrightarrow & \overline{\mathcal{M}}_{0,1}(X/C, 1), \\
\Phi_{\mathrm{forget}} : & \mathrm{Porcupine}^{e,n+m}(X/C/\kappa) & \longrightarrow & \mathrm{Porcupine}^{e,n}(X/C/\kappa)
\end{array}$$

the obvious forgetful morphisms where Φ_{forget} forgets about $(r_i \in L_i(k), L_i \to X_{q_i})$ and q_i for $i > n$. Note that Φ_{forget} is smooth.

Lemma **8.3.** — *In Situation 8.1. Let $n \geq 1$. If the geometric generic fibre of f is rationally connected then there exists a n-free section of f. If the set of peaceful points is nonempty then we may assume the section meets this locus.*

Proof. — By Lemma 4.12 it suffices to construct a section that is D-free for some divisor D of degree $2g(C) + n$. Set $m = 2g(C) + n$. Consider the subsets W^m_e of $H^m_{X,f}$ introduced in Proposition 4.15. Since W^m_e is constructible and dense in $H^m_{X,f}$ there exists a nonempty open subset $U^m_e \subset H^m_{X,f}$, such that $U^m_e \subset W^m_e$. Moreover the proposition implies that a section $\tau : C \to X_{f,\text{smooth}}$ will be D-free for some effective divisor $D \subset C$ of degree m provided that the restriction $\tau|_D$ is in $\bigcap_e U^m_e$. By Lemma 4.17 (applied with $T = \operatorname{Spec}(k)$, $m = m$, $U = Y = C$, and $V = X_{f,\text{smooth}}$ and $W = X$) we can find a family of sections $\tau' : T' \times C \to X$ such that the rational map $\tau'_m : T' \times \operatorname{Sym}^m(C) \dashrightarrow H^m_{X,f}$ is dominant. Hence, for a very general $t' \in T'(k)$ and very general divisor $D \subset C$ of degree m we have $\tau|_D \in \bigcap U^m_e$ with $\tau = \tau'_{t'}$. We omit the proof of the last statement. □

Before we state the next lemma, we wish to remind the reader that an irreducible scheme is necessarily nonempty. Also, see Remark 8.10 below for a more "lightweight" variant of some of the following lemmas.

Lemma **8.4.** — *In Situation 8.1, assume in addition Hypothesis 7.8 for the restriction of f to some nonempty open subscheme $S \subset C$. Let e_0 be an integer and let Z be an irreducible component of $\Sigma^{e_0}(X/C/k)$ whose general point parametrizes a free section of f. For every integer $e \geq e_0$ there exists a unique irreducible component Z_e of $\Sigma^e(X/C/k)$ such that every porcupine with body in Z and with $e - e_0$ quills is parametrized by Z_e.*

Proof. — Take any point $[s]$ of Z corresponding to a section s which is free. By Proposition 4.13, the space $\Sigma^{e_0}(X/C/k)$ is smooth at the point $[s]$. Thus this point is contained in a unique irreducible component of $\Sigma^{e_0}(X/C/k)$, namely Z. By Lemma 4.13 free sections deform to contain a general point of X. The set of peaceful points is nonempty by Lemma 7.9. Hence there are also points $[s]$ of Z which correspond to free sections s which intersect the peaceful locus. For every such section s, the inverse image $s^{-1}(X_{f,\text{pax}})$ is a dense open subset of C. Thus the variety parametrizing $(e - e_0)$-tuples of closed points $(q_1, \ldots, q_{e-e_0})$ in $s^{-1}(X_{f,\text{pax}})$ is smooth and irreducible. By definition of peaceful the fibre of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/C, 1) \to X$$

over $s(q_i)$ is smooth and by Lemma 7.11 it is irreducible. Putting the pieces together, the variety $W_{Z,e}$ parametrizing degree e porcupines with body in Z is smooth and irreducible. Moreover, because the body is unobstructed and because the teeth are all free, each such porcupine is unobstructed. Thus $\Sigma^e(X/C/k)$ is smooth at every point of $W_{Z,e}$. Thus $W_{Z,e}$ is contained in a unique irreducible component Z_e of $\Sigma^e(X/C/k)$. ☐

Lemma **8.5.** — *Notation and assumptions as in Lemma 8.4. Let $e \geq e_0$. A general point of Z_e corresponds to a section of $X \to C$ which is free and meets $X_{f,\mathrm{pax}}$.*

Proof. — Another way to state this lemma is that (1) we can smooth porcupines, and (2) a smoothing of a porcupine is a porcupine. Consider a porcupine $(s, r_i \in L_i(k), L_i \to X)$. Let $h : C \cup \bigcup L_i \to X$ be the corresponding stable map. We noted in the proof of Lemma 8.4 above that the map h is unobstructed. Hence any smoothing of the comb $C \cup \bigcup L_i \to C$ can be followed by a deformation of h. We can realize $C \cup \bigcup L_i$ as the special fibre of a family of curves over C whose general fibre is C, simply by blowing up $C \times \mathbf{P}^1$ in suitable points of the fibre $C \times \{0\}$. Thus we can smooth the stable map h.

Consider a general smoothing $s' : C \to X$ of h. The statement on the intersection with $X_{f,\mathrm{pax}}$ holds since this is clearly an open condition on all of Z_e. To see that the smoothing s' of $(s, r_i \in L_i(k), L_i \to X)$ is free we will use Lemma 3.13. It suffices to show for any effective Cartier divisor $D \subset C$ of degree 1 that $H^1(C \bigcup L_i, h^*T_f(-D)) = 0$. And this is immediate from the assumption that $H^1(C, s^*T_f(-D)) = 0$ and $T_f|_{L_i}$ is globally generated. ☐

Lemma **8.6.** — *Notation and assumptions as in Lemma 8.4. For any $e' \geq e \geq e_0$, any porcupine of total degree e' with body in Z_e and $e' - e$ quills corresponds to a point of $Z_{e'}$.*

Proof. — Let $[s] \in Z_e$ be the moduli point corresponding to a section of $X \to C$ which meets $X_{f,\mathrm{pax}}$ and is free. Because Z_e is irreducible, and because of the construction of Z_e we can find an irreducible curve T, a morphism $h : T \to Z_e$, and points $0, 1 \in T(k)$ such that $h(0) = [s]$ and $h(1)$ corresponds to a point of $W_{Z,e}$ (see proof of Lemma 8.4). After deleting finitely many points $\neq 0, 1$ from T we may assume every point $t \in T(k)$, $t \neq 1$ corresponds to a section s_t which is free (see Lemma 4.8) and meets $X_{f,\mathrm{pax}}$. Thus every point $h(t) \in Z_e$ corresponds to a porcupine. Consider the scheme $T' \to T$ parametrizing choices of $e' - e$ quills attached to the body of the porcupine $h(t)$, $t \in T(k)$ in pairwise distinct points of $X_{f,\mathrm{pax}}$. As in the proof of Lemma 8.4 there is an irreducible parameter space of choices, in other words T' is irreducible. Also $T' \subset \Sigma^{e'}(X/C/k)$ lies in the smooth locus, because each point of T' corresponds to a porcupine. The result follows because $T'_1 \subset Z_{e'}$ by definition of $Z_{e'}$ and T'_0 parametrizes porcupines with body s. ☐

Lemma **8.7.** — *Notation and assumptions as in Lemma 8.4. Let $e \geq e_0$. Let $[s] \in Z_e(k)$ be a point corresponding to a 0-free section $s : C \to X$. Let $(h_i : C_i \to X, p_i \in C_i(k))$ be a finite number of 1-pointed chains of free lines in fibres of f. Assume $h_i(p_i) \in s(C)$ and assume $h_i(p_i)$ pairwise distinct. Then $s(C) \cup \bigcup C_i$ defines an unobstructed nonstacky point of $\Sigma^{e'}(X/C/k)$ which lies in $Z_{e'}$ with $e' = e + \sum \deg(C_i)$.*

Proof. — There are no automorphisms of the stable curve $s(\mathrm{C}) \cup \bigcup \mathrm{C}_i$, see our discussion of lines in Section 7. Since s is 0-free and each line in each chain C_i is free, there are no obstructions to deforming $s(\mathrm{C}) \cup \bigcup \mathrm{C}_i$, hence $\Sigma^{e'}(\mathrm{X}/\mathrm{C}/k)$ is smooth at the corresponding point. In particular, this point lies in a unique irreducible component. The next goal is to prove that this irreducible component equals $\mathrm{Z}_{e'}$.

Because the evaluation morphisms $\mathrm{FreeChain}_1(\mathrm{X}/\mathrm{C}, n) \to \mathrm{X}$ are smooth (Lemma 7.2), we may replace s by any of its deformations. And by Lemma 8.5, after deforming we may assume that $s^{-1}(\mathrm{X}_{f,\mathrm{pax}})$ is nonempty and that s is free. By the smoothness of $\mathrm{FreeChain}_1(\mathrm{X}/\mathrm{C}, n) \to \mathrm{X}$ again we may deform the 1-pointed chains $(\mathrm{C}_i \to \mathrm{X}, p_i)$ and assume $p_i \in s(\mathrm{C}) \cap \mathrm{X}_{f,\mathrm{pax}}$.

The lemma follows from Lemma 8.6 by induction on the length of the chains of free lines attached to the body. Namely, let $\mathrm{L}_i \subset \mathrm{C}_i$ be the first line of the chain, i.e., the line that contains p_i. Let C'_i be the rest of the chain, and let $p'_i \in \mathrm{C}'_i$ be the attachment point (where it is attached to L_i). Any deformation of $s(\mathrm{C}) \cup \bigcup \mathrm{L}_i$ to a section $s' : \mathrm{C} \to \mathrm{X}$ lies in Z_{e_0+m} by construction. Here m is the number of chains C_i. By the smoothness of $\mathrm{FreeChain}_1(\mathrm{X}/\mathrm{C}, n) \to \mathrm{X}$ again we may deform the 1-pointed chains $(\mathrm{C}'_i \to \mathrm{X}, p'_i)$ along with the given deformation to obtain $p'_i \in s'(\mathrm{C})$. By induction the resulting deformation of $s'(\mathrm{C}) \cup \bigcup \mathrm{C}'_i$ is in $\mathrm{Z}_{e'}$ as desired. □

Lemma **8.8.** — *Notation and assumptions are as in Lemma 8.4. For every integer $m \geq 0$ there exists an integer $\mathrm{E}(m)$ such that for every integer $e \geq \mathrm{E}(m)$, Z_e contains a point corresponding to a section which is m-free and meets $\mathrm{X}_{f,\mathrm{pax}}$.*

Proof. — Let $\mathrm{N} = m + 2g(\mathrm{C}) + 1$. Let n be an integer such that condition (2) of Definition 7.4 holds for all peaceful points and this choice of n, see Lemma 7.5. By Lemma 8.5, for every integer $e_1 \geq e_0$ there exists $[s] \in \mathrm{Z}_{e_1}$, a moduli point corresponding to a section $s : \mathrm{C} \to \mathrm{X}$ which is free and meets $\mathrm{X}_{f,\mathrm{pax}}$. Consider pairwise distinct points $c_1, \ldots, c_\mathrm{N} \in \mathrm{C}(k)$ such that $s(c_i) \in \mathrm{X}_{f,\mathrm{pax}}$. Choose 2-pointed free chains of lines $(h_i : \mathrm{C}_i \to \mathrm{X}, p_i, q_i \in \mathrm{C}_i(k))$ of length n such that $s(c_i) = h(p_i)$ and such that, with $z_i = h(q_i)$, the point $(z_1, \ldots, z_\mathrm{N})$ is a general point of $\mathrm{X}_{c_1} \times \cdots \times \mathrm{X}_{c_\mathrm{N}}$. This is possible by our choice of n above. By Lemma 8.7 any smoothing of the stable N-pointed map $s(\mathrm{C}) \cup \bigcup \mathrm{C}_i \to \mathrm{X}$ (marking given by the points q_i) defines a point of Z_e, where $e = e_1 + \mathrm{N}n$. The marking of a general smoothing will still determine a general point of $\mathrm{X}_{c_1} \times \cdots \times \mathrm{X}_{c_\mathrm{N}}$. Whence applying Proposition 4.15 we get that the resulting section is D-free with $\mathrm{D} = \sum c_i$ a divisor of degree N. By Lemma 4.12 we conclude with $\mathrm{E}(m) = e_0 + n\mathrm{N}$. □

Lemma **8.9.** — *Notation and assumptions are as in Lemma 8.4. For every $e \geq e_0 + g(\mathrm{C})$, the Abel map*

$$\alpha_{\mathcal{L}}|_{\mathrm{Z}_e} : \mathrm{Z}_e \longrightarrow \underline{Pic}^e_{\mathrm{C}/k}$$

is dominant with irreducible geometric generic fibre.

Proof. — Let n be an integer with $n \geq g(C)$ and let $e = e_0 + n$. Let $W_{Z,e}$ be as in the proof of Lemma 8.4. As in the proof of Lemma 8.4, every point of $W_{Z,e}$ is a smooth point of Z_e. Therefore to prove that

$$\alpha_{\mathcal{L}}|_{Z_e} : Z_e \longrightarrow \underline{\mathrm{Pic}}^e_{C/k}$$

is dominant with irreducible geometric generic fibre, it suffices to prove that the restriction

$$\alpha_{\mathcal{L}}|_{W_{Z,e}} : W_{Z,e} \longrightarrow \underline{\mathrm{Pic}}^e_{C/k}$$

is dominant with irreducible geometric generic fibre. And by Lemma 6.7, the morphism $\alpha_{\mathcal{L}}|_{W_{Z,e}}$ factors as follows

$$W_{Z,e} \to Z \times C^{e-e_0} \to Z \times \mathrm{Pic}^{e-e_0}_{C/k} \to \mathrm{Pic}^{e_0}_{C/k} \times \mathrm{Pic}^{e-e_0}_{C/k} \to \mathrm{Pic}^e_{C/k}$$

where each of the component maps is the obvious morphism. The first map has smooth, geometrically irreducible fibres as was explained in the proof of the Lemma 8.4 above. And since $e - e_0$ is $\geq g(C)$, the second is dominant with irreducible geometric generic fibre.

The trick is to factor the composition β of the last two maps. Namely, it is the composition of the map

$$(\mathrm{Id}_Z, \beta) : Z \times_k \mathrm{Pic}^{e-e_0}_{C/k} \to Z \times_k \mathrm{Pic}^e_{C/k}, \quad (s, [D]) \mapsto (s, \alpha_{\mathcal{L},Z}(s) + [D])$$

and the projection

$$\mathrm{pr}_2 : Z \times_k \mathrm{Pic}^e_{C/k} \to \mathrm{Pic}^e_{C/k}.$$

Since $(s, [D]) \mapsto (s, -\alpha_{\mathcal{L},Z}(s) + [D])$ is an inverse of (Id_Z, β), we see (Id_Z, β) is an isomorphism of schemes. Moreover, considering the source as a scheme over $\mathrm{Pic}^e_{C/k}$ via β and considering the target as a scheme over $\mathrm{Pic}^e_{C/k}$ via pr_2, it is an isomorphism of schemes over $\mathrm{Pic}^e_{C/k}$. Therefore every geometric fibre of β is isomorphic to Z, which is irreducible by hypothesis. So β is dominant with irreducible geometric generic fibre. Since a composition of dominant morphisms with irreducible geometric generic fibres is of the same sort, $\alpha_{\mathcal{L}}|_{W_{Z,e}}$, and thus $\alpha_{\mathcal{L}}|_{Z_e}$, are dominant with irreducible geometric generic fibres. □

Remark **8.10.** — The proof of Lemmas 8.4 and 8.9 works in a more general setting. Namely, suppose that k, $X \to C$ and $\mathcal{L}$ are as in 8.1. Redefine a porcupine temporarily by replacing condition (a) with the condition: all lines on X_{q_i} through $s(q_i)$ are free and they form an irreducible variety. Assume that for a general $c \in C(k)$ and a general $x \in X_c(k)$ the space of lines through x is nonempty and irreducible. Then the conclusions of Lemmas 8.4 and 8.9 hold.

9. Pencils of porcupines

In this section we start using ruled surfaces to connect Abel-equivalent porcupines by chains of rational curves in $\Sigma^e(X/C/k)$. In brief, two porcupines which are linearly equivalent Cartier divisors in a common ruled surface R determine a pencil of divisors on R which determines a morphism from the base of the pencil to $\Sigma^e(X/C/k)$. The main result is Proposition 9.7. A key step is that if the general fibre is rationally simply connected by chains of free lines, then a pair of sections is contained in a 2-dimensional scheme over C whose fibres are chains of free lines.

Let $k, f : X \to C$ and $\mathcal{L}$ be as in Hypothesis 8.1. A *ruled surface* or a *scroll* in X will be a morphism $R \to X$ such that $R \to C$ is proper smooth and all fibres are lines in X. The following will be used over and over again.

Lemma **9.1.** — *In Situation 8.1 any ruled surface* $R \to X$ *lies in the smooth locus of* $X \to C$.

Proof. — Since by assumption the total space of X is smooth any section of $X \to C$ lies in the smooth locus of $X \to C$. On the other hand, weak approximation holds for the ruled surface $R \to C$, a fortiori for any point $r \in R(k)$ we can find a section of $R \to C$ passing through r. The lemma follows. □

Let $h : C' = C \cup \bigcup L_i \to X$ be a porcupine as in Definition 8.2.

Definition **9.2.** — *Notations as above. A* pen *for* C' *is a ruled surface* R *such that* $C' \to X$ *factors through* $R \to X$.

We will always assume a pen for C' comes with a factorization. If lines in fibres of $X \to C$ are automatically smooth (e.g. if $\mathcal{L}$ is very ample on the fibres of $X \to C$) then R and C' are closed subschemes of X and the definition just means $C' \subset R$.

Lemma **9.3.** — *Let* C'_0 *and* C'_∞ *be porcupines whose bodies* $s_0(C)$ *and* $s_\infty(C)$ *are penned in a common ruled surface* R. *Let* (e_0, n_0), *resp.* (e_∞, n_∞) *be the numerical invariants of* C'_0, *resp.* C'_∞ *and assume* $e_0 + n_0 = e_\infty + n_\infty$. *Denote the attachment points of* C'_0, *resp. of* C'_∞, *by* $q_{0,i}$, *resp.* $q_{\infty,j}$. *The Abel images* $\alpha_{\mathcal{L}}([C'_0])$ *and* $\alpha_{\mathcal{L}}([C'_\infty])$ *are equal if and only if* $D_0 \sim D_\infty$ *in the divisor class group of* R *where*

$$D_0 := s_0(C) + \sum_{i=1}^{n_0} R_{q_{0,i}} \quad \textit{and} \quad D_\infty := s_\infty(C) + \sum_{j=1}^{n_\infty} R_{q_{\infty,j}}.$$

Here $R_{q_{0,i}}$, $R_{q_{\infty,j}}$ *denote the fibres of* R *over the points* $q_{0,i}$, $q_{\infty,j}$ *in* C.

Proof. — Note that Pic(R) is isomorphic to $\mathbf{Z} \oplus \mathrm{Pic}(C)$. As generator for the summand $\mathbf{Z}$ we take the class of $\mathcal{L}$. The divisors R_q on R correspond to the elements $\mathcal{O}_C(q)$ in Pic(C). Let K_{rel} denote the relative canonical divisor of $R \to C$. The divisor $s_0(C)$ corresponds to the element $-K_R + (-1, s_0^*\mathcal{L})$, by a calculation left to the reader. Hence

$D_0 \sim -K_r + (-1, s_0^*\mathcal{L}(\sum q_{0,i}))$ The argument is similar for s_∞. The result now follows from Lemma 6.7. □

In the above lemma, if D_0 and D_∞ are linearly equivalent, denote by $(D_\lambda)_{\lambda\in\Pi}$ the pencil of effective Cartier divisors on R spanned by D_0 and D_∞. At this point we start constructing families of porcupines over rational curves. In order to do this we state and prove a few lemmas.

Lemma **9.4.** — *In Situation 8.1 assume the restriction of f to some nonempty open subscheme* $S \subset C$ *satisfies Hypothesis 7.8. Suppose that* C' *and* C'' *are porcupines whose bodies and attachment points agree, but which may have different quills. Then there exists a rational curve in the smooth nonstacky locus of* $\Sigma^{e+n}(X/C/k)$ *connecting the corresponding points (where e is the degree of the body and n is the number of quills).*

Proof. — A porcupine always represents a smooth nonstacky point of $\Sigma^{e+n}(X/C/k)$. The space parametrizing choice of quills given the body s and the attachment points $q_1, \ldots, q_n$ is the product of n fibres of the evaluation map $\overline{\mathcal{M}}_{0,1}(X/C, 1) \to X$ at peaceful points. But these fibres are smooth projective rationally connected varieties by definition of peaceful and Lemma 7.11. □

In the following lemma and below we will say that the porcupine C'' is an *extension* of C' if you get C' from C'' by deleting some of its quills.

Lemma **9.5.** — *Same assumptions as in Lemma 9.4. Let* $s_0, s_\infty : C \to X$ *be sections. Assume* s_0 *and* s_∞ *are free, and that* $V = s_0^{-1}(X_{f,\mathrm{pax}}) \cap s_\infty^{-1}(X_{f,\mathrm{pax}})$ *is not empty. Assume* s_0 *and* s_∞ *are penned in a common ruled surface* R. *Then there exists a nonnegative integer* E *with the following property: for all* e *with* $e \geq \min\{E, \deg(s_0), \deg(s_\infty)\}$, *for any pairwise distinct points* $q_{0,1}, \ldots, q_{0,e-\deg(s_0)} \in C(k)$, *for any pairwise distinct points* $q_{\infty,0}, \ldots, q_{\infty,e-\deg(s_\infty)} \in C(k)$ *and for every extension* C'_0, *resp.* C'_∞, *of* s_0, *resp.* s_∞, *to a porcupine with quills over the points* $q_{0,1}, \ldots, q_{0,e-\deg(e_0)}$, *resp.* $q_{\infty,0}, \ldots, q_{\infty,e-\deg(e_\infty)}$ *if* $s_0^*(\mathcal{L})(\sum q_{0,i}) \cong s_\infty^*(\mathcal{L})(\sum q_{\infty,i})$ *as invertible sheaves on* C, *then the points* $[C'_0]$ *and* $[C'_\infty]$ *are connected in the stack* $\Sigma^e(X/C/k)$ *by a chain of rational curves whose nodes parametrize unobstructed, non-stacky stable sections. Moreover, the integer* E *is bounded above by* $\varphi(\deg(R), \deg(s_0))$ *where* $\varphi : \mathbf{Z}^2 \to \mathbf{Z}_{\geq 0}$ *is a function such that* $B \geq B' \Rightarrow \varphi(B, e) \geq \varphi(B', e)$.

Proof. — We first describe our choice of E. For any effective Cartier divisor $D \subset C$ denote $R_D \subset R$ the corresponding divisor in R. Note that there exists an integer E_1 such that the linear system $|s_0(C) + R_D|$ is base point free and very ample for all D with $\deg_C(D) \geq E_1$. This is true because $s_0(C)$ is relatively very ample for $R \to C$. We may also choose E_2 such that codimension 1 points of $|s_0(C) + R_D|$ parametrize nodal curves for all D with $\deg_C(D) \geq E_2$. This follows from a simple Bertini type argument once we have chosen E_2 big enough: namely, so big that

$$H^0(R, \mathcal{O}_R(s_0(C) + R_D)) \longrightarrow H^0(R_Z, \mathcal{O}_{R_Z}(s_0(C)))$$

is surjective for $Z \subset C$ any length 2 closed subscheme. As above, such an E_2 exists since $s_0(C)$ is relatively very ample. We will take $E = E_2$. The easiest way to see that $E \leq \varphi(\deg(R), \deg(s_0))$ with φ as in the lemma is to remark that the family of pairs $s_0 : C \to X$, $R \to X$ with bounded degrees is bounded.

Choose $e \geq E$. By Lemma 9.4 it suffices to prove the lemma assuming all the quills are in the surface R. Let $q_{i,j}$ be as in the statement of the lemma. Note that $s_0(C) + \sum R_{q_{0,j}} \sim s_\infty(C) + \sum R_{q_{\infty,j}}$ as divisors on R by Lemma 9.3.

Let us connect our points $s_0(C) + \sum R_{q_{0,j}} \in |s_0(C) + \sum R_{q_{0,j}}|$ and $s_\infty(C) + \sum R_{q_{\infty,j}} \in |s_0(C) + \sum R_{q_{0,j}}|$ by a pair of general lines Π_1, Π_2. In other words, Π_1 and Π_2 intersect, Π_1 contains the first point and Π_2 contains the second, but otherwise the lines are general. We claim that this chain of two rational curves in $\Sigma^e(X/C/k)$ satisfies the stated properties. Since the curves $s_0(C) + \sum R_{q_{0,j}}$ and $s_\infty(C) + \sum R_{q_{\infty,j}}$ are nodal and by our choice of e we see that all points of Π_1 and Π_2 correspond to nodal curves, hence automatically Π_1 and Π_2 correspond to families of stable sections. Finally, since the point $s_0(C) + \sum R_{q_{0,j}} \in \Pi_1$ corresponds to a smooth point of $\Sigma^e(X/C/k)$ we see that a general point of Π_1 corresponds to a smooth point as well. And since $\Pi_1 \cap \Pi_2$ is just a general point on Π_1 this node corresponds to a smooth point of $\Sigma^e(X/C/k)$ as desired. □

Here is one of the key technical lemmas of this paper.

Lemma **9.6.** — *In Situation 8.1 assume the restriction of f to some nonempty open $S \subset C$ satisfies Hypothesis 7.8. Let D be an effective Cartier divisor of degree $2g(C) + 1$ on C. Suppose we have sections s_0, s_∞ of $X \to C$ which are D-free.*[3] *Let T_0, resp. T_∞ be the unique irreducible component of Sections$(X/C/k)$ of which s_0, resp. s_∞ is a smooth point (see Proposition 4.13). There exists a dense open $W \subset T_0 \times T_\infty$, an integer $n \geq 1$, integers $e_1, \ldots, e_{n+1}$, B such that for every point $(\tau_0, \tau_\infty) \in W$ we have the following:*

1. *there exist sections $\tau_0 = \tau_1, \tau_2, \ldots, \tau_{n+1} = \tau_\infty$,*
2. *the degree of τ_i is e_i for $i = 1, \ldots, n+1$,*
3. *$\tau_i^{-1}(X_{f,\mathrm{pax}}) \neq \emptyset$ for $i = 1, \ldots, n+1$,*
4. *τ_i is free for $i = 1, \ldots, n+1$,*
5. *there exist ruled surfaces $R_i \to X$, $i = 1, \ldots, n$,*
6. *the degree of R_i (with respect to the relatively ample invertible sheaf $\mathcal{L}$) is at most B, and*
7. *τ_i, τ_{i+1} factor through R_i for $i = 1, \ldots, n$.*

Proof. — We will write $m = 2g(C) + 1$ in order to ease notation. Let $W_0 \subset T_0 \times \mathrm{Sym}^m(C)$, be the open subset corresponding to pairs (τ_0, D_0), such that τ_0 is D_0-free. Similarly we have $W_\infty \subset T_\infty \times \mathrm{Sym}^m(C)$. Our assumption is that W_0 and W_∞ are not

[3] It is not necessary to have the same divisor for the two sections.

empty. Recall the spaces $H^m_{X,f}$ defined above Proposition 4.15. By Lemma 4.14 the maps

$$res_0 : W_0 \to H^m_{X,f}, \quad (\tau_0, D_0) \mapsto (\tau_0|_{D_0} : D_0 \hookrightarrow X),$$
$$res_\infty : W_\infty \to H^m_{X,f}, \quad (\tau_\infty, D_\infty) \mapsto (\tau_\infty|_{D_\infty} : D_\infty \hookrightarrow X)$$

are smooth. Let $e_0 = \deg(s_0)$ and $e_\infty = \deg(s_\infty)$.

Let $S \subset C$ be the nonempty open over which Hypothesis 7.8 holds. Denote $U = X_{f,\mathrm{pax}}$. Consider the morphism

$$ev_{1,n+1} : \mathrm{FreeChain}^U_2(X/C, n) \longrightarrow U \times_S U \subset X \times_C X$$

we studied in Section 7. By Proposition 7.12 there exists an n, and an open $V \subset \mathrm{FreeChain}^U_2(X/C, n)$ such that $ev_{1,n+1}|_V : V \to U \times_S U$ is surjective, smooth, with irreducible and birationally rationally connected fibres. Also, the morphisms $ev_i|_V : V \to U$ are smooth, see Lemma 7.11.

The commutative diagram

$$\begin{array}{ccccc} C \times T_0 \times T_\infty & \xrightarrow{\tau_0 \times \tau_\infty} & X \times_C X & \longleftarrow & U \times_S U \\ & \searrow & \downarrow & \swarrow & \\ & & C & & \end{array}$$

is a diagram as in Situation 4.16. The associated rational map

$$(\tau_0 \times \tau_\infty)_m : \mathrm{Sym}^m(C) \times T_0 \times T_\infty \dashrightarrow H^m_{U\times_S U} = H^m_U \times_{\mathrm{Sym}^m(C)} H^m_U,$$

see (4.4), is the product map

$$\mathrm{Sym}^m(C) \times T_0 \times T_\infty \supset W_0 \times_{\mathrm{Sym}^m(C)} W_\infty \longrightarrow H^m_U \times_{\mathrm{Sym}^m(C)} H^m_U$$

and by the remarks above is even smooth. The morphism $\Phi = ev_{1,n+1}|_V : V \to U \times_S U$ satisfies the assumptions of Lemma 4.17. This lemma implies that there exists a variety T', a dominant morphism $T' \to T_0 \times T_\infty$, a compactification $V \subset \overline{V}$ over $X \times_C X$ and a morphism $\tau : C \times T' \to \overline{V}$ such that $C \times T' \to \overline{V} \to X \times_C X$ equals $C \times T' \to C \times T_0 \times T_\infty \to X \times_C X$, and such that the induced rational map $\tau_m : \mathrm{Sym}^m(C) \times T' \dashrightarrow H^m_V$ is dominant. We may also assume the compactification $\overline{V}$ dominates the compactification

$$V \subset \mathrm{FreeChain}^U_2(X/C, n) \subset \mathrm{Chain}_2(X/C, n)$$

by the space $\mathrm{Chain}_2(X/C, n)$ of 2-pointed chains of (not necessarily free) lines in fibres of X/C.

At this point, every $t \in T'(k)$ gives rise to a morphism $\tau_t : C \to \mathrm{Chain}_2(X/C, n)$, which in turn gives rise to n ruled surfaces $R_1, \dots, R_n$ in X over C, and $n+1$ sections $\tau_1, \dots, \tau_{n+1}$ of X/C such that

(1) τ_1 corresponds to a point of T_0,
(2) τ_{n+1} corresponds to a point of T_∞,
(3) τ_i, τ_{i+1} factor through R_i for $i = 1, \dots, n$.

By shrinking T' we may assume that both τ_1 and τ_{n+1} meet U and are D-free for some divisor D of degree m.

Let $e_i = \deg(\tau_i)$ for $i = 2, \dots, n$. Recall the constructible sets $W^m_{e_i} \subset H^m_{X,f}$ constructed in Proposition 4.15, in particular recall that each $W^m_{e_i}$ contains a dense open of $H^m_{X,f}$, and hence of H^m_U. Because the associated rational map from $\mathrm{Sym}^m(C) \times T'$ to H^m_V is dominant, and because each $\mathrm{ev}_i : V \to U$ is smooth we see that after replacing T' by a nonempty open subvariety, for each $t \in T'(k)$ the associated maps τ_i map a general divisor $D \subset C$ of degree m to a point of $W^m_{e_i}$. In particular, τ_i meets $U = X_{f,\mathrm{pax}}$ and there exists a divisor D of degree m such that τ_i is D-free. By Lemma 4.12 all the sections τ_i are 1-free, i.e., free. Since $T' \to T_0 \times T_\infty$ is dominant its image contains an open subset and the lemma is proved. □

Let $P \subset \mathrm{Porcupine}^{e,n}(X/C/k)$ be an irreducible component. Since the space $\mathrm{Porcupine}^{e,n}(X/C/k)$ is smooth this is also a connected component. And since the forgetful morphisms $\Phi_{\mathrm{forget}} : \mathrm{Porcupine}^{e,n+m}(X/C/k) \to \mathrm{Porcupine}^{e,n}(X/C/k)$ are smooth with geometrically irreducible fibres, we see that P determines a sequence $(P_m)_{m\geq 1}$ of irreducible components, $P_m \subset \mathrm{Porcupine}^{e,n+m}(X/C/k)$. Note that points of P_{d-e-n} correspond to porcupines with total degree d.

Proposition **9.7.** — *In Situation 8.1 assume the restriction of f to some nonempty open set $S \subset C$ satisfies Hypothesis 7.8. Let $D \subset C$ be an effective Cartier divisor of degree $2g(C) + 1$. Let e_0, e_∞ be integers and let n_0, n_∞ be nonnegative integers. Let $P \subset \mathrm{Porcupine}^{e_0,n_0}(X/C/k)$ be an irreducible component, and let $P' \subset \mathrm{Porcupine}^{e_\infty,n_\infty}(X/C/k)$ be an irreducible component. Assume that a general point of P corresponds to a porcupine whose body is D-free, and similarly for P'. There exists an integer E such that for every $e \geq E$ there exists a dense open subscheme*

$$U \subset P_{e-e_0-n_0} \times_{\alpha_{\mathcal{L}}, \underline{Pic}^e_{C/k}, \alpha_{\mathcal{L}}} P'_{e-e_\infty-n_\infty}$$

such that for any pair of points $(p, q) \in U$ the points p and q are connected in $\Sigma^e(X/C/k)$ by a chain of rational curves whose marked points and nodes parametrize unobstructed, non-stacky stable sections.

Proof. — Let T_0, resp. T_∞ denote the irreducible components of $\mathrm{Sections}(X/C/k)$ to which the bodies of porcupines in P, resp. P' belong. Let $n, e_1, \dots, e_{n+1}$, B and $W \subset T_0 \times T_\infty$ be the data found in Lemma 9.6. Note that $e_0 = e_1$ and $e_\infty = e_{n+1}$. Set $E = \max\{e_i\} + \max\{0, \varphi(B, e_i)\} + 2g(C) + 999$ where $\varphi(-,-)$ is the function mentioned in Lemma 9.5. Pick $e \geq E$. The open subset U will correspond to the pairs (p, q) in the fibre product with $p = (s_0, q_{0,i}, r_{0,i} \in L_{0,i}(k), L_{0,i} \to X)$, $q = (s_\infty, q_{\infty,i}, r_{\infty,i} \in L_{\infty,i}(k), L_{\infty,i} \to X)$ such that the pair $(s_0, s_\infty) \in W$.

Namely, let $R_1, \ldots, R_n, \tau_1, \ldots, \tau_\infty$ be as in Lemma 9.6 adapted to the pair (s_0, s_∞). Denote $\xi = \alpha_{\mathcal{L}}(p) = \alpha_{\mathcal{L}}(q) \in \underline{\mathrm{Pic}}^e_{C/k}(k)$ the corresponding degree e divisor class on C. For each $i = 2, \ldots, n$ it is possible to find a reduced effective divisor $Q_i \subset C$, $Q_i \subset \tau_i^{-1}(X_{f,\mathrm{pax}})$ of degree $e - e_i \geq 2g(C) + 999$ such that $\tau_i^*(\mathcal{L})(Q_i)$ is ξ. Now we will repeatedly use Lemmas 9.4 and 9.5 to find our chain of rational curves. Let p_1 be the point of $\mathrm{Porcupine}^{e_0, e-e_0}(X/C/k)$ gotten from p by moving all $e - e_0 = e - e_1$ quills attached to $s_0 = \tau_1$ into the ruled surface R_1. Then p is connected to p_1 by a suitable chain of rational curves by Lemma 9.4. Let $p_2 \in \mathrm{Porcupine}^{e_2, e-e_2}(X/C/k)$ be the porcupine with body τ_2 and quills attached to the points of the divisor Q_2 lying in the ruled surface R_1. By Lemma 9.5 we see that p_1 is connected to p_2 by a suitable chain of rational curves. Let $p_3 \in \mathrm{Porcupine}^{e_2, e-e_2}(X/C/k)$ be the porcupine with body τ_2 and quills attached to the points of the divisor Q_2 lying in the ruled surface R_2. By Lemma 9.4 the points p_2 and p_3 are connected by a suitable chain of rational curves. And so on and so forth. □

In particular, this implies the following corollary, that the sequence of irreducible components is "asymptotically unique". And it does imply that Abel-equivalent points of some boundary strata are connected by chains of rational curves in a fiber of the Abel map. But it does not yet imply that the general fiber of an Abel map is rationally connected, since we do not yet know that a *general* pair of Abel-equivalent points are connected by a chain of rational curves (this requires much more work).

Corollary **9.8.** — *In Situation 8.1, assume in addition Hypothesis 7.8 for the restriction of* f *to some nonempty open* $S \subset C$. *Let* e_0, e_∞ *be integers and let* Z, Z' *be irreducible components of* $\Sigma^{e_0}(X/C/k)$, $\Sigma^{e_\infty}(X/C/k)$ *whose general point parametrizes a free section of* f. *Consider the families of components* Z_e, Z'_e *constructed in Lemma 8.4. Then for* $e \gg 0$ *we have* $Z_e = Z'_e$.

Proof. — By Lemmas 8.8 and 8.6 we may assume that Z and Z' each contain a section which meets $X_{f,\mathrm{smooth}}$ and is $2g(C) + 1$-free. By Proposition 9.7 there exists an E such that for all $e \geq E$ there are points $p \in Z_e$ and $q \in Z'_e$ corresponding to porcupines with bodies in Z and Z' such that p can be connected to q by a chain of rational curves in $\Sigma^e(X/C/k)$ whose marked points correspond to unobstructed, non-stacky stable sections. In particular such a chain lies in a unique irreducible component! □

10. Varieties and peaceful chains

Previously we studied moduli of sections of X/C. Here we apply this to the study of rational curves on a smooth projective variety Y. In particular, if Y is rationally simply connected by chains of free lines, then the sequence Z_e from Section 8 gives a canonical sequence of irreducible components of the moduli space $\overline{\mathcal{M}}_{0,0}(Y, e)$.

Hypothesis **10.1.** — *Let k be an uncountable algebraically closed field of characteristic* 0. *Let* Y *be a smooth projective variety over k. Let $\mathcal{L}$ be an ample invertible sheaf on* Y. *We assume that* $Y \to Spec(k)$ *satisfies Hypothesis 7.8 (where* S *is* $Spec(k)$ *and* $f : X \to S$ *is* $Y \to Spec(k)$*).*

Assume that Hypothesis 10.1 holds. Let $X = \mathbf{P}^1 \times Y$ and

$$f = \mathrm{pr}_1 : X = \mathbf{P}^1 \times Y \longrightarrow \mathbf{P}^1.$$

The first thing to remark is that a morphism $g : \mathbf{P}^1 \to Y$ gives rise to a section $s : \mathbf{P}^1 \to X$, and conversely. Also, the section s is 0-free, i.e., unobstructed (see Definition 4.7), if and only if the rational curve $g : \mathbf{P}^1 \to Y$ is unobstructed (i.e., every invertible sheaf summand in the direct sum decomposition of g^*T_Y has degree at least -1). Similarly, s is 1-free, i.e., free (see Definition 4.7), if and only if the rational curve $g : \mathbf{P}^1 \to Y$ is free. Arguing in this way we see that we can interpret many of the previous results for the family $f : X \to \mathbf{P}^1$ in terms of the moduli spaces of lines on Y.

Instead of reproving everything from scratch in this setting we make a list of statements and we point out the corresponding lemmas, and propositions in the more general treatement above. We denote $\overline{\mathcal{M}}_{0,0}(Y, e)$ the Kontsevich moduli space of degree e rational curves in Y. Also, let Y_{pax} denote the set of peaceful points (w.r.t. $Y \to \mathrm{Spec}(k)$), see Definition 7.4.

(1) There is a sequence of irreducible components $Z_e \subset \overline{\mathcal{M}}_{0,0}(Y, e)$, $e \geq 1$ uniquely characterized by the property that every comb in Y whose handle is contracted to a point of Y_{pax} and which has e teeth mapping to lines in Y is in Z_e. This is the exact analogue of Lemma 8.4, starting with the fact that a constant map $\mathbf{P}^1 \to Y$ is free!

(2) A general point of Z_e corresponds to a map $\mathbf{P}^1 \to Y$ which is free and meets Y_{pax}. This is the exact analogue of Lemma 8.5.

(3) Consider a comb in Y whose handle has degree e with $e' - e$ teeth which are lines. Such a comb is in $Z_{e'}$ if the handle is free, is in Z_e, and the attachment points map to Y_{pax}. This is the exact analogue of Lemma 8.6.

(4) Any stable map $f : C \to Y$ in $\overline{\mathcal{M}}_{0,0}(Y, e')$ which has exactly one component a free curve corresponding to a point in Z_e and all other components free lines corresponds to a smooth nonstacky point of $Z_{e'}$. This is the exact analogue of Lemma 8.7.

(5) Given any integer $m \geq 0$ there exists $e \gg 0$ such that the general point of Z_e corresponds to a map $f : \mathbf{P}^1 \to Y$ with $H^1(\mathbf{P}^1, f^*T_Y(-m)) = 0$. This is the exact analogue of Lemma 8.8.

(6) Let $Z' \subset \overline{\mathcal{M}}_{0,0}(Y, e_0)$ be an irreducible component whose general point corresponds to a free rational curve. There exists an integer $E \gg 0$ such that for all $e \geq E$ we have the following property: Consider a comb in Y whose handle has degree e_0 with $e - e_0$ teeth which are lines. Such a comb is in Z_e if the handle is free, is in Z', and the attachment points map to Y_{pax}. (And in particular

any smoothing of this comb defines a point of Z_e.) This is the exact analogue of Corollary 9.8, once you observe that such a comb defines an unobstructed nonstacky point of $\overline{\mathcal{M}}_{0,0}(Y, e)$.

It is perhaps not necessary, but we point out that instead of defining Z_e as in (1) above, we could just define Z_e as the unique irreducible component of $\overline{M}_{0,0}(Y, e)$ which contains all trees of free lines. This follows from (4) (with $e' = 0$).

Lemma **10.2.** — *In Situation 10.1 above. Let* $Z_{e,1} \subset \overline{\mathcal{M}}_{0,1}(Y, e)$ *be the unique irreducible component dominating* $Z_e \subset \overline{\mathcal{M}}_{0,0}(Y, e)$. *The general fibre of* $\mathrm{ev} : Z_{e,1} \to Y$ *is irreducible.*

Proof. — Let us temporarily denote by $Z^\circ_{e,1} \subset Z_{e,1}$ the dense open locus of stable 1-pointed maps $\bigcup C_i \to Y$ all of whose irreducible components $C_i \to Y$ are free and whose nodes map to points of Y_{pax}. The morphism $\mathrm{ev} : Z^\circ_{e,1} \to Y$ is smooth. Consider the locus $W_{e,1} \subset Z^\circ_{e,1}$ whose points correspond to those combs whose handles are contracted to a point of Y_{pax} and whose teeth are (automatically) free lines and whose marked point is on the handle. By definition of the irreducible components Z_e the space $W_{e,1}$ is nonempty. It is smooth and $W_{e,1} \to Y$ is still smooth onto Y_{pax}. Clearly, Hypothesis 7.8 for $Y \to \mathrm{Spec}(k)$ implies the general fibre of $W_{e,1} \to Y$ is irreducible. From this, and the irreducibility of $Z_{e,1}$ it follows that the general fibre of $Z_{e,1} \to Y$ is irreducible, for example by [Gro67, Proposition 4.5.13]. □

Lemma **10.3.** — *In Situation 10.1 above. Let* $Z' \subset \overline{\mathcal{M}}_{0,0}(Y, e_0)$ *and* E *be as in (6) above. Suppose that* $g : C \to Y$ *is a genus* 0 *Kontsevich stable map such that* $C = \bigcup_{i=1}^{A} C_i \cup \bigcup_{j=1}^{B} L_j$ *is a decomposition into irreducible components with the following properties: (1) each* $C_i \to Y$ *is free and corresponds to a point of* Z', *(2) each* $L_i \to Y$ *is a free line and a "leaf of the tree", i.e., it intersects the rest of the curve in a single node, and (3)* $B \geq E - e_0$. *Then* g *defines an unobstructed point of* Z_{Ae_0+B}.

Proof. — We will show there is a connected chain of curves contained in the unobstructed locus of $\overline{\mathcal{M}}_{0,0}(Y, Ae_0 + B)$ which connects g to a point of Z_{Ae_0+B}. As the unobstructed locus is smooth this will imply that the entire curve is contained in Z_{Ae_0+B} and hence g is in it. We may first move the map a little bit such that all the nodes of C are mapped into Y_{pax}. Since the lines are leaves, we may 1 by 1 slide all the lines L_i along the curves C_i and onto one of the curves C_{i_0} which is a leaf of the tree $\bigcup C_i$. Say this curve is C_1, and is attached to $C'' = \bigcup_{i\neq 1} C_i$ at the point $p \in C_1$ and $q \in C''$. Set $C' = C_1 \cup \bigcup L_i$, so $C = C' \cup_{p\sim q} C''$. By (6) above we see that $C' \to Y$ defines an unobstructed point of Z_{e_0+B}.

We may assume (after possibly moving the map a little bit) that the attachment point p is mapped to a point $y \in Y$ such that the fibre of $Z_{e_0+B,1} \to Y$ is irreducible over this point, see Lemma 10.2. Hence we may connect the 1-pointed stable map $(C', p) \to Y$ inside the fibre of $Z_{e_0+B,1} \to Y$ over y to a curve which is made out of a comb C''' whose handle is contracted and whose teeth are free lines. Since the marked point is throughout mapped to the same $y \in Y(k)$ this connects the original stable map $C' \cup C'' \to Y$ to a stable map $C''' \cup C'' \to Y$ where the number of lines in C''' is now $e_0 + B$. We may again slide these lines over to some irreducible component C_i of C'' and continue until all the C_i are gone, and so obtain a point of Z_{Ae_0+B}. This proves the lemma. □

The notation in the section conflicts with the notation introduced in Lemma 8.4 in case Y is a fibre of a family as in Situation 8.1 such that Hypothesis 7.8 holds over a nonempty open of C. In that situation we will use the notation $Z_e(X_t)$ to denote the irreducible component defined in this section for the irreducible fibre X_t.

Lemma **10.4.** — *In Situation 8.1, assume in addition Hypothesis 7.8 for the restriction of f to some nonempty open $S \subset C$. Suppose that $s : C \to X$ is a free section, and let $Z \subset \Sigma(X/C/k)$ be the unique irreducible component containing the moduli point $[s]$. Let Z_e as in Lemma 8.4. For any $t_1, \ldots, t_\delta \in S(k)$, for any free rational maps $s_i : \mathbf{P}^1 \to X_{t_i}$ such that $s_i(0) = s(t_i)$, if s_i corresponds to $Z_{e_i}(X_{t_i})$, then the comb $C \cup \bigcup \mathbf{P}^1 \to X/C$ defines an unobstructed point of $Z_{\deg(s)+\sum e_i}$.*

Proof. — This is very similar to the proof of Lemma 10.3 above. Namely, we first move the comb (as a comb) such that the points $s(t_i)$ are in the locus where the fibres of $Z_{e_i,1} \to X_{t_i}$ are irreducible. then we connect the s_i in these fibres to chains of lines. After this we can apply Lemma 8.7 for example. □

11. Families of varieties and lines in fibres

This section contains two results that did not seem to fit well in other sections. Lemma 11.1 proves, under our hypotheses, the smooth locus is dense in every fibre and is strongly rationally connected in the sense of [HT06]. Lemma 11.4 gives a criterion for when the limit of a family of free lines is also free.

Lemma **11.1.** — *In Situation 8.1, assume in addition Hypothesis 7.8 for the restriction of f to some nonempty open $S \subset C$. Every geometric fibre X_t of f is integral and every pair of points of the smooth locus X_t° of the fibre is contained in (1) a chain of lines contained in X_t°, and (2) a very free rational curve in the smooth locus.*

Proof. — Although the conclusions are stated entirely in terms of the fiber, we see no way to prove any of the conclusions without using the hypotheses on the global family. In fact one can easily find families failing some of the hypotheses where also the conclusions of this lemma fail.

Let $t \in C(k)$. Situation 8.1 requires X_t to be irreducible. Since X is smooth, by [GHS03], the fibre X_t intersects the smooth locus of f. Hence X_t is generically reduced and Cohen-Macaulay, and hence is reduced.

Let $x, y \in X_t(k)$ be smooth points of X_t. We will construct a very free rational curve in X_t passing through x and y. There exists a finite cover $C' \to C$ etale over t and over the discriminant of $f : X \to C$, a point $t' \in C'(k)$ and sections $a, b : C' \to X_{c'}$ such that $a(t') = x$, $b(t') = y$. To construct C' choose two complete intersection curves $C_x, C_y \subset X$ general apart from the condition $x \in C_x$ and $y \in C_y$ and let C' be an irreducible component of the normalization of $C_x \times_C C_y$ (details left to the reader).

Set $X' = X_{C'}$ and $f' : X' \to C'$ the base change of f. By construction, all conditions of Situation 8.1, and Hypothesis 7.8 remain satisfied for $X' \to C'$. Hence, upon replacing $X \to C$ by $X' \to C'$, and t by t' we may assume there are sections s_0, s_∞ such that $s_0(t) = x$ and $s_\infty(t) = y$. Attaching very free rational curves in fibres of f and deforming we may assume that s_0 and s_∞ are sufficiently general such that $(s_0, s_\infty) : C \to X \times_C X$ meets the open set $V \subset X_S \times_S X_S \subset X \times_C X$ from Hypothesis 7.8.

By [GHS03] we can find a morphism $\phi : C \to \mathrm{Chain}_2(X/C, n)$ which meets $\mathrm{FreeChain}_2(X_S/S, n)$ and such that $ev_{1,n+1} \circ \phi = (s_0, s_\infty)$. As in the proof of Lemma 9.6 this translates into a collection of n ruled surfaces $R_1, \ldots, R_n$ in X/C, $n + 1$ sections $\tau_1, \ldots, \tau_{n+1}$ of X/C such that $\tau_1 = s_0$, $\tau_{n+1} = s_\infty$ and such that τ_i, τ_{i+1} factor through R_i.

Note that the fibres $R_{1,t}, R_{2,t}, \ldots, R_{n,t}$ form a chain of lines connecting x to y in the fibre X_t. By Lemma 9.1 this chain lies in the smooth locus X_t° of $X \to C$. Thus any pair of points of X_t° may be connected by a chain of lines in X_t°. The result follows upon applying [Kol96, IV Theorem 3.9.4] for example. □

Remark **11.2.** — The Lemmas 9.1 and 11.1 could be stated with slightly weaker hypotheses. Namely, we could remove the assumption (in 8.1) that all the fibres of $X \to C$ are geometrically irreducible. This does not pose a problem for Lemma 9.1. For 11.1 it means that one has to show there are no irreducible components of X_t which have multiplicity > 1. We may try to prove this by specializing the "chain of ruled surfaces" $R_1, \ldots, R_n, \tau_1, \ldots, \tau_{n+1}$ as you move one of the two points x, y into a presumed higher multiplicity component. What might happen is that the ruled surfaces may break, and we do not see how to conclude the proof. However, if $\mathcal{L}$ is very ample on all the fibres (an assumption that always holds in practice), then this argument works: in this case any line that meets a higher multiplicity component must be contained in it and it is easy to conclude from this.

Lemma **11.3.** — *Let X be a nonsingular projective variety over an algebraically closed field k of characteristic 0. Let $\mathcal{L}$ be an ample invertible sheaf on X.*

(1) *The locus $T \subset X$ swept out by non free lines is a proper closed subset. Let $D'_1, \ldots, D'_r$ be desingularizations (see [Hir64, Hir64a]) of the irreducible components $D_1, \ldots, D_r$ of T which have codimension 1 in X.*
(2) *There exists a closed codimension 2 subset $T' \subset X$ containing the singular locus of T and all codimension ≥ 2 components such that any non free line not contained in T' is the image of a free rational curve on some D'_i.*

Proof. — The first assertion is basic, see [Kol96, II Theorem 3.11]. For the second, first let $T' \subset T$ be the closed subset over which $\sqcup D'_i \to T$ is not an isomorphism. Any line in T not contained in T′ is the image of a unique rational curve on some D'_i. Finally we apply the general fact that since these curves cover D'_i the general one is free. □

Lemma **11.4.** — *Notations and assumptions as in Lemma 11.3. Let* C *be a nonsingular curve, and let* $f : \mathbf{P}^1 \times C \to X$ *be a family of lines in* X. *Assume that* $\gamma = f|_{\{0\}\times C} : C \to X$ *does not meet* T′, *and meets* T *transversally at all of its points of intersection. Then for each* $c \in C(k)$ *the line* $L_c \to X$ *is free.*

Proof. — By the definition of T, for every point $t \in C(k)$ with $f_t(0) \notin T$, e.g., for a general point of $C(k)$, the corresponding line $f_t : \mathbf{P}^1 \to X$ is free. Thus let $t \in C$ be a point such that $f_t(0) \in T$. Again by the definition of T, if $f_t(\mathbf{P}^1) \not\subset T$ then this is a free line. Thus assume that $f_t(\mathbf{P}^1) \subset T$.

The goal is to prove that $f_t^*T_X$ is globally generated. By Lemma 11.3 the line $f_t : \mathbf{P}^1 \to X$ is the image of a free rational curve $g : \mathbf{P}^1 \to D'$, where D′ is a resolution of some $D = D_i$. In particular $g^*T_{D'}$ is globally generated. Consider the maps

$$g^*T_{D'} \to f_t^*T_X \to f_t^*N_DX.$$

Note that the composition is zero. Moreover, the sequence is exact on the open complement of $f_t^{-1}(T')$ (which is dense since it contains $0 \in \mathbf{P}^1$ by hypothesis). So the quotient of the kernel by the image is a torsion sheaf, which is thus globally generated. Since $g^*T_{D'}$ is globally generated, also its image in $f_t^*T_X$ is globally generated. So the kernel sheaf is an extension of globally generated coherent sheaves. On $\mathbf{P}^1$, such an extension is itself globally generated, i.e., the kernel of $f_t^*T_X \to f_t^*N_DX$ is globally generated. Again using that an extension of globally generated, coherent sheaves on $\mathbf{P}^1$ is itself globally generated, $f_t^*T_X$ is globally generated provided that the image of $f_t^*T_X \to f_t^*N_DX$ is globally generated, i.e., so long as the image is not a negative invertible sheaf.

However, the given deformation $f : \mathbf{P}^1 \times C \to X$ gives rise to a vector $\theta \in H^0(\mathbf{P}^1, f_t^*T_X)$. And $\theta(0)$ corresponds to the tangent vector $d\gamma \in T_{\gamma(0)}X$ which points out of D by the hypothesis that γ is transverse to T. Thus the sheaf homomorphism $f_t^*T_X \to f_t^*N_DX$ induces a nonzero map on H^0. Therefore $f_t^*T_X \to f_t^*N_DX$ cannot factor through a negative invertible sheaf. So $f_t^*T_X$ is globally generated, i.e., $f_t : \mathbf{P}^1 \to X$ is a free line. □

12. Perfect pens

In this section we introduce the notion of twisting and very twisting scrolls, see Definition 12.3. The main consequence of this definition occurs in Lemma 12.5, which shows that a porcupine contained in a twisting scroll gives a moduli point t in $\Sigma(X/C/k)$ which is contained in a $\mathbf{P}^1$ whose general point parameterizes a section. In other words, the $\mathbf{P}^1$ connects t, a point in a boundary stratum, to a moduli point in the interior, the complement of the boundary divisor. It is important to have a geometric criterion for twisting and very twisting surfaces, which is the purpose of Lemma 12.6. The main result of this section, Proposition 12.12, is that a family $X \to C$ contains very twisting scrolls, if the geometric generic fiber Y has a very twisting scroll. The proof uses lemmas about gluing twisting and very twisting scrolls and then deforming to produce new twisting and very twisting scrolls. We also prove that there are good parameter spaces for twisting and very twisting scrolls. And we prove that there are good parameter spaces for those porcupines "penned" by a very twisting scroll.

Recall the definition of a ruled surface R in $X \to C$ with respect to $\mathcal{L}$ was given at the start of Section 9. The morphism $R \to X$ maps into the smooth locus of $X \to C$, see Lemma 9.1. Our first task is to show that (many) free sections lie on scrolls of free lines.

Lemma **12.1.** — *In Situation 8.1 assume Hypothesis 7.8 holds for the restriction of f to some nonempty open set $S \subset C$. Let e_0 be an integer and let Z be an irreducible component of $\Sigma^{e_0}(X/C/k)$ whose general point parametrizes a free section. Then there exists a nonempty open set $U \subset Z$ such that every $u \in U(k)$ corresponds to a section s such that (a) it is penned by a ruled surface $R \to X$, and (b) any ruled surface R penning s has the property that all of its fibres are free lines in X.*

Proof. — We are going to use Lemma 11.4. Note that although $\mathcal{L}$ is f-relatively ample, $\mathcal{L}$ may not be "absolutely" ample, i.e., it may not be ample on X. However, to prove the lemma we may replace $\mathcal{L}$ by $\mathcal{L} \otimes f^*\mathcal{N}$ for some suitable very ample sheaf $\mathcal{N}$ on C. Thus, without loss of generality, we may assume that $\mathcal{L}$ is ample on the total space X. Similarly, if $\mathcal{N}$ is sufficiently ample, then we may also assume that any rational curve in X of degree 1 is in a fibre of $f : X \to C$.

Let $T' \subset T \subset X$, D_i be as in Lemma 11.3. Because $T' \subset X$ has codimension ≥ 2 there is a nonempty open $U' \subset Z$ such that every $u \in U'(k)$ corresponds to a section s which is disjoint from T'. Furthermore, we then pick a nonempty open $U'' \subset U'$ such every $u \in U''(k)$ corresponds to a section s that meets each irreducible component D_i transversally in smooth points, see [Kol96, II Proposition 3.7]. Finally, by part (1) of Hypothesis 7.8, we may find a further nonempty open $U \subset U''$ such that each section $s : C \to X$ corresponding to a point of U meets the locus over which the evaluation morphism $\text{ev} : \overline{\mathcal{M}}_{0,1}(X/C, 1) \to X$ has irreducible rationally connected fibres.

Let $s : C \to X$ correspond to a k-point of U. By [GHS03] we can find a morphism $g : C \to \overline{\mathcal{M}}_{0,1}(X/C, 1)$ such that $\text{ev} \circ g = s$. This corresponds to a ruled surface $R \to X$ which pens s. This proves (a). Next, let R be any ruled surface penning s. By Lemma 9.1 it lies in the smooth locus of $X \to C$. By Lemma 11.4 and our choice of s all fibres of $R \to C$ are free curves in X. This proves (b). □

To define the notion of a twisting scroll, we introduce some notation. Let $f : X \to C$, $\mathcal{L}$ be as in Situation 8.1. Consider a ruled surface $h : R \to X$. The following commutative diagram of coherent sheaves on R with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}_{\mathrm{R/C}} & \longrightarrow & h^*\mathrm{T}_{\mathrm{X/C}} & \longrightarrow & \mathrm{N}_{\mathrm{R/X}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{T}_{\mathrm{R}} & \longrightarrow & h^*\mathrm{T}_{\mathrm{X}} & \longrightarrow & \mathrm{N}_{\mathrm{R/X}} & \longrightarrow & 0 \end{array}$$

defines the coherent sheaf $\mathrm{N}_{\mathrm{R/X}}$, which we call the *normal bundle*. Even when the morphism from R to X fails to be a closed immersion, the normal bundle is of fundamental importance cf. its use in [GHS03]. Moreover, if the sheaf $\mathcal{L}$ is relatively very ample, which is satisfied in important cases, then the map $h : \mathrm{R} \to \mathrm{X}$ will be a closed immersion and $\mathrm{N}_{\mathrm{R/X}}$ will be a locally free sheaf.

Remark **12.2.** — There are two remarks about the case where $\mathcal{L}$ is relatively ample, but perhaps not very ample. The first is that $\mathrm{N}_{\mathrm{R/X}}$ is always flat over C, even when it is not locally free over R. This follows as the maps $\mathrm{T}_{\mathrm{R/C}}|_{\mathrm{R}_t} \to h^*\mathrm{T}_{\mathrm{X/C}}|_{\mathrm{R}_t}$ are injective, and [Mat80, Section (20.E)]. In particular it has depth ≥ 1, its torsion is supported in codimension ≥ 1, and any fibre meets the torsion locus in at most finitely many points. This also shows that $\mathrm{N}_{\mathrm{R/X}}|_{\mathrm{R}_t}$ equals $\mathrm{N}_{\mathrm{R}_t/\mathrm{X}_t}$, the normal bundle of the map $\mathrm{R}_t \to \mathrm{X}_t$ (defined similarly). The second is that the deformation theory of the morphism $\mathrm{R} \to \mathrm{X}$, with X held fixed, is given by $\mathrm{H}^0(\mathrm{R}, \mathrm{N}_{\mathrm{R/X}})$ (infinitesimal deformations) and $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R/X}})$ (obstruction space). The deformation theory is described in [Ill71, Section 2.1.6, pp. 191–192], especially [Ill71, Théorème 2.1.7]. In particular, the cotangent complex of $h : \mathrm{R} \to \mathrm{X}$ is given by $h^*\Omega^1_{\mathrm{X}} \to \Omega^1_{\mathrm{R}}$ which is quasi-isomorphic to $h^*\Omega^1_{\mathrm{X/C}} \to \Omega^1_{\mathrm{R/C}}$. It follows that $\mathbf{E}\mathrm{xt}^i(h^*\Omega^1_{\mathrm{X/C}} \to \Omega^1_{\mathrm{R/C}}, \mathcal{O}_{\mathrm{R}}) = \mathrm{H}^{i-1}(\mathrm{R}, \mathrm{N}_{\mathrm{R/X}})$ as usual.

Definition **12.3.** — *In Situation 8.1. Let* $\mathrm{R} \to \mathrm{X}$ *be a ruled surface in* X *and let* D *be a Cartier divisor on* R. *For every nonnegative integer m, we say* (R, D) *is m*-twisting *if*

1. *the complete linear system* $|\mathrm{D}|$ *is basepoint free,*
2. *the cohomology group* $\mathrm{H}^1(\mathrm{R}, \mathcal{O}_{\mathrm{R}}(\mathrm{D}))$ *is* 0,
3. D *has relative degree* 1 *over* C,
4. *the normal bundle* $\mathrm{N}_{\mathrm{R/X}}$ *is globally generated,*
5. $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R/X}})$ *equals* 0, *and*
6. *we have* $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R/X}}(-\mathrm{D} - \mathrm{A})) = 0$ *for every divisor* A *which is the pullback of any divisor on* C *of degree* $\leq m$.

This only depends on the Cartier divisor class *of the Cartier divisor* D.

This is a "relative" definition—it is defined with respect to the morphism f. An important special case is an "absolute" variant, see Definition 12.7 below.

Suppose we are in Situation 8.1 and suppose that (R, D) is an m-twisting scroll in X/C. By assumption $|D|$ is nonempty, base point free, and of relative degree 1. Hence a general element is smooth and defines a section $\sigma : C \to R$. We will often say "let (R, σ) be an m-twisting scroll" to denote this situation. Having chosen σ we can think of $R \to X$ as a family of stable 1-pointed lines. Let $g = g_{(R,\sigma)} : C \to \overline{\mathcal{M}}_{0,1}(X/C, 1)$ denote the associated morphism.

Lemma **12.4.** — *In the situation above:*

1. *The image of* $g = g_{(R,\sigma)}$ *lies in the unobstructed (and hence smooth) locus of the morphism* $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X/C, 1) \to X$.
2. *We have* $H^1(C, g^*T_{\mathrm{ev}}(-A)) = 0$ *for every divisor* A *of degree* $\leq m$ *on* C.
3. *Let* $\Phi : \overline{\mathcal{M}}_{0,1}(X/C, 1) \to \overline{\mathcal{M}}_{0,0}(X/C, 1)$ *be the forgetful morphism (which is smooth). Then* g^*T_Φ *is globally generated, and* $H^1(C, g^*T_\Phi) = 0$.
4. *The section* $h \circ \sigma : C \to X$ *is free, see Definition 4.7.*

Proof. —

1. The fact that $N_{R/X}$ is globally generated implies that for each $t \in C(k)$ the fibre $R_t \to X$ is free. This follows upon considering the exact sequences $0 \to T_R|_{R_t} \to h^*T_X|_{R_t} \to N_{R/X}|_{R_t} \to 0$ (exact by flatness of $N_{R/X}$ over C), and the fact that $T_R|_{R_t}$ is globally generated. This implies the image of g is in the unobstructed locus for ev.
2. Let $\pi : R \to C$ be the structural morphism. The pullback of the relative tangent bundle T_{ev} by g is canonically identified with $\pi_* N_{R/X}(-\sigma)$. This is true because the normal bundle of a fibre $R_t \to X$ is an extension of the restriction $N_{R/X}|_{R_t}$ by a rank 1 trivial sheaf on $R_t \cong \mathbf{P}^1$. The assumptions of Definition 12.3 imply that $R^1\pi_* N_{R/X}(-\sigma) = 0$. Hence by the Leray spectral sequence for $N_{R/X}(-\sigma - \pi^*A)$, $H^1(C, g^*T_{\mathrm{ev}}(-A)) = H^1(R, N_{R/X}(-\sigma - \pi^*A))$. Thus we get the desired vanishing from the definition of twisting scrolls.
3. The pullback g^*T_Φ is canonically identified with $\sigma^*\mathcal{O}_R(\sigma)$. The global generation of the sheaf g^*T_Φ is therefore a consequence of the base point freeness of $\mathcal{O}_R(\sigma)$ of Definition 12.3. The vanishing of $H^1(C, g^*T_\Phi)$ follows on considering the long exact cohomology sequence associated to $0 \to \mathcal{O}_R \to \mathcal{O}_R(\sigma) \to \sigma_*\sigma^*\mathcal{O}_R(\sigma) \to 0$ and the vanishing of $H^1(R, \mathcal{O}_R(\sigma))$ in Definition 12.3.
4. Consider the exact sequence $\sigma^*T_{R/C} \to \sigma^*h^*T_{X/C} \to \sigma^*N_{R/X} \to 0$. Note that $\sigma^*T_{R/C} = \sigma^*\mathcal{O}_R(\sigma)$. By Definition 12.3 both $\sigma^*\mathcal{O}_R(\sigma)$ and $\sigma^*N_{R/X}$ are globally generated. Thus by the exact sequence $\sigma^*h^*T_{X/C}$ is globally generated. This is obvious if the first map is zero, otherwise the exact sequence is a short exact sequence, and in the previous paragraph we showed that $H^1(C, \sigma^*\mathcal{O}_R(\sigma)) = 0$. There is an exact sequence $N_{R/X}(-\sigma) \to N_{R/X} \to \sigma_*\sigma^*N_{R/X} \to 0$. By Definition 12.3, $H^1(R, N_{R/X}) = 0$. The group

$H^2(R, N_{R/X}(-\sigma))$ vanishes as its Serre dual $\mathrm{Hom}_R(N_{R/X}(-\sigma), \omega_R)$ is zero (hint: consider restriction to fibres). Together these imply that $H^1(C, \sigma^* N_{R/X}) = 0$. Thus the first exact sequence of this paragraph implies that $H^1(C, \sigma^* h^* T_{X/C}) = 0$. Thus $(h \circ \sigma)^* T_{X/C}$ is globally generated with trivial H^1 and we conclude that $h \circ \sigma$ is 1-free, i.e., free. ☐

The following innocuous looking lemma is why we introduce twisting scrolls. It will eventually show that for every canonical irreducible component of the moduli space of stable sections with sufficiently positive degree, for the associated "boundary" stratification of the moduli space (according to the dual graph of the domain curve), a general point of a particular boundary stratum is connected by a $\mathbf{P}^1$ to a point in the "interior", i.e., the open complement of the boundary divisor.

Lemma **12.5.** — *In Situation 8.1, let* (R, σ) *be m-twisting with* $m \geq 1$. *Let* $t \in C(k)$. *The stable map* $\sigma(C) \cup R_t \to X$ *defines a nonstacky unobstructed point of* $\Sigma(X/C/k)$ *which is connected by a rational curve in* $\Sigma(X/C/k)$ *to a free section of* $X \to C$.

Proof. — The fact that the point is nonstacky comes from the fact that sections and lines have no automorphisms. The fact that the point is unobstructed follows from Lemma 12.4 above. Consider the linear system $|\sigma + R_t|$. Since $H^1(C, \mathcal{O}_R(\sigma)) = 0$ the map $H^0(R, \mathcal{O}_R(\sigma + R_t)) \to H^0(R_t, \mathcal{O}_{R_t}(1))$ is surjective. This implies that $|\sigma + R_t|$ is base point free. A general member of $|\sigma + R_t|$ is a section $\sigma' : C \to R$. It is trivial to show that (R, σ') is $(m-1)$-twisting. Hence by Lemma 12.4 we see that σ' is free. The result is clear now by considering the pencil of curves on R connecting $\sigma + R_t$ to σ'. ☐

It is useful to have a criterion that guarantees the existence of a twisting surface. In particular, we would like a condition formulated in terms of the map $g : C \to \overline{\mathcal{M}}_{0,1}(X/C, 1)$. We do not know a good way to do this unless $g(C) = 0$.

Lemma **12.6.** — *In Situation 8.1 assume* $C = \mathbf{P}^1$. *Let* $g : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(X/\mathbf{P}^1, 1)$ *be a section. Let* $m \geq 1$. *Assume we have:*

(1) *The pullback* $g^* T_\Phi$ *has degree* ≥ 0.
(2) *The image of g is contained in the unobstructed locus of* ev.
(3) *The cohomology group* $H^1(\mathbf{P}^1, g^* T_{ev}(-m))$ *is zero.*
(4) *The composition* $ev \circ g : \mathbf{P}^1 \to X$ *is a free section of* $X \to \mathbf{P}^1$.

Then the associated pair (R, σ) *is a m-twisting scroll in* X.

Proof. — We will use the identifications of $g^* T_\Phi = \sigma^* \mathcal{O}_R(\sigma)$ and $g^* T_{ev} = \pi_* N_{R/X}(-\sigma)$ from the proof of Lemma 12.4. Note that R is a Hirzebruch surface, and $H^1(R, \mathcal{O}_R) = H^2(R, \mathcal{O}_R) = 0$. Combined with the fact that $\sigma^2 = \deg(g^* T_\Phi) \geq 0$

we see that $|\sigma|$ is base point free. Also, during the course of the proof we may assume that σ is general in its linear system on R. In particular this means that the section $1 \in \Gamma(\mathrm{R}, \mathcal{O}_{\mathrm{R}}(\sigma))$ is regular for the coherent sheaf $\mathrm{N}_{\mathrm{R}/\mathrm{X}}$. (Note that this is automatic in the case, which always holds in practice, that $\mathrm{R} \to \mathrm{X}$ is a closed immersion.)

The fact that $\mathrm{ev} \circ g : \mathbf{P}^1 \to \mathrm{X}$ is 1-free means that $g^*\mathrm{ev}^*\mathrm{T}_{\mathrm{X}/\mathbf{P}^1}$ is globally generated. The fact that $\mathrm{H}^1(\mathbf{P}^1, g^*\mathrm{T}_{\mathrm{ev}}) = 0$ means that any infinitesimal deformation of the morphism $\mathrm{ev} \circ g : \mathbf{P}^1 \to \mathrm{X}$ can be followed by an infinitesimal deformation of $g : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(\mathrm{X}/\mathrm{C}, 1)$. In terms of the pair (R, σ) this means that the image of $\alpha : \mathrm{H}^0(\mathrm{R}, \mathrm{N}_{\mathrm{R}/\mathrm{X}}) \to \mathrm{H}^0(\mathbf{P}^1, \sigma^*\mathrm{N}_{\mathrm{R}/\mathrm{X}})$ contains the image of $\beta : \mathrm{H}^0(\mathbf{P}^1, (\mathrm{ev} \circ g)^*\mathrm{T}_{\mathrm{X}/\mathbf{P}^1}) \to \mathrm{H}^0(\mathbf{P}^1, \sigma^*\mathrm{N}_{\mathrm{R}/\mathrm{X}})$. In this way we conclude that $\mathrm{N}_{\mathrm{R}/\mathrm{X}}$ is at least globally generated over the image of σ.

This weak global generation result in particular implies that $\mathrm{R}^1\pi_*\mathrm{N}_{\mathrm{R}/\mathrm{X}}(-\sigma) = 0$, and $\mathrm{R}^1\pi_*\mathrm{N}_{\mathrm{R}/\mathrm{X}} = 0$; we can for example see this by computing the cohomology on the fibres. Thus we see that $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R}/\mathrm{X}}(-\sigma - \pi^*\mathrm{A})) = \mathrm{H}^1(\mathbf{P}^1, g^*\mathrm{T}_{\mathrm{ev}}(-\mathrm{A}))$. This gives us the vanishing of the cohomology group $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R}/\mathrm{X}}(-\sigma - \pi^*\mathrm{A}))$ for any divisor A of degree $\leq m$ on $\mathbf{P}^1$. We also get $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R}/\mathrm{X}}) = \mathrm{H}^1(\mathbf{P}^1, \pi_*\mathrm{N}_{\mathrm{R}/\mathrm{X}})$, and an exact sequence $0 \to \pi_*\mathrm{N}_{\mathrm{R}/\mathrm{X}}(-\sigma) \to \pi_*\mathrm{N}_{\mathrm{R}/\mathrm{X}} \to \sigma^*\mathrm{N}_{\mathrm{R}/\mathrm{X}} \to 0$. The first sheaf being identified with $g^*\mathrm{T}_{\mathrm{ev}}$ and the second being globally generated we conclude that $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R}/\mathrm{X}}) = 0$. Note that, with $m > 0$ this argument actually also implies that $\mathrm{H}^1(\mathrm{R}, \mathrm{N}_{\mathrm{R}/\mathrm{X}}(-\mathrm{R}_t)) = 0$ for any $t \in \mathrm{C}(k)$. At this point, what is left, is to show that $\mathrm{N}_{\mathrm{R}/\mathrm{X}}$ is globally generated. It is easy to show that a coherent sheaf on $\mathbf{P}^1$ which is globally generated at a point is globally generated. Since $m \geq 1$ the map $\mathrm{H}^0(\mathrm{R}, \mathrm{N}_{\mathrm{R}/\mathrm{X}}) \to \mathrm{H}^0(\mathrm{R}_t, \mathrm{N}_{\mathrm{R}/\mathrm{X}}|_{\mathrm{R}_t})$ is surjective by the vanishing of cohomology we just established. Combined these imply that $\mathrm{N}_{\mathrm{R}/\mathrm{X}}$ is globally generated. ☐

Here is the definition we promised above.

Definition **12.7.** — *Let k be an algebraically closed field of characteristic* 0. *Let* Y *be projective smooth over k, and let $\mathcal{L}$ be an ample invertible sheaf on* Y. *A* scroll *on* Y *is a scroll for the morphism* $\mathrm{pr}_1 : \mathbf{P}^1 \times \mathrm{Y} \to \mathbf{P}^1$, *i.e., it is given by morphisms* $\mathbf{P}^1 \leftarrow \mathrm{R} \to \mathrm{Y}$ *such that (a)* R *is a smooth projective surface, (b) all fibres* R_t *of* $\mathrm{R} \to \mathbf{P}^1$ *are nonsingular rational curves, and (c) the induced maps* $\mathrm{R}_t \to \mathrm{Y}$ *are lines in* Y. *Let* R *be a scroll in* Y *and suppose* $\mathrm{D} \subset \mathrm{R}$ *is a Cartier divisor. For every nonnegative integer m, we say* (R, D) *is an m-*twisting scroll *in* Y *if the diagram*

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \mathbf{P}^1 \times \mathrm{Y} \\ \downarrow & & \downarrow \mathrm{pr}_1 \\ \mathbf{P}^1 & = & \mathbf{P}^1 \end{array}$$

and the divisor D *form an m-twisting scroll with respect to the invertible sheaf* $\mathrm{pr}_2^*\mathcal{L}$. *If m is at least* 2, *an m-twisting scroll will also be called a* very twisting scroll.

The main observation of this section is that if $X \to C$ is as in Situation 8.1, and Hypothesis 7.8 holds over an open part of C, and if the geometric generic fibre has a very twisting scroll, then $X \to C$ also has very twisting scrolls.

Lemma **12.8.** — *In Situation 10.1, assume that* Y *has a very twisting scroll. Then there exist m-twisting scrolls* $(\pi : R \to \mathbf{P}^1, h : R \to Y, \sigma)$ *such that* $\pi_*\mathcal{O}_R(D)$ *and* $\pi_* N_{R/\mathbf{P}^1 \times Y}$ *are ample locally free sheaves on* $\mathbf{P}^1$ *with m arbitrarily large. Moreover, for a sequence* $(Z_e)_{e \geq e_0}$ *as defined in Section 10, we can further arrange it so that* $h \circ \sigma$ *corresponds to an unobstructed point of one of the irreducible components* Z_e.

Proof. — First we have a simple construction which begins with m-twisting scrolls for an integer m, and produces m'-twisting scrolls for a larger integer m'. Let (R, D) be an m-twisting scroll on Y. Choose a section σ of R in $|D|$ and think of the associated morphism $g : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$ as in Lemmas 12.4 and 12.6. In fact Lemma 12.4 implies that the morphism g satisfies the assumptions of Lemma 12.6. It is clear that the assumptions of Lemma 12.6 hold on an open subspace of $\mathrm{Mor}(\mathbf{P}^1, \overline{\mathcal{M}}_{0,1}(Y, 1))$. Also, a morphism g satisfying those assumptions is free. Given two morphisms $g_i : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$, $i = 1, 2$ corresponding to m_i-twisting surfaces with $g_1(\infty) = g_2(0)$ there exists a smoothing (see [Kol96, II Definition 7.1, Theorem 7.6]) of $g_1 \cup g_2 : \mathbf{P}^1 \cup_{\infty \sim 0} \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$ whose general fibre is a morphism $g : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$. As explained above, such a morphism gives a $\mathbf{P}^1$-bundle R over $\mathbf{P}^1$, a Cartier divisor D on R (the "marked point" divisor), and a morphism from R to $\mathbf{P}^1 \times Y$ compatible with the projection to $\mathbf{P}^1$. The hypotheses in Definition 12.3 are each open in families. Thus using that g_1 is m_1-twisting and that g_2 is m_2-twisting, it follows directly that for the general smoothing g, also g is an $(m_1 + m_2 - 1)$-twisting scroll.

By hypothesis, there exist 2-twisting scrolls. Using the construction above, we conclude that there exist 3-twisting scrolls. Also if $(R \to \mathbf{P}^1, R \to Y, D)$ is m-twisting with $m \geq 3$, then $(R \to \mathbf{P}^1, R \to Y, D + R_0)$ is $(m-1)$-twisting and has the property that $\pi_*\mathcal{O}_R(D + R_0)$ is ample.

Fix $m \geq 3$ such that an m-twisting scroll exists. Consider a large integer N and consider maps $g_i : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$, $i = 1, \ldots, N$ each corresponding to an m-twisting scroll $(\pi_i : R_i \to \mathbf{P}^1, h_i : R_i \to X, \sigma_i)$, and such that $g_i(\infty) = g_{i+1}(0)$ for $i = 1, \ldots, N-1$. We may assume that all the free rational curves $h_i \circ \sigma_i$ lie in the same irreducible component Z' of $\overline{\mathcal{M}}_{0,0}(Y, e_0)$, for example by taking the same twisting scroll for each i (with coordinate on $\mathbf{P}^1$ reversed for odd indices i). We may also assume that $N \geq E - e_0$ where E is as in assertion (6) in Section 10. Since each g_i is free we can find a smoothing of the stable map $g_1 \cup \ldots \cup g_N : \mathbf{P}^1 \bigcup_{\infty \sim 0} \cdots \bigcup_{\infty \sim 0} \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$. In fact, we may moreover assume the smoothing of $\mathbf{P}^1 \bigcup_{\infty \sim 0} \cdots \bigcup_{\infty \sim 0} \mathbf{P}^1$ has N sections z_i with z_i limiting to a point t'_i on the ith component of the initial chain.

Let $(\pi : R \to \mathbf{P}^1, h : R \to Y, \sigma)$ correspond to a general point of the smoothing, and let $t_1, \ldots, t_N \in \mathbf{P}^1(k)$ be the values of the sections z_i. We claim that $(R \to \mathbf{P}^1, h : R \to$

$Y, \sigma + \sum R_{t_i})$ is a twisting surface which has two of the three desired properties, i.e., $\pi_* \mathcal{O}_R(\sigma + \sum R_{t_i})$ is ample and the section curve can be chosen to be an unobstructed point of some Z_e. The third property, ampleness of $\pi_* N_{R/X}$, will require more work.

First, since (R, σ) is Nm-twisting we see immediately that $(R, \sigma + \sum R_{t_i})$ is $N(m-1)$ twisting.

Second, the Cartier divisor $\sigma + \sum R_{t_i}$ is a deformation of a divisor on the surface

$$R_1 \bigcup_{R_{1,\infty} \cong R_{2,0}} \cdots \bigcup_{R_{N-1,\infty} \cong R_{N,0}} R_N$$

which restricts to the divisor class of $\sigma_i + R_{i,t'_i}$ on every R_i. By our discussion above we conclude that $\pi_* \mathcal{O}_R(\sigma + \sum R_{t_i})$ is a deformation of a locally free sheaf on a chain of $\mathbf{P}^1$'s which is ample on each link, hence ample.

Third, we show that a general element of $|\sigma + \sum R_{t_i}|$ corresponds to a point of Z_e with $e = Ne_0 + N$. Consider the comb consisting of $h : \sigma(\mathbf{P}^1) \cup \bigcup R_{t_i} \to Y$. It suffices to show that this defines a unobstructed point of $\overline{\mathcal{M}}_{0,0}(Y, e)$ which is in Z_e. It is unobstructed because (R, σ) is a twisting scroll. By construction our comb is a "partial smoothing" of the tree of rational curves

$$(\mathbf{P}^1 \cup R_{1,t'_i}) \bigcup\nolimits_{\infty \sim 0} \cdots \bigcup\nolimits_{\infty \sim 0} (\mathbf{P}^1 \cup R_{N,t'_N}) \xrightarrow{h_i \circ \sigma_i \cup h_i|_{R_{i,t'_i}}} Y.$$

Again since all fibres of twisting scrolls are free lines, and since each σ_i is free this stable map is unobstructed. At this point we may apply Lemma 10.3 to conclude that this stable map is in Z_e.

At this point it is not yet clear that $\pi_* N_{R/X}$ is ample. Let σ' be a section of R representing a general element of $\sigma + \sum R_{t_i}$. We just proved that σ' is a point of Z_e. Let $g' : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(Y, 1)$ be the morphism corresponding to the twisting scroll (R, σ'). We saw above that g' corresponds to a twisting scroll. Since $(g')^* T_{ev}$ has no H^1 there are no obstructions to lifting a given deformation of $ev \circ g' = \sigma'$ to a deformation of g'. Hence we may assume that σ' is a general point of Z_e and in particular we may assume that σ' is very free, see Section 10 (6). At this point consider the exact sequence of locally free sheaves on $\mathbf{P}^1$

$$0 \to \pi_* N_{R/\mathbf{P}^1 \times Y}(-\sigma') \to \pi_* N_{R/\mathbf{P}^1 \times Y} \to \sigma^* N_{R/\mathbf{P}^1 \times Y} \to 0.$$

The sheaf on the left hand side is ample because (R, σ') is $N(m-1)$-twisting. Combined with the surjective map $(\sigma')^* T_Y \to \sigma^* N_{R/\mathbf{P}^1 \times Y}$ and the ampleness of $(\sigma')^* T_Y$ this proves the result. $\square$

In order to formulate the next lemma, we need a definition. A very twisting scroll $(\pi : R \to \mathbf{P}^1, h : R \to Y, \sigma)$ as in the lemma above is called *wonderful* if the pushforward sheaf $\pi_* N_{R/\mathbf{P}^1 \times Y}$ (which is locally free by Remark 12.2) is ample, if $\pi_* \mathcal{O}_R(\sigma)$ is ample, and if the section $h \circ \sigma$ belongs to the canonical irreducible component Z_e defined in Section 10 for some integer e (of course the integer e equals the degree of $h \circ \sigma$).

Lemma **12.9.** — *In Situation 8.1, assume that Hypothesis 7.8 holds over a nonempty open* $S \subset C$. *Assume that for some* $t \in C(k)$ *the fibre* X_t *has a very twisting scroll. Then there exists a family of wonderful, very twisting scrolls, in the following precise sense. There exist*

(1) *a smooth variety* B *over* k,
(2) *a flat morphism* $\underline{t} : B \to C$,
(3) *a smooth projective family of surfaces* $\mathcal{R} \to B$,
(4) *a morphism* $\pi : \mathcal{R} \to \mathbf{P}^1$,
(5) *a morphism* $h : \mathcal{R} \to X$ *such that* $f \circ h = \underline{t}$, *and*
(6) *a morphism* $\sigma : \mathbf{P}^1 \times B \to \mathcal{R}$ *over* B *such that* $\pi \circ \sigma = \mathrm{pr}_1$.

These data satisfy:

(1) *for each* $b \in B(k)$ *the fibre* $(\pi_b : \mathcal{R}_b \to \mathbf{P}^1, h_b : \mathcal{R}_b \to X_{\underline{t}(b)}, \sigma_b)$ *is a wonderful very twisting scroll in* $X_{\underline{t}(b)}$, *and*
(2) *the image of the map*

$$fib_0 : B \longrightarrow \overline{\mathcal{M}}_{0,1}(X/C, 1),$$

which assigns to $b \in B(k)$ *the* 1*-pointed free line* $\sigma_b(0) \in \pi_b^{-1}(0) \to X_{\underline{t}(b)}$ *contains a nonempty open* $V \subset \overline{\mathcal{M}}_{0,1}(X/C, 1)$.

Remark **12.10.** — The formulation above is just one possible formulation of "family of wonderful very twisting scrolls". The important part of the lemma is once there is a single such a scroll, then there are many.

Proof. — Let $(\mathbf{P}^1 \leftarrow R \to X_t, D)$ be a scroll in a fiber. As in the proof of Lemma 12.8, we consider this as a morphism $\mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(\mathbf{P}^1 \times X_t, 1) \subset \overline{\mathcal{M}}_{0,1}(\mathbf{P}^1 \times X, 1)$. And this gives a point in $\mathrm{Mor}(\mathbf{P}^1, \overline{\mathcal{M}}_{0,1}(\mathbf{P}^1 \times X, 1))$. For a suitably chosen R, B will be an open subset of $\mathrm{Mor}(\mathbf{P}^1, \overline{\mathcal{M}}_{0,1}(\mathbf{P}^1 \times X, 1))$ containing this point.

Let $(\mathbf{P}^1 \leftarrow R \to X_t, D)$ be an m-twisting scroll in a fibre. There are no obstructions to deforming the morphism $R \to \mathbf{P}^1 \times X_t$ because $H^1(R, N_{R/\mathbf{P}^1 \times X_t}) = 0$, see Remark 12.2. Also there are also no obstructions to deforming the morphism $R \to \mathbf{P}^1 \times X$: the normal bundle of this morphism is a direct sum of $N_{R/X}$ and a trivial summand $T_t C \otimes \kappa(t) \mathcal{O}_R \cong \mathcal{O}_R$, where the splitting is induced by the derivative df of the morphism $f : X \to C$ (which is constant on R by hypothesis). Since $H^1(R, N_{R/\mathbf{P}^1 \times X_t})$ equals 0 and since also $H^1(R, \mathcal{O}_R)$ equals 0 (as R is a rational surface), also $H^1(R, N_{R/\mathbf{P}^1 \times X})$ equals 0. A deformation of $R \to \mathbf{P}^1 \times X$ is a scroll $R' \to \mathbf{P}^1 \times X$ which is also contained in a fiber $\mathbf{P}^1 \times X_{t'}$ of f over some point t' of C. Also since $\mathcal{O}_R(D)$ is globally generated on R and since $h^1(R, \mathcal{O}_R(D))$ equals 0, we can deform D to a divisor D' on R'. Since the hypotheses of Definition 12.3 are open conditions, a general deformation (R', D') of (R, D) is still m-twisting. And since the normal bundle of $R \to \mathbf{P}^1 \times X$ is globally generated, we see that we may deform $R \to \mathbf{P}^1 \times X_t$ to a morphism $R' \to \mathbf{P}^1 \times X_{t'}$ with t' general and $R' \to X_{t'}$ passing through a general point of $X_{t'}$. So this gives very twisting scrolls in fibres over S. Since Hypothesis 7.8 holds over S, these scrolls in fibres over S are in Situation 10.1. Thus we may apply Lemma 12.8 to these scrolls. Therefore we may now assume that R is a wonderful twisting scroll.

Moreover, let $\sigma : \mathbf{P}^1 \to R$ be a section of $R \to \mathbf{P}^1$ such that $D \sim \sigma$ are rationally equivalent. Pick a point $p \in \mathbf{P}^1(k)$. Since $m > 0$ the map $H^0(R, N_{R/\mathbf{P}^1 \times X_t}(-\sigma)) \to H^0(R_p, N_{R_p/X_t}(-\sigma(p)))$ is surjective. We see that given any infinitesimal deformation of the pointed map $(R_p, \sigma(p)) \to (X_t, h(\sigma(p)))$ (from a pointed line to X_t pointed by the image of the point) in X_t we can find an (unobstructed) infinitesimal deformation of $R \to \mathbf{P}^1 \times X_t$ that induces it. Combined with the fact, proven above, that we can pass a deformation of R through a general point of X this shows that we can deform $R \to X_t$ such that a given fibre of $R \to \mathbf{P}^1$ is a general line in X/C.

Now we consider the point of $\mathrm{Mor}(\mathbf{P}^1, \overline{\mathcal{M}}_{0,1}(\mathbf{P}^1 \times X, 1))$ corresponding to this scroll, and we define B to be a suitable open neighborhood. Finally, what is left is to show that in a family of morphisms of surfaces $\mathcal{R} \to \mathbf{P}^1 \times X$ the locus where the surface is a wonderful m-twisting scroll is open. This follows from semi-continuity of cohomology and can safely be left to the reader, although a very similar and more difficult case is handled in Lemma 12.11 below. □

In Situation 8.1, let $R \to X$ be a ruled surface all of whose fibres are free lines in X/C. Let D be a Cartier divisor on R of degree 1 on the fibres of $R \to C$. Let $t_1, \ldots, t_\delta \in C(k)$ be pairwise distinct points. Let $\mathbf{P}^1 \leftarrow S_i \to X_{t_i}$ be scrolls. Let D_i be a Cartier divisor on S_i of degree 1 on the fibres of $S_i \to \mathbf{P}^1$. Assume given isomorphisms $R_{t_i} \cong S_{i,0}$ of the fibre of R over t_i with the fibre of S_i over 0 compatible with the maps into X_{t_i}. Let $C' = C \cup \bigcup_i \mathbf{P}^1$ be a copy of C with δ copies of $\mathbf{P}^1$ glued by identifying 0 in the ith copy with $t_i \in C(k)$. Let $R' = R \cup \bigcup_i S_i \to C'$ be the ruled surface over C' gotten by gluing R_{t_i} to $S_{i,0}$ using the given isomorphisms. Let $h' : R' \to C' \times_C X$ be the obvious morphism. Note that there is a Cartier divisor D' on R' whose restriction to R is rationally equivalent to D and whose restriction to each S_i is rationally equivalent to D_i. Moreover this Cartier divisor is unique up to rational equivalence.

In the following we are interested in smoothings of situations as above. This means that we have an irreducible variety T over k, and a commutative diagram of varieties

$$\begin{array}{ccccccc}
R' = R \cup \bigcup S_i & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{C} \times_{C \times T} (X \times T) & \longrightarrow & X \times T \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} & & \downarrow & \swarrow & \downarrow \\
C' = C \cup \bigcup \mathbf{P}^1 & \longrightarrow & \mathcal{C} & \longrightarrow & C \times T & \longrightarrow & T
\end{array}$$

satisfying the following conditions: (1) $\mathcal{C} \to T$ and $\mathcal{R} \to T$ are flat and proper, (2) every fibre $\mathcal{C}_t$ of $\mathcal{C} \to T$ over $t \in T(k)$ is a nodal curve of genus $g(C)$ and $\mathcal{C}_t \to C$ has degree 1 (see discussion in Section 6), (3) the morphism $\mathcal{R} \to \mathcal{C}$ is smooth, (4) every fibre $\mathcal{R}_t \to \mathcal{C}_t \times_C X$ is a ruled surface over every irreducible component of $\mathcal{C}_t$, (5) for some point $0 \in T(k)$ the fibre $\mathcal{R}_0 \to \mathcal{C}_0 \times_C X$ is isomorphic to our map $h' : R' \to C' \times_C X$ above, and finally (6) for some point $t \in T(k)$ the fibre $\mathcal{C}_t = C$. In addition we assume given a Cartier divisor $\mathcal{D}$ on $\mathcal{R}$ restricting to the divisor D' on $R' = \mathcal{R}_0$.

Lemma **12.11.** — *In the situation above.*

(1) *If* (R, D) *is m-twisting and* $(S_i, D_i)_{i=1\dots\delta}$ *are* m_i*-twisting, with* $m_i \geq 2$*, then for* $t \in T(k)$ *general the ruled surface* $(\mathcal{R}_t, \mathcal{D}_t)$ *in* X *is* $(m+\delta)$*-twisting.*

(2) *Let* n_i*,* $i = 1, 2, 3$ *be integers such that for all Cartier divisors* A *of degree on* C *we have*
   (a) $\deg(A) \geq n_1 \Rightarrow H^1(C, (\pi_* N_{R/X})(A)) = 0$,
   (b) $\deg(A) \geq n_2 \Rightarrow H^1(C, (\pi_* N_{R/X}(-D))(A)) = 0$, *and*
   (c) $\deg(A) \geq n_3 \Rightarrow H^1(C, (\pi_* \mathcal{O}_R(D))(A)) = 0$.

   If $(S_i, D_i)_{i=1\dots\delta}$ *are wonderful very twisting scrolls and* $\delta \geq \max\{n_1+1, n_2+1, n_3\}$*, then for* $t \in T(k)$ *general the ruled surface* $(\mathcal{R}_t, \mathcal{D}_t)$ *in* X *is* $(\delta - n_3)$*-twisting.*

Proof. — The main part is (2). Part (1) follows by applying the proof of (2) and the proof of Lemma 12.8. Regarding the proof of (1), we would like to remark that the increase in the twisting comes from the fact that the sheaves $(S_i \to \mathbf{P}^1)_*\big(N_{S_i/\mathbf{P}^1 \times X_{t_i}}(-D_i)\big)$ are ample vector bundles on $\mathbf{P}^1$: they are locally free by Remark 12.2, and ampleness follows from the hypothesis that each $m_i \geq 2$, combined with [Kol96, II Lemma 7.10.1].

We begin the proof of (2). Because the morphism $\mathcal{R} \to \mathcal{C}$ is smooth we can define $N_{\mathcal{R}/X}$ by the short exact sequence

$$0 \to T_{\mathcal{R}/\mathcal{C}} \to (\mathcal{R} \to X)^* T_{X/C} \to N_{\mathcal{R}/X} \to 0.$$

As before this is a sheaf on $\mathcal{R}$ which is flat over $\mathcal{C}$. For every point $c \in \mathcal{C}(k)$ lying over $t \in C(k)$ the restriction of $N_{\mathcal{R}/X}$ to the fibre $\mathcal{R}_c$ is the normal bundle of the line $\mathcal{R}_c \to X_t$ (compare with Remark 12.2). For all points $c \in \mathcal{C}(k)$ lying over 0 the sheaf $N_{\mathcal{R}/X}|_{\mathcal{R}_c}$ is globally generated. Hence after replacing T by an open subset containing 0 we may assume this is true for all points c.

Consider the coherent sheaves of $\mathcal{O}_{\mathcal{C}}$-modules $\mathcal{E}_1 = \pi_* \mathcal{O}_R(\mathcal{D})$, $\mathcal{E}_2 = \pi_* N_{\mathcal{R}/X}$ and $\mathcal{E}_3 = \pi_* N_{\mathcal{R}/X}(-\mathcal{D})$. By the above the corresponding higher direct images are zero (as H^1 of a globally generated sheaf on $\mathbf{P}^1$ is zero). The semicontinuity theorem implies $\mathcal{E}_i$ is a locally free sheaf on $\mathcal{C}$. Let $U \subset T$ be a nonempty open such that $\mathcal{C}_t = C$ for all $t \in U(k)$ (this exists because we started with a smoothing). For $t \in U(k)$ we can and do think of $\mathcal{E}_{i,t}$ as a locally free sheaf on C. Let A be a Cartier divisor of degree $\leq \delta - n_3$ on C, and let $p \in C(k)$ be any point. We would like to show that $H^1(C, \mathcal{E}_{1,t}(-p)) = 0$, $H^1(C, \mathcal{E}_{2,t}(-p)) = 0$, and $H^1(C, \mathcal{E}_{3,t}(-A)) = 0$, for t general in $U(k)$. This now follows from [Kol96, II Lemma 7.10.1], the definition of n_i in the lemma and the fact that $\delta \geq n$.

Namely, these vanishings imply that both $\mathcal{E}_{1,t}$, and $\mathcal{E}_{2,t}$ are globally generated and have no H^1, which implies that $\mathcal{O}_{\mathcal{R}_t}(\mathcal{D}_t)$ and $N_{\mathcal{R}_t/X}$ are globally generated and have no H^1. The statement on $\mathcal{E}_{3,t}$ gives the desired vanishing of $H^1(\mathcal{R}_t, N_{\mathcal{R}_t/X}(-\mathcal{D}_t - \pi_t^*A))$. □

Proposition **12.12.** — *In Situation 8.1, assume Hypothesis 7.8 holds over a dense open subset* S *of* C. *Suppose there exists a geometric fibre of* $X \to C$ *which has a very twisting scroll. Then there exists an irreducible component* $Z \subset \Sigma(X/C/k)$ *containing a free section (and thus containing a dense open subset parameterizing free sections), such that for all* $e \gg 0$ *the irreducible component* Z_e *defined in Lemma 8.4 contains a point* $[s]$ *such that* $s = h \circ \sigma$ *for some very twisting scroll* $(h : R \to X, \sigma)$ *in* X.

Proof. — The proof of this proposition is in two stages. In the first stage we will find some sufficiently twisting scroll (R, σ). The irreducible component Z will be the unique one containing $[h \circ \sigma]$. This then determines a sequence $(Z_e)_{e \geq e_0}$ by gluing free lines, as in Section 8. In the second stage we will prove that for all Z_e, a generic section is penned by a very twisting scroll obtained from the initial scroll (R, σ) by attaching wonderful, very twisting scrolls in fibers, and then deforming.

Let B, $\underline{t} : B \to C$, $\mathcal{R} \to B$, $\pi : \mathcal{R} \to \mathbf{P}^1$, $h : \mathcal{R} \to X$, $\sigma : \mathbf{P}^1 \times B \to \mathcal{R}$ be the family of wonderful very twisting scrolls in fibres of X/C we found in Lemma 12.9. Let $V \subset \overline{\mathcal{M}}_{0,1}(X/C, 1)$ be the nonempty open set swept out by fib_0 as in that lemma. In addition, let e_0 be the degree of the rational curves $h_b \circ \sigma_b : \mathbf{P}^1 \to X_{\underline{t}(b)}$.

By Lemma 8.3 there exists a free section s of $X \to C$ meeting $X_{f,\text{pax}}$. We may deform s such that it also meets the open set ev(V). We may also deform s such that s defines a point of the open U of Lemma 12.1. In the proof of Lemma 12.1 we used the main result of [GHS03] to establish the existence of a ruled surface R penning s all of whose fibres are free lines. Hence we may actually choose R such that for some $t \in C(k)$ general the one pointed line $s(t) \in R_t$ defines a point of the open V. Here we use that the moduli space of lines in X_S/S meeting $X_{f,\text{pax}}$ is irreducible by Hypothesis 7.8. Thus we may assume that s is penned by a ruled surface R all of whose fibres are free lines in X/C, by (b) of Lemma 12.1, and such that there exists an open $W \subset C$ with the property that $s(t) \in R_t \to X_t$ defines a point of V for all $t \in W(k)$.

In other words, at every point of W we may attach a wonderful very twisting scroll out of the family $\mathcal{R}/B$ to R. We reinterpret this in terms of the corresponding map $g = g_{(R,\sigma)} : C \to \overline{\mathcal{M}}_{0,1}(X/C, 1)$. In terms of this it means that we can find arbitrarily many free maps $g_\alpha : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(X/C, 1)$ corresponding to wonderful very twisting scrolls in fibres that we may attach to the morphism g to get combs $C \cup \bigcup_\alpha \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(X/C, 1)$. By [Kol96, II Theorem 7.9] an arbitrarily large subcomb will smooth. Now we reinterpret this back into a smoothing of the corresponding glueing of ruled surfaces. It says, via part (2) of Lemma 12.11, that a general point of the base of this smoothing will correspond to a $(e_0 + 1)$-twisting surface $(\pi : R \to C, h : R \to X, \sigma)$. This finishes the proof of the first stage.

Let e_1 be the degree of $h\circ\sigma$, and let $Z\subset\Sigma^{e_1}(X/C/k)$ (as promised) be the unique irreducible component containing the free section $h\circ\sigma$. Fix an integer $i\in\{0,1,\ldots,e_0-1\}$. Pick some effective Cartier divisor $\Delta_i\subset C$ of degree i, and pick a section σ_i of $R\to\mathbf{P}^1$ which is a member of the (base point free) linear system $|\sigma+\pi^*\Delta_i|$. Then (R,σ_i) is (e_0+1-i)-twisting. Thus the associated morphism $g_i: C\to\overline{\mathcal{M}}_{0,1}(X/C,1)$ is free, see Lemma 12.4. In particular after deforming g_i a bit we may assume that g_i meets the open dense subset V above.

For any $\delta>0$ consider pairwise distinct point $t_1,\ldots,t_\delta\in C(k)$ such that $g_i(t_j)\in V$ for all $j=1,\ldots,\delta$. In addition let $g_{ij}:\mathbf{P}^1\to\overline{\mathcal{M}}_{0,1}(X_{t_j},1)$ be morphisms corresponding to wonderful very twisting scrolls out of our family $\mathcal{R}/B$ above with the property that $g_{ij}(0)=g_i(t_j)$ (this is possible). In other words, now we have a comb $\tilde{g}_i: C\cup\bigcup_j\mathbf{P}^1\to\overline{\mathcal{M}}_{0,1}(X/C,1)$. Note that the resulting stable section $\mathrm{ev}\circ\tilde{g}_i$ of X/C has degree $e_1+i+\delta e_0$. Since both g_i and all the g_{ij} are free there exists a smoothing of this comb (for all $\delta\geq 0$). By part (1) of Lemma 12.11 a general point of the base of this smoothing corresponds to a $(e_0+1-i+\delta)$-twisting scroll.

It remains to show that (for every i) the stable section $\mathrm{ev}\circ\tilde{g}_i$ corresponds to a point of $Z_{e_1+i+\delta e_0}$ (since then the same will be true after smoothing the surface). This follows from Lemma 10.4. □

Corollary **12.13.** — *Assumptions and notations are as in the Proposition 12.12 above, with* Z, Z_e *as in that proposition. Let* $P_{e,1}$ *be the moduli space of porcupines* $(s:C\to X, q\in C(k), r\in L(k), L\to X_q)$ *with one quill and body s whose moduli point is in* Z_e. *There exists a dense open* $U_{e,1}\subset P_{e,1}$ *such that if* $[(s:C\to X, q\in C(k), r\in L(k), L\to X_q)]$ *lies in* $U_{e,1}$ *then there exists a ruled surface* R *in* X *penning* $s(C)\cup L$ *(see Definition 9.2) such that* (R,s) *is very twisting.*

Proof. — Although the formulation of this corollary is cumbersome the proof is not. Namely, we already know that for all $e\gg 0$ there exists a very twisting scroll $(\pi:R\to C, h:R\to X,\sigma)$ such that $h\circ\sigma$ lies in Z_e. Pick a general $q\in C(k)$ and consider the datum $(q\in C(k),\pi:R\to C, h:R\to X,\sigma)$. After moving the ruled surface (R,σ) a bit we may assume $h(\sigma(q))$ is a point of $X_{f,\mathrm{pax}}$ such that $(h\circ\sigma)(C)\cup h(R_q)$ is a porcupine. (This step is not strictly necessary for the proof.) At this point it suffices to show that any deformation of the porcupine can be followed by a deformation of the datum $(q\in C(k),\pi:R\to C, h:R\to X,\sigma)$. This is true because $H^1(R,N_{R/X}(-\sigma-R_q))=0$ as (R,σ) was assumed very twisting. □

13. Main theorem

Here is the main theorem of the paper, as well as its important Corollary 13.2.

Theorem **13.1.** — *In Situation 8.1, assume Hypothesis 7.8 holds over an open part* S *of* C. *Suppose there exists a geometric fibre of* $X\to C$ *which has a very twisting scroll. In this case there exist* *an integer* ϵ *and a sequence* $(Z_e)_{e\geq\epsilon}$ *of irreducible components* Z_e *of* $\Sigma^e(X/C/\kappa)$ *as in Lemma 8.4 such that*

(1) *a general point of* Z_e *parametrizes a free section of* f,
(2) *each Abel map*

$$\alpha_{\mathcal{L}}|_{Z_e} : Z_e \to Pic^e_{C/\kappa}$$

has nonempty rationally connected geometric generic fibre, and
(3) *for every free section* s *of* f *there exists an* $E = E(s) > 0$ *such that for all* $e \geq E$, Z_e *is the unique irreducible component of* $\Sigma^e(X/C/\kappa)$ *containing every comb whose handle is* s *and whose teeth are free lines in fibres of* f.

Proof. — We will show the sequence of irreducible components we found in Proposition 12.12 satisfies the conclusions of the theorem. By Lemma 8.5 the irreducible components satisfy the first condition. As a second step we remark that $\{Z_e\}$ satisfies the third part by Corollary 9.8. Also, the geometric generic fibres of $\alpha_{\mathcal{L}}|_{Z_e}$ are irreducible by Lemma 8.9 (for large enough e).

So what is left is to show that these fibres are rationally connected. Since they are proper it is the same as showing they are birationally rationally connected. We will prove this for a general fibre rather than the geometric generic fibre, which is anyway the same thing since our field is uncountable. To show that a variety W/k is birationally rationally connected it suffices to show that for a general pair of points $(p, q) \in W \times W$ there exist open subsets $V_i \subset \mathbf{P}^1$, morphisms $f_i : V_i \to W$, for $i = 1, \ldots, N$ such that (1) each V_i nonempty open, (2) $p \in f_1(V_1)$, (3) $q \in f_N(V_N)$ and (4) $f_i(V_i) \cap f_{i+1}(V_{i+1})$ contains a *smooth* point of W for $i = 1, \ldots, N-1$. See [Kol96, Theorem IV.3.10.3]. This is the criterion we will use. During the proof of the theorem we will call such a chain of rational curves a "useful chain". We apply this criterion to the coarse moduli scheme for the stack Z_e. The proof is an accumulation of intermediate steps proving that various subvarieties of the coarse moduli space are connected to each other by chains of rational curves. And the useful chain is an accumulation of these subchains: we will show there are sufficiently may chains of rational curves to do the job.

A first point is to note that since the target of $\alpha_{\mathcal{L}}$ is an abelian variety every rational curve in Z_e is automatically contracted to a point under $\alpha_{\mathcal{L}}$. Hence we do not have to worry about our curves "leaving" the fibre.

Second, what does Corollary 12.13 say? Let E be the implied (large) constant of Corollary 12.13. It says that for all $e - 1 \geq E$ there is a nonempty open $U_{e-1,1} \subset Z_e$ of the space of porcupines of total degree 1 and one quill which are penned by a very twisting scroll. By Lemma 12.5, all points of $U_{e-1,1}$ are connected by a rational curve to a point in the "interior" of Z_e, i.e., the dense open subset parameterizing free sections. The subset of Z_e which is the union of these rational curves is a constructible subset whose closure properly contains $U_{e-1,1}$. Note that Z_e is irreducible and $U_{e-1,1}$ is a Cartier divisor in Z_e. Thus the only closed subset of Z_e which properly contains $U_{e-1,1}$ is all of Z_e. Therefore this constructible subset is dense in Z_e, i.e., a general point of Z_e is contained in one of these rational curves intersecting $U_{e-1,1}$ in a general point. Now since the points of $U_{e-1,1}$ parameterize unobstructed curves, we conclude: (A) a general point of Z_e is connected by a rational point to a general point of $U_{e-1,1}$.

Thirdly, we repeat the previous argument several times, as follows. Consider the forgetful map $U_{e-1,1} \to Z_{e-1} \times C$ which omits the quill but remembers the attachment point. It has rationally connected smooth projective fibres at least over a suitable open of $Z_{e-1} \times C$, by Hypothesis 7.8 which holds over $S \subset C$. By (A), if $e - 2 \geq E$, then we can find a rational curve connecting a general point of Z_{e-1} to a general point of $U_{e-2,1}$, and by the previous remark we can find a rational curve in $U_{e-1,1}$ connecting a general point of $U_{e-1,1}$ to a point of Z_e which corresponds to a general porcupine with 2 quills. In other words, if $e - 2 \geq E$, we can find a useful chain of rational curves connecting a general point of Z_e to a general porcupine with two quills.

Continuing in this manner, we see that if $e - \delta \geq E$, then we can connect a general point of Z_e by a useful chain of rational curves to a general porcupine with δ quills. It is mathematically more correct to say that we proved that useful chains emanating from points corresponding to porcupines whose body is in $Z_{e-\delta}$ and with δ quills sweep out an open in Z_e.

Fix e_0 an integer such that (a) $e_0 > E$, and (b) the general point of Z_{e_0} corresponds to a $2g(C)+1$-free section, see Lemma 8.8. Let $P \subset Z_{e_0}$ be the set of points corresponding to porcupines without quills. Recall (see just above Proposition 9.7) that P_{e-e_0} indicates porcupines with $e - e_0$ quills and body in P, in other words porcupines with body in Z_{e_0} and $e - e_0$ quills. We apply Proposition 9.7 to the pair $(P, P' = P)$; let E_1 be the constant (which is $\geq e_0$) found in that proposition. The proposition says exactly that for $e \geq E_1$, a general point of

$$P_{e-e_0} \times_{\alpha_{\mathcal{L}}, \underline{\mathrm{Pic}}^e_{C/k}, \alpha_{\mathcal{L}}} P_{e-e_0}$$

can be connected by useful chains in Z_e. Combined with the assertion above that a general point of Z_e may be connected by useful chains to a point of P_{e-e_0}, the proof is complete. □

The impetus of a lot of the research in this paper comes from the following application of the main theorem.

Corollary **13.2.** — *Let k be an algebraically closed field of characteristic* 0. *Let* S *be a smooth, irreducible, projective surface over k. Let*

$$f : X \to S$$

be a flat proper morphism. Assume there exists a Zariski open subset U *of* S *whose complement has codimension* 2 *such that the total space $f^{-1}(U)$ is smooth and such that $f : f^{-1}(U) \to U$ has geometrically irreducible fibres. Let $\mathcal{L}$ be an f-ample invertible sheaf on $f^{-1}(U)$. Assume in addition that the geometric* *generic fibre $X_{\overline{\eta}} \to \overline{\eta}$ satisfies Hypothesis 7.8 and has a very twisting scroll. Then there exists a rational section of f.*

Proof. — Right away we may assume that k is an uncountable algebraically closed field. Also, if $X_{\overline{\eta}} \to \overline{\eta}$ satisfies 7.8 and has a very twisting scroll, then for some nonempty open $V \subset U$ the morphism $X_v \to V$ satisfies 7.8 and the fibres contain very twisting surfaces. In addition we may blow up S (at points of U) and assume we have a morphism

$$b : S \longrightarrow \mathbf{P}^1$$

whose general fibre is a smooth irreducible projective curve. Let $W \subset \mathbf{P}^1$ be a nonempty Zariski open such that all fibres of b over W are smooth, and meet the open V. Consider the space and moduli map

$$\Sigma^e(X_W/S_W/W) \longrightarrow \underline{\mathrm{Pic}}^e_{S_W/W}$$

introduced in Definition 6.2 and Lemma 6.7.

Let $\xi \in W$ be the generic point so its residue field $\kappa(\xi)$ is equal to the function field $k(W)$ of W. Let $\overline{\xi} = \mathrm{Spec}(\overline{k(W)}) \to W$ be a geometric point over ξ corresponding to an algebraic closure of $k(W)$. Note that by our choice of W and the assumption of the corollary the hypotheses of the main theorem (13.1) are satisfied for the morphisms $X_w \to S_w$ for every geometric point w of W. Let us apply this to the geometric generic point $\overline{\xi}$ of W. This gives a sequence of irreducible components $Z_e \subset \Sigma^e(X_W/S_W/W)_{\overline{\xi}}$ say for all $e \geq e_0$. Note that Z_{e_0} can be defined over a finite Galois extension $\kappa(\xi) \subset L$. By construction of the sequence $\{Z_e\}_{e \geq e_0}$, see Lemma 8.4, we see that each Z_e is defined over L. Next, suppose that $\sigma \in \mathrm{Gal}(L/\kappa(\xi))$. Then we similarly have a sequence $\{Z_e^\sigma\}_{e \geq e_0}$ deduced from the sequence by applying σ to the coefficients of the equations of the original Z_e. Of course the sequence $\{Z_e^\sigma\}_{e \geq e_0}$ is the sequence of components deduced from $Z_{e_0}^\sigma$ by Lemma 8.4. Thus $\{Z_e\}_{e \geq e_0}$ and $\{Z_e^\sigma\}_{e \geq e_0}$ satisfy the hypotheses of Corollary 9.8. So, by Corollary 9.8, there exists an integer $e_1 \geq e_0$ such that Z_e equals Z_e^σ for all $e \geq e_1$. Since there are only finitely many elements σ in $\mathrm{Gal}(L/\kappa(\xi))$, in fact there exists a single integer $e_1 \geq e_0$ such that for every $e \geq e_1$ and for every σ in $\mathrm{Gal}(L/\kappa(\xi))$, Z_e equals Z_e^σ. Thus Z_e is Galois invariant. Thus it is the base change of an irreducible component defined over $\kappa(\xi)$. So for $e \gg 0$ the irreducible components $Z_e \subset \Sigma^e(X_W/S_W/W)_{\overline{\xi}}$ are the geometric fibres of irreducible components $Z_{W,e} \subset \Sigma^e(X_W/S_W/W)$!

The conclusion of the above is that we know that for all $e \gg 0$ there exist irreducible components $Z_{W,e} \subset \Sigma^e(X_W/S_W/W)$ such that the induced morphisms

$$Z_{W,e} \longrightarrow \underline{\mathrm{Pic}}^e_{S_W/W}$$

have birationally rationally connected nonempty geometric generic fibre. But note that since $\Sigma^e(X_W/S_W/W)$ is actually proper over W, so is $Z_{W,e}$ and hence its fibres over

$\underline{\mathrm{Pic}}^e_{S_W/W}$ are actually rationally connected (more precisely the underlying coarse moduli spaces of the fibres are rationally connected).

At this point we are done by the lemmas below. Namely, Lemma 13.3 says it is enough to construct a rational section of $Z_{W,e} \to W$, and even such a rational section to its coarse moduli space $Z^{\mathrm{coarse}}_{W,e}$. Since the original surface S was projective, we can find for arbitrarily large e divisors D on S which give rise to sections of $\underline{\mathrm{Pic}}^e_{S_W/W} \to W$. On the other hand, since $\underline{\mathrm{Pic}}^e_{S_W/W}$ is smooth, Lemma 13.4 guarantees, because of the result on the geometric generic fibre above, that we can lift this up to a section of $Z^{\mathrm{coarse}}_{W,e} \to W$. Thus the proof of the corollary is finished. □

Lemma **13.3.** — *Let k be an algebraically closed field of characteristic zero. Let* S *be a variety over k. Let* $C \to S$ *be a family of smooth projective curves. Let* $X \to C$ *be a proper flat morphism. Let* $\mathcal{L}$ *be an invertible sheaf on* X *ample w.r.t.* $X \to C$. *If* $\Sigma^e(X/C/S) \to S$ *(see Definition 6.2) has a rational section for some e, then* $X \to C$ *has a rational section. The same conclusion holds if we replace* $\Sigma^e(X/C/S)$ *by its coarse moduli space.*

Proof. — After shrinking S we have to show that if there is a section, then $X \to C$ has a rational section. Let $\tau : S \to \Sigma(X/C/S)$ be a section. This corresponds to a proper flat family of curves $\mathcal{C} \to S$, and a morphism $\mathcal{C} \to X$ such that $\mathcal{C} \to C$ is a degree 1 map from a nodal, genus 0 curve to C. See Section 6 and the initial discussion in that section. In particular there is a nonempty Zariski open $\mathcal{U} \subset \mathcal{C}$ such that $\mathcal{U} \to \mathcal{C} \to X \to C$ is an open immersion. (For example, $\mathcal{U}$ is the open part where the morphism $\mathcal{C} \to C$ is flat, or it is the part obtained by excising the vertical pieces of the stable maps.) This proves the first assertion of the lemma.

To prove the final assertion, after shrinking S as before, let $t : S \to \Sigma(X/C/S)^{\mathrm{coarse}}$ be a map to the coarse moduli space. After shrinking S some more we may assume there is a surjective finite étale Galois morphism $\nu : S' \to S$ such that $t \circ \nu$ corresponds to a morphism $\tau' : S' \to \Sigma(X/C/S)'_S = \Sigma(X_{S'}/C_{S'}/S')$. By the arguments in the first part of the proof this corresponds to a section of $X_{S'} \to C_{S'}$. To see that it descends to a rational section of $X \to C$ it is enough to see that it is Galois invariant. This is clear because the only stackiness in the spaces of stable maps comes from the contracted components. □

Lemma **13.4.** — *Let* $\alpha : Z \to P$ *be a proper morphism of varieties over an algebraically closed field k of characteristic zero. Assume the geometric generic fibre is rationally connected, and* P *is quasi-projective. Let* $C \subset P$ *be any curve meeting the nonsingular locus of* P. *Then there exists a rational section of* $\alpha^{-1}(C) \to C$.

Proof. — (Sketch only, see also similar arguments in [GHMS05].) Curve means irreducible and reduced closed subscheme of dimension 1. Thus C is generically nonsingular and meets the nonsingular locus of P. This means C is locally at some point a complete intersection in P. This means that globally C is an irreducible component of multiplicity 1 of a global complete intersection curve on P (here we use that P is quasi-projective).

This means that there exists a 1-parameter flat family of complete intersections $\mathcal{C} \subset \mathrm{P} \times \mathrm{T}$ parametrized by a smooth irreducible curve T/k such that C is an irreducible component of multiplicity 1 of a fibre $\mathcal{C}_t$ for some $t \in \mathrm{T}(k)$ and such that the general fibre is a smooth irreducible curve passing through a general point of P. In other words, by [GHS03] and the assumption of the lemma, we can find a lift of $\mathcal{C}_{\overline{k(\mathrm{T})}} \to \mathrm{P}$ to a morphism $\gamma : \mathcal{C}_{\overline{k(\mathrm{T})}} \to \mathrm{X}$ into X. Note that if $\mathrm{T}' \to \mathrm{T}$ is a nonconstant morphism of nonsingular curves over k and $t' \in \mathrm{T}'(k)$ maps to $t \in \mathrm{T}(k)$ then the pullback family $\mathcal{C}' = \mathcal{C} \times_{\mathrm{T}} \mathrm{T}'$ still has C as a component of multiplicity one of the fibre over t'. Using this, we may assume that γ is defined over $k(\mathrm{T})$. In other words we obtain a rational map $\mathcal{C} \supset \mathcal{W} \xrightarrow{\gamma} \mathrm{X}$ lifting the morphism $\mathcal{C} \to \mathrm{P}$. Since T is nonsingular, and since C is a component of multiplicity one of the fibre of $\mathcal{C} \to \mathrm{T}$ we see that the generic point of C corresponds to a codimension 1 point ξ of $\mathcal{C}$ whose local ring is a DVR. Because $\mathrm{X} \to \mathrm{P}$ is proper we see, by the valuative criterion of properness that we may assume that γ is defined at ξ. This gives the desired rational map from C into X. □

14. Special rational curves on flag varieties

In this section we show that flag varieties contain many good rational curves. In characteristic $p > 0$ there exist closed subgroup schemes H of reductive groups G such that G/H is projective, but H not reduced. To avoid dealing with these, we say $\mathrm{P} \subset \mathrm{G}$ is a *parabolic subgroup* if P is geometrically reduced and G/P is projective.

Theorem **14.1.** — *Let k be an algebraically closed field. Let* G *be a connected, reductive algebraic group over k. Let* P *be a parabolic subgroup of* G. *There exists a k-morphism $f : \mathbf{P}^1 \to \mathrm{G}/\mathrm{P}$ such that for every parabolic subgroup* Q *of* G *containing* P, *denoting the projection by*

$$\pi : \mathrm{G}/\mathrm{P} \to \mathrm{G}/\mathrm{Q},$$

we have that the pullback of the relative tangent bundle $f^*\mathrm{T}_\pi$ is ample on $\mathbf{P}^1$.

Remark **14.2.** — Note that for any *fixed* Q, since the fibres of π are rationally connected there exists a morphism $f : \mathbf{P}^1 \to \mathrm{G}/\mathrm{P}$ such that $f^*\mathrm{T}_\pi$ is ample. The importance of Theorem 14.1 is that f has this property for *all* parabolics Q simultaneously.

The purpose of this section is to prove Theorem 14.1. We use some results in the theory of reductive linear algebraic groups. It turns out that the proof in the case $\mathrm{X} = \mathrm{G}/\mathrm{B}$ is somewhat easier. Thus we first reduce the general case to this case by a simple geometric argument.

Lemma **14.3.** — *It suffices to prove Theorem 14.1 when* P *is a Borel subgroup.*

Proof. — If P is not a Borel, then pick any $B \subset P$ Borel contained in P. Let $g : \mathbf{P}^1 \to G/B$ be a morphism as in Theorem 14.1. We claim that the composition f of g with the projection map $G/B \to G/P$ has the necessary property. Namely, if $P \subset Q$ is another parabolic, then we have a commutative diagram

$$\begin{array}{ccc} G/B & \longrightarrow & G/P \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi} \\ G/Q & = & G/Q \end{array}$$

which proves that f^*T_π is a quotient of g^*T_τ, and hence is ample. □

The outline of the proof of Theorem 14.1 in the case of G/B is as follows. We will construct a curve $C \cong \mathbf{P}^1$ in G/B which is the closure of the orbit of a $\mathbf{G}_m$-action, such that all the weights of the induced action of $\mathbf{G}_m$ at T_0G/B, resp. $T_\infty G/B$ are positive, resp. negative.

We review some of the notation of [Spr98], see also [MR04, Section 2]. Let G be a connected reductive algebraic group over an algebraically closed field k. Choose a maximal torus $T \subset G$ and let $T \subset B$ be a Borel subgroup of G containing T. Let B^- be the opposite Borel subgroup, so that $T = B \cap B^-$. Let U (resp. U^-) be the unipotent radical of B (resp. B^-). Let $X = X^*(T) = \mathrm{Hom}\,(T, \mathbf{G}_m)$ and $Y = X_*(T) = \mathrm{Hom}\,(\mathbf{G}_m, T)$ be the character and cocharacter groups of T. Let $(\,,) : X \times Y \to \mathbf{Z}$ be the pairing defined by $x(y(t)) = t^{(x,y)}$ for $t \in \mathbf{G}_m$. The choice of $T \subset B \subset G$ determines the set of roots $\Phi \subset X$, the set of positive roots Φ^+, as well as its complement Φ^-. For $\alpha \in \Phi$, let U_α be the root subgroup corresponding to α. Our normalization is that $\alpha \in \Phi^+ \Leftrightarrow U_\alpha \subset B$. We denote the simple roots $\Delta = \{\alpha_i \mid i \in I\}$. So here I is implicitly defined as the index set for the simple roots. For any $J \subset I$, let Φ_J be the set of roots spanned by simple roots in J and $\Phi_J^+ = \Phi_J \cap \Phi^+$, $\Phi_J^- = \Phi_J \cap \Phi^-$. Let P_J be the parabolic subgroup corresponding to J which is characterized by $U_\alpha \subset P_J \Leftrightarrow (\alpha \in \Phi_J \text{ or } \alpha \in \Phi^+)$. Let $X_J = G/P_J$ the corresponding flag variety (there will be no subscripts to $X = X^*(T)$ and so we will avoid any possible confusion). Note that $X_\emptyset = G/B$ classifies Borel subgroups of G and is sometimes written as $\mathcal{B}$.

For each $i \in I$ there is a morphism of algebraic groups

$$h_i : SL_{2,k} \to G$$

which maps the diagonal torus into T, such that $a \mapsto h_i\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ is an isomorphism $x_i : \mathbf{G}_a \to U_{\alpha_i}$, and such that $a \mapsto h_i\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$ is an isomorphism $y_i : \mathbf{G}_a \to U_{-\alpha_i}$. It follows that $\alpha_i\big(h_i\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\big)$ is equal to t^2. The cocharacter $\mathbf{G}_m \to T$, $t \mapsto h_i\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ is the coroot $\alpha_i^\vee$ dual to the simple root α_i. The datum $(B, T, x_i, y_i; i \in I)$ is called an *épinglage* or a *pinning* of G.

For $i \in \mathrm{I}$, let s_i be the corresponding simple reflection in the Weyl group $\mathrm{W} = \mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$. The element $\dot{s}_i = h_i\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{N}_{\mathrm{G}}(\mathrm{T})$ is a representative of s_i. Then

(**a**) $$y_i(1) = x_i(1)\dot{s}_i x_i(1).$$

For $w \in \mathrm{W}$, we denote by $l(w)$ its length and we set $\dot{w} = \dot{s}_{i_1}\dot{s}_{i_2}\cdots\dot{s}_{i_n}$, where $s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression of w. This expression is independent of the choice of the reduced expression for w.

For any subgroup H of G, we denote by Lie(H) its Lie algebra. (We will think of this as the spectrum of the symmetric algebra on the cotangent space of G at the identity.) Let us recall the following result (see [Slo80, page 26, Lemma 4]).

Lemma **14.4.** — *Let* H *be a closed subgroup scheme of* G. *Let* X *be a scheme of finite type over* k *endowed with a left* G*-action. Let* $f : \mathrm{X} \to \mathrm{G}/\mathrm{H}$ *be a* G*-equivariant morphism from* X *to the homogeneous space* G/H. *Let* $\mathrm{E} \subset \mathrm{X}$ *be the scheme theoretic fibre* $f1(\mathrm{H}/\mathrm{H})$. *Then* E *inherits a left* H*-action and the map* $\mathrm{G} \times_{\mathrm{H}} \mathrm{E} \to \mathrm{X}$ *sending* (g, e) *to* $g \cdot e$ *defines a* G*-equivariant isomorphism of schemes.*

Let $\mathrm{J} \subset \mathrm{I}$ and $e = \mathrm{P}_{\mathrm{J}}/\mathrm{P}_{\mathrm{J}} \in \mathrm{X}_{\mathrm{J}} = \mathrm{G}/\mathrm{P}_{\mathrm{J}}$. We are going to apply the lemma to the tangent bundle $\mathrm{T}_{\mathrm{X}_{\mathrm{J}}}$ of X_{J} and the G-equivariant map $f : \mathrm{T}_{\mathrm{X}_{\mathrm{J}}} \to \mathrm{X}_{\mathrm{J}}$. Note that $f1(e) = \mathrm{T}_{\mathrm{X}_{\mathrm{J}},e} = \mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{P}_{\mathrm{J}})$ because $1 \in \mathrm{G}$ lifts $e \in \mathrm{X}_{\mathrm{J}}$. For an element $p \in \mathrm{P}_{\mathrm{J}}$ the left multiplication $p : \mathrm{X}_{\mathrm{J}} \to \mathrm{X}_{\mathrm{J}}$ lifts to $\mathrm{inn}_p : \mathrm{G} \to \mathrm{G}$, $g \mapsto pgp1$ because $pgp1\mathrm{P}_{\mathrm{J}} = pg\mathrm{P}_{\mathrm{J}}$. Since inn_p fixes $1 \in \mathrm{G}$ and acts by the adjoint action on Lie(G) it follows that the P_{J}-action on $f1(e) = \mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{P}_{\mathrm{J}})$ is the adjoint action of P_{J} on $\mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{P}_{\mathrm{J}})$. We define an action of P_{J} on $\mathrm{G} \times \mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{P}_{\mathrm{J}})$ by $p \cdot (g, k) = (gp1, \mathrm{Ad}(p) \cdot k)$. Let $\mathrm{G} \times_{\mathrm{P}_{\mathrm{J}}} \mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{P}_{\mathrm{J}})$ be the quotient space. Then by the lemma, we have

$$\mathrm{T}_{\mathrm{X}_{\mathrm{J}}} \cong \mathrm{G} \times_{\mathrm{P}_{\mathrm{J}}} \mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{P}_{\mathrm{J}}).$$

In particular, when $\mathrm{J} = \emptyset$ we get that the tangent bundle of $\mathrm{X}_{\emptyset} = \mathrm{G}/\mathrm{B}$ is identified with $\mathrm{G} \times_{\mathrm{B}} \mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{B})$.

What about the relative tangent bundles? Consider a subset $\mathrm{J} \subset \mathrm{I}$ as before. Then $\mathrm{B} \subset \mathrm{P}_{\mathrm{J}}$ and we obtain a canonical morphism $p_{\mathrm{J}} : \mathrm{X}_{\emptyset} \to \mathrm{X}_{\mathrm{J}}$. Note that $\mathrm{Lie}(\mathrm{P}_{\mathrm{J}})/\mathrm{Lie}(\mathrm{B})$ is a B-stable subspace of Lie(G)/Lie(B). Again by the above lemma, we see that the vertical tangent bundle $\mathrm{T}_{p_{\mathrm{J}}} \subset \mathrm{T}_{\mathrm{X}_{\emptyset}}$ of the projection map p_{J} is isomorphic to

$$\mathrm{G} \times_{\mathrm{B}} \mathrm{Lie}(\mathrm{P}_{\mathrm{J}})/\mathrm{Lie}(\mathrm{B}) \subset \mathrm{G} \times_{\mathrm{B}} \mathrm{Lie}(\mathrm{G})/\mathrm{Lie}(\mathrm{B}).$$

Consider the adjoint group $ad(\mathrm{G})$ of G together with the image $ad(\mathrm{T})$ of T. The root lattice $\mathrm{X}^*(ad(\mathrm{T}))$ is the free $\mathbf{Z}$-module $\bigoplus_{i \in \mathrm{I}} \mathbf{Z}\alpha_i \subset \mathrm{X} = \mathrm{X}^*(\mathrm{T})$. Let $\rho^\vee \in \mathrm{X}_*(ad(\mathrm{T}))$ be the cocharacter such that $(\alpha_i, \rho^\vee) = 1$ for all $i \in \mathrm{I}$. In other words,

$$\rho^\vee : \mathbf{G}_m \to ad(\mathrm{G})$$

is a morphism into $ad(\mathrm{T})$ such that $\rho^\vee(t)$ acts via multiplication by t on U_{α_i} for all $i \in \mathrm{I}$. From this we conclude that, for any $u \in \mathrm{U}$ we have

$$\lim_{t\to 0}\ \mathrm{inn}_{\rho^\vee(t)}(u) = 1$$

and, for any $u^- \in \mathrm{U}^-$ we have

$$\lim_{t\to\infty}\ \mathrm{inn}_{\rho^\vee(t)}(u^-) = 1$$

Here, by abuse of notation, for an element $\bar{g} \in ad(\mathrm{G})$ we denote $\mathrm{inn}_{\bar{g}}$ the associated inner automorphism of G; in other words, $\mathrm{inn}_{\bar{g}} = \mathrm{inn}_g$ for any choice of $g \in \mathrm{G}$ mapping to $\bar{g} \in ad(\mathrm{G})$. Since each $\mathrm{P_J}$ contains T and hence the center of G we see that $\mathbf{G}_m$ acts on $\mathrm{X_J} = \mathrm{G}/\mathrm{P_J}$ via $\rho^\vee$. This action induces a $\mathbf{G}_m$-action on the bundles $\mathrm{T}_{\mathrm{G/B}}$ and $\mathrm{T}_{p_\mathrm{J}}$. Under the isomorphism $\mathrm{T}_{p_\mathrm{J}} = \mathrm{G} \times_\mathrm{B} \mathrm{Lie}(\mathrm{P_J})/\mathrm{Lie}(\mathrm{B})$ this action is given by $t \cdot (g, k) = (\mathrm{inn}_{\rho^\vee(t)}(g), \mathrm{A}d_{\rho^\vee(t)}(k))$. We leave the verification to the reader.

Let w be the maximal element in W and let $w = s_{i_1}s_{i_2}\cdots s_{i_n}$ be a reduced expression of w. Set $u = x_{i_1}(1)\dot{s}_{i_1}x_{i_2}(1)\dot{s}_{i_2}\cdots x_{i_n}(1)s_{i_n}$. By [MR04, Proposition 5.2], $u \cdot \mathrm{B/B} \in (\mathrm{B}\dot{w}\cdot \mathrm{B/B}) \cap (\mathrm{B}^- \cdot \mathrm{B/B})$. In particular, $u \cdot \mathrm{B/B} = u_1\dot{w}\cdot \mathrm{B/B} = u_2 \cdot \mathrm{B/B}$ for some $u_1 \in \mathrm{U}$ and $u_2 \in \mathrm{U}^-$. This is the only property of u that we will use. Another way to find an element with this property is as follows: Since w is the longest element we have $\dot{w}\mathrm{B}\dot{w}^{-1} = \mathrm{B}^-$. Thus finding a u as above amounts to showing that $\mathrm{B}\dot{w}\mathrm{B} \cap \mathrm{B}^-\mathrm{B}$ is nonempty, which is clear since both are open in G (because $\mathrm{B}w\mathrm{B} = w\mathrm{B}^-\mathrm{B}$).

Let C be the closure of $\{\mathrm{inn}_{\rho^\vee(t)}(u)\cdot \mathrm{B/B};\, t \in \mathbf{k}^*\}$ in $\mathrm{X}_\emptyset$. In other words, C is the closure of the $\mathbf{G}_m$-orbit of the point $u_\emptyset := u \cdot \mathrm{B/B}$ of $\mathrm{X}_\emptyset$ with $\mathbf{G}_m$ acting via $\rho^\vee$. Then

$$\lim_{t\to\infty} t\cdot u_\emptyset = \lim_{t\to\infty} \mathrm{inn}_{\rho^\vee(t)}(u_2)\cdot \mathrm{B/B} = \mathrm{B/B},$$
$$\lim_{t\to 0} t\cdot u_\emptyset = \lim_{t\to 0} \mathrm{inn}_{\rho^\vee(t)}(u_1)\dot{w}\cdot \mathrm{B/B} = \dot{w}\cdot \mathrm{B/B}.$$

Then $\mathrm{C} = \{\mathrm{inn}_{\rho^\vee(t)}(u_\emptyset);\, t \in \mathbf{k}^*\} \sqcup [0] \sqcup [\infty]$, where $[0] = \dot{w}\cdot \mathrm{P_J/P_J}$ and $[\infty] = \mathrm{P_J/P_J}$. Let

$$h : \mathbf{P}^1 \to \mathrm{C} \subset \mathrm{X_J}$$

be the $\mathbf{G}_m$-equivariant morphism with $h(t) = t\cdot u$, $h(0) = [0]$, and $h(\infty) = [\infty]$. (Warning: h needs not be birational.)

With this notation $h^*\mathrm{T}_{p_\mathrm{J}}$ is a $\mathbf{G}_m$-equivariant vector bundle over $\mathbf{P}^1$. It is an exercise to show that an $\mathbf{G}_m$-equivariant bundle on $\mathbf{P}^1$ is a direct sum of $\mathbf{G}_m$-equivariant line bundles. Thus we have that $h^*\mathrm{T}_{p_\mathrm{J}} = \bigoplus \mathrm{L}_l$ for some $\mathbf{G}_m$-equivariant line bundles L_l over $\mathbf{P}^1$. We would like to compute the degrees of the line bundles L_l. For any $\mathbf{G}_m$-equivariant line bundle L on $\mathbf{P}^1$ we define two integers $n(\mathrm{L}, 0)$, and $n(\mathrm{L}, \infty)$ as the weight of the

$\mathbf{G}_m$-action on the fibre at 0, respectively ∞. We leave it to the reader to prove the formula

$$\deg \mathrm{L} = n(\mathrm{L}, 0) - n(\mathrm{L}, \infty),$$

see also Remark 14.5 below. In order to compute the degrees of the L_l we compute the integers $n(\mathrm{L}_l, 0)$ and $n(\mathrm{L}_l, \infty)$. And for this in turn we compute the fibre of $\mathrm{T}_{p_\mathrm{J}}$ at the points $[0]$ and $[\infty]$ as $\mathbf{G}_m$-representations. In fact both points are T-invariant and hence we can think of $\mathrm{T}_{p_\mathrm{J}}|_{[0]}$ and $\mathrm{T}_{p_\mathrm{J}}|_{[\infty]}$ as representations of T.

The fibre of $\mathrm{T}_{p_\mathrm{J}}$ at the point $[\infty] = \mathrm{B}/\mathrm{B}$ is just the vector space $\mathrm{Lie}(\mathrm{P_J})/\mathrm{Lie}(\mathrm{B})$ with T acting via the adjoint action. We have $\mathrm{Lie}(\mathrm{P_J}) = \mathrm{Lie}(\mathrm{B}) \oplus \bigoplus_{\alpha\in\Phi_\mathrm{J}^-} \mathrm{Lie}(\mathrm{U}_\alpha)$. Thus

$$\mathrm{Lie}(\mathrm{P_J})/\mathrm{Lie}(\mathrm{B}) \cong \bigoplus_{\alpha\in\Phi_\mathrm{J}^-} \mathrm{Lie}(\mathrm{U}_\alpha).$$

How does $\mathbf{G}_m$ act on this? It acts via the coroot $\rho^\vee$. For each $\alpha \in \Phi_\mathrm{J}^-$ we have $(\alpha, \rho^\vee) < 0$, hence all the weights are negative, in other words all the $n(\mathrm{L}_l, \infty)$ are negative.

The fibre of $\mathrm{T}_{p_\mathrm{J}}$ at the point $[0] = \dot{w}\cdot \mathrm{B}/\mathrm{B}$ is the space $\{\dot{w}\} \times \mathrm{Lie}(\mathrm{P_J})/\mathrm{Lie}(\mathrm{B})$ inside $\mathrm{G} \times_\mathrm{B} \mathrm{Lie}(\mathrm{P_J})/\mathrm{Lie}(\mathrm{B})$. For an element $t \in \mathrm{T}$ and an element $k \in \mathrm{Lie}(\mathrm{P_J})/\mathrm{Lie}(\mathrm{B})$ we have (see above)

$$\begin{aligned} t\cdot(\dot{w}, k) &= (\mathrm{inn}_t(\dot{w}), \mathrm{A}d_t(k)) \\ &= (t\dot{w}t^{-1}, \mathrm{A}d_t(k)) \\ &= (\dot{w}t^w t^{-1}, \mathrm{A}d_t(k)) \\ &= (\dot{w}, \mathrm{A}d_{t^w}(\mathrm{A}d_{t^{-1}}(\mathrm{A}d_t(k)))) \\ &= (\dot{w}, \mathrm{A}d_{t^w}(k)). \end{aligned}$$

Thus the weights for T on $\mathrm{T}_{p_\mathrm{J}}|_{[0]}$ are the $w(\alpha)$, where $\alpha \in \Phi_\mathrm{J}^-$. How does $\mathbf{G}_m$ act on this? It acts via the coroot $\rho^\vee$. For each $\alpha \in \Phi_\mathrm{J}^-$ we have $(w(\alpha), \rho^\vee) > 0$, because the longest element w exchanges positive and negative roots. This follows directly from $\dot{w}\mathrm{B}\dot{w}^{-1} = \mathrm{B}^-$. Hence all the weights are positive, in other words all the $n(\mathrm{L}_l, 0)$ are positive.

From this we conclude that all the line bundles L_l have positive degree. In other words we have shown that $h^*\mathrm{T}_{p_\mathrm{J}}$ is a positive vector bundle for all $\mathrm{J} \subset \mathrm{I}$. This proves Theorem 14.1.

Remark **14.5.** — The sign of the formula for the degree of an equivariant line bundle on $\mathbf{P}^1$ may appear to be wrong due to the fact that we are working with line bundles and not invertible sheaves (which is dual). The actual sign of the formula for an equivariant line bundle on $\mathbf{P}^1$ does not matter for the argument however. Namely, in either case the rest of the arguments show that all the line bundles L_l have the same sign. And since we know that G/B is Fano we know that the sum of all the degrees of the line bundles in the pullback of $\mathrm{T}_{\mathrm{G/B}}$ has to be positive.

15. Rational simple connectedness of homogeneous spaces

In this section we show that a projective homogeneous space Z of Picard number 1 has a very twisting scroll, see Corollary 15.4. We also prove Z is rationally simply connected by chains of free lines, see Lemmas 15.7 and 15.8.

Let k be an algebraically closed field of characteristic 0. Let X and Y be smooth, connected, quasi-projective k-schemes. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let $u : \mathrm{Y} \to \overline{\mathcal{M}}_{0,0}(\mathrm{X}, 1)$ be a morphism. This corresponds to the X, $\mathcal{C}$, Y, p, q in the following diagram of k-schemes

$$\begin{array}{ccccc} \mathbf{P}^1 & \xrightarrow{\sigma} & \mathcal{C} & \xrightarrow{p} & \mathrm{X} \\ & \searrow^{\tau} & \downarrow{\scriptstyle q} & & \\ & & \mathrm{Y}. & & \end{array}$$

Automatically q is a smooth, projective morphism whose geometric fibres are isomorphic to $\mathbf{P}^1$ which map to lines in X via p. There is an associated 1-morphism

$$v : \mathcal{C} \to \overline{\mathcal{M}}_{0,1}(\mathrm{X}, 1).$$

Proposition **15.1.** — *In the situation above with $\sigma : \mathbf{P}^1 \to \mathcal{C}$ as in the displayed diagram. Assume that*

(1) *all lines parametrized by* Y *are free,*
(2) *p is smooth,*
(3) *u is unramified,*
(4) *either τ is a free rational curve on* Y*, or $p \circ \sigma : \mathbf{P}^1 \to \mathrm{X}$ is free,*
(5) *$\sigma^*\mathrm{T}_p$ and $\sigma^*\mathrm{T}_q$ are ample.*

Then X *has a very twisting scroll.*

Proof. — We will show the pair (R, s) where the scroll R is defined as $\mathrm{R} = \mathbf{P}^1 \times_{q\circ\sigma, \mathrm{Y}, q} \mathcal{C}$ and with section $s : \mathbf{P}^1 \to \mathrm{R}$ given by σ satisfies all the properties of Definition 12.7. It is enough to show this after replacing k by any algebraically closed extension so we may assume k is uncountable. Product will mean product over Spec(k).

We will use the following remarks frequently below. Denote the projection to $\mathbf{P}^1$ by $q_\mathrm{R} : \mathrm{R} \to \mathbf{P}^1$ and the projection to $\mathcal{C}$ by $\sigma_\mathrm{R} : \mathrm{R} \to \mathcal{C}$. For every coherent sheaf $\mathcal{F}$ on $\mathcal{C}$, cohomology and base change implies the natural map

$$\tau^*\mathrm{R}^1 q_* \mathcal{F} \to \mathrm{R}^1 q_{\mathrm{R},*}(\sigma_\mathrm{R}^* \mathcal{F})$$

is an isomorphism. If the sheaves $\mathrm{R}^1 q_\mathrm{R}(\mathcal{F})$ are zero (along the image of τ) then

$$\tau^* q_* \mathcal{F} \to q_{\mathrm{R},*}(\sigma_\mathrm{R}^* \mathcal{F})$$

is an isomorphism.

Note that $N_{R/\mathbf{P}^1\times X}$ equals the pullback of $N_{\mathcal{C}/Y\times X}$. Because $T_{X\times Y} = \mathrm{pr}_1^* T_X \oplus \mathrm{pr}_2^* T_Y$, and because q is smooth there is a short exact sequence of locally free sheaves $0 \to T_p \to q^*T_Y \to N_{\mathcal{C}/Y\times X} \to 0$ on $\mathcal{C}$. There is an associated long exact sequence

$$0 \to q_*T_p \to q_*q^*T_Y \xrightarrow{\alpha} q_*N_{\mathcal{C}/Y\times X} \to R^1q_*T_p \to 0$$

using the fact that $R^1q_*q^*T_Y$ is zero. We can say more: (a) $q_*q^*T_Y$ is canonically isomorphic to T_Y, (b) $q_*N_{\mathcal{C}/Y\times X}$ is canonically isomorphic to $u^*T_{\overline{\mathcal{M}}_{0,0}(X,1)}$ (see Remark 12.2), and (c) the sheaf homomorphism α above is canonically isomorphic to du. By hypothesis u is unramified, in other words α is injective. Thus $q_*T_p = 0$, and we get an isomorphism $R^1q_*T_p = N_{Y/\overline{\mathcal{M}}_{0,0}(X,1)}$. We conclude that $R^1q_*T_p$ is locally free, and formation of $R^1q_*T_p$ and q_*T_p commutes with arbitrary change of base. By relative duality, we get that $R^1q_*T_p$ is dual to $q_*(\Omega_p \otimes_{\mathcal{O}_\mathcal{C}} \Omega_q)$, and that $R^1q_*(\Omega_p \otimes_{\mathcal{O}_\mathcal{C}} \Omega_q) = 0$.

Claim **15.2.** — *The sheaf $\tau^*q_*(\Omega_p \otimes_{\mathcal{O}_\mathcal{C}} \Omega_q)$ is anti-ample, and thus $\tau^*N_{Y/\overline{\mathcal{M}}_{0,0}(X,1)}$ is ample.*

The surface R is a Hirzebruch surface. Denote by $\mathcal{E}$ the pullback of $\Omega_p \otimes_{\mathcal{O}_\mathcal{C}} \Omega_q$ to R. By our remarks above the claim is equivalent to the assertion that $(q_R)_*\mathcal{E}$ is anti-ample. The normal bundle $N_{s(\mathbf{P}^1)/R}$ is canonically isomorphic to σ^*T_q, which is ample by hypothesis. Therefore the divisor $s(\mathbf{P}^1)$ on R moves in a basepoint free linear system. This implies the section s deforms to a family $\{s_t\}_{t\in\Pi}$ of sections whose images cover a dense open subset of R. Of course the pullback of $\Omega_p \otimes_{\mathcal{O}_\mathcal{C}} \Omega_q$ to R is a locally free sheaf $\mathcal{E}$ whose dual $\mathcal{E}^\vee$ is canonically isomorphic to the pullback of $T_p \otimes_{\mathcal{O}_\mathcal{C}} T_q$. And $s^*\mathcal{E}^\vee$ equals $\sigma^*T_p \otimes_{\mathcal{O}_{\mathbf{P}^1}} \sigma^*T_q$. By hypothesis, each of σ^*T_p and σ^*T_q is ample on $\mathbf{P}^1$. Thus $s^*\mathcal{E}^\vee$ is ample on $\mathbf{P}^1$. Since ampleness is an open condition, after shrinking Π we may assume $s_t^*\mathcal{E}^\vee$ is ample for $t \in \Pi$. Therefore, $s_t^*\mathcal{E}$ is anti-ample.

For every t there is an evaluation morphism

$$e_t : (q_R)_*\mathcal{E} \to s_t^*\mathcal{E}.$$

Of course, since the curves $s_t(\mathbf{P}^1)$ cover a dense open subset of R, the only local section of $(q_R)_*\mathcal{E}$ in the kernel of *every* evaluation morphism e_t is the zero section. Since $(q_R)_*\mathcal{E}$ is a coherent sheaf, in fact for $N \gg 0$ and for $t_1, \ldots, t_N$ a general collection of closed points of Π, the morphism

$$(e_{t_1}, \ldots, e_{t_n}) : (q_R)_*\mathcal{E} \longrightarrow \bigoplus_{i=1}^{N} s_{t_i}^*\mathcal{E}$$

is injective. Since every summand $s_{t_i}^*\mathcal{E}$ is anti-ample, the direct sum is anti-ample. And a locally free sheaf admitting an injective sheaf homomorphism to an anti-ample sheaf is itself anti-ample. Therefore $(q_R)_*\mathcal{E}$ is anti-ample, proving Claim 15.2.

As usual, we denote by $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ the evaluation morphism. Thus $\mathrm{ev} \circ v = p$ as morphisms $\mathcal{C} \to X$. Consider the diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{v} & \overline{\mathcal{M}}_{0,1}(X, 1) & \xrightarrow{\mathrm{ev}} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \text{forget}} & & \\ Y & \xrightarrow{u} & \overline{\mathcal{M}}_{0,0}(X, 1) & & \end{array}$$

whose square is Cartesian. Since we assumed that all lines parametrized by Y are free it follows that $\overline{\mathcal{M}}_{0,1}(X, 1)$ is nonsingular at every point of $v(\mathcal{C})$. As p is assumed smooth, it follows that $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is smooth at all points of $v(\mathcal{C})$ (by Jacobian criterion for example). Also, the diagram shows v is unramified and $N_{\mathcal{C}/\overline{\mathcal{M}}_{0,1}(X,1)}$ is canonically isomorphic to $q^*N_{Y/\overline{\mathcal{M}}_{0,0}(X,1)}$. Smoothness of ev and p implies we have a short exact sequence

$$0 \to T_p \to v^*T_{\mathrm{ev}} \to N_{\mathcal{C}/\overline{\mathcal{M}}_{0,1}(X,1)} \to 0$$

By hypothesis, σ^*T_p is ample. Claim 15.2 implies $\tau^*N_{Y/\overline{\mathcal{M}}_{0,0}(X,1)} = \sigma^*N_{\mathcal{C}/\overline{\mathcal{M}}_{0,1}(X,1)}$ is ample. Therefore $(v \circ \sigma)^*T_{\mathrm{ev}}$ is ample.

At this point we can apply Lemma 12.6 to the surface $R \to \mathbf{P}^1 \times X$ and the section s, which combine to give the morphism $g = v \circ \sigma : \mathbf{P}^1 \to \overline{\mathcal{M}}_{0,1}(\mathbf{P}^1 \times X/\mathbf{P}^1, 1)$. Condition (1) corresponds to the fact that s moves on R. Condition (2) holds because the lines parametrized by Y are free. Condition (3) with $m = 2$ holds because, as we just saw, the sheaf $(v \circ \sigma)^*T_{\mathrm{ev}}$ is ample. It remains to see that the morphism $p \circ \sigma : \mathbf{P}^1 \to X$ is free. This follows either by assumption, or if τ is free we argue as follows: The pullback $(p \circ \sigma)^*T_X$ is a quotient of $\sigma^*T_{\mathcal{C}}$. The sheaf $\sigma^*T_{\mathcal{C}}$ is sandwiched between σ^*T_q and $\sigma^*q^*T_Y$ and by assumption both are globally generated. Hence $\sigma^*T_{\mathcal{C}}$ and also $(p \circ \sigma)^*T_X$ is globally generated. ☐

Remark **15.3.** — In the "abstract" situation of Proposition 15.1 we can often deduce that the fibres of $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ are rationally connected using [Sta04, Proposition 3.6] (the proof of this uses a characteristic 0 hypothesis). We will avoid this below to keep the paper more self contained; instead we will use a little more group theory.

The new result in the following corollary is the statement about very twisting scrolls.

Corollary **15.4.** — *Let* G *be a connected semi-simple group over* k *and let* $P \subset G$ *be a maximal parabolic subgroup. Let* $Z = G/P$. *Let* $\mathcal{L}$ *be an ample generator for* $Pic(Z) \cong \mathbf{Z}$. *Then the evaluation morphism*

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(Z, 1) \to Z$$

is surjective, smooth and projective. In addition, there exists a very twisting scroll in Z.

Proof. — The action of G on Z determines a sheaf homomorphism

$$\mathrm{T}_e\mathrm{G} \otimes_k \mathcal{O}_\mathrm{Z} \to \mathrm{T}_\mathrm{Z}$$

which is surjective because the action is separable and X is homogeneous. Thus T_Z is globally generated. Therefore every smooth, rational curve in Z is free. This implies

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(\mathrm{Z}, 1) \to \mathrm{Z}$$

is smooth. It is always projective, although at this point the space of lines on Z could be empty.

Let B be a Borel subgroup of G contained in P, and let T be a maximal torus in B. The data (G, B, T) determine a root system $(\mathrm{X}, \Phi, \mathrm{Y}, \Phi^\wedge)$ as in Section 14; we will use the notation introduced there. As P is a maximal parabolic subgroup, P equals the parabolic subgroup P_J where $\mathrm{J} = \mathrm{I} - \{j\}$ for an element j of I. Thus $\mathrm{Z} = \mathrm{G}/\mathrm{P}_\mathrm{J} = \mathrm{X}_\mathrm{J}$ in the notation of Section 14. It is proved in [Coh95, § 4.20] and [CC98, Lemma 3.1] that the subvariety

$$\mathrm{L} := \mathrm{P}_{\{j\}} \cdot \mathrm{P}_\mathrm{J}/\mathrm{P}_\mathrm{J}$$

is a line in $\mathrm{G}/\mathrm{P}_\mathrm{J}$ with respect to the ample sheaf $\mathcal{L}$. (See also Remark 15.5 below.)

The action of G on Z induces an action of G on $\overline{\mathcal{M}}_{0,0}(\mathrm{Z}, 1)$ and $\overline{\mathcal{M}}_{0,1}(\mathrm{Z}, 1)$. Let $p = \mathrm{P}_\mathrm{J}/\mathrm{P}_\mathrm{J} \in \mathrm{G}/\mathrm{P}_\mathrm{J}$. This is a point on our line L. The stabilizer of (L, p) in $\overline{\mathcal{M}}_{0,1}(\mathrm{Z}, 1)$ contains the Borel subgroup B, and thus is of the form P_K for a subset $\mathrm{K} \subset \mathrm{J} = \mathrm{I} - \{j\}$. Since P_K is parabolic, the orbit $\mathcal{C}$ of (L, p) is a projective (hence closed) G-orbit. The image Y of $\mathcal{C}$ in $\overline{\mathcal{M}}_{0,0}(\mathrm{Z}, 1)$ is also a projective G-orbit. Observe that P_j acts transitively on the subset $\{(\mathrm{L}, q) | q \in \mathrm{L}\}$ of $\overline{\mathcal{M}}_{0,1}(\mathrm{Z}, 1)$ by construction of L. Thus the fibre of the forgetful morphism $\overline{\mathcal{M}}_{0,1}(\mathrm{Z}, 1) \to \overline{\mathcal{M}}_{0,0}(\mathrm{Z}, 1)$ over [L] equals the fibre of $\mathcal{C} \to \mathrm{Y}$ over [L]. By homogeneity it follows that

$$\mathcal{C} = \overline{\mathcal{M}}_{0,1}(\mathrm{Z}, 1) \times_{\overline{\mathcal{M}}_{0,0}(\mathrm{Z},1)} \mathrm{Y}.$$

So the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & \mathrm{Z} \\ {\scriptstyle q}\downarrow & & \\ \mathrm{Y} & & \end{array}$$

is a diagram of smooth, projective morphisms where every geometric fibre of q is a smooth, rational curve.

By Theorem 14.1, there exists a morphism,

$$\sigma : \mathbf{P}^1 \to \mathcal{C}$$

such that $\sigma^* T_p$ and $\sigma^* T_q$ are both ample sheaves. The morphism p is smooth by homogeneity. And the morphism $u : Y \to \overline{\mathcal{M}}_{0,0}(Z, 1)$ is a closed immersion hence unramified. Thus, by Proposition 15.1, there is a very twisting scroll in Z. □

Remark **15.5.** — The variety G/P can be constructed as the orbit of the highest weight vector in $\mathbf{P}(V_{\omega_j})$ where ω_j is the jth fundamental weight (this is what we called β in the proof above). Here V_{ω_j} is the representation with highest weight ω_j. Under this embedding $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$ and lines are lines in $\mathbf{P}(V_{\omega_j})$. In particular $\mathcal{L}$ is very ample.

Lemma **15.6.** — *Let* G *be a linear algebraic group over* k. *Let* X, Y *be proper* G*-varieties over* k. *Let* $Y \to X$ *be a* G*-equivariant morphism. Suppose that* Y *is rationally connected and that* X *has an open orbit whose stabilizer is a connected linear algebraic group. Then the geometric generic fibre of* $Y \to X$ *is rationally connected.*

Proof. — Let $x \in X(k)$ be a point in the open orbit. Let H be the stabilizer of x in G, which is a connected linear algebraic group by assumption. Let $F = Y_x$ be the fibre of $Y \to X$ over x. Consider the morphism $\Psi : G \times F \to Y$, $(g, f) \mapsto g \cdot f$. This is a dominant morphism of varieties over k. By construction the fibres of Ψ are isomorphic to the birationally rationally connected variety H. It follows from the main result of [GHS03] that $G \times F$ is birationally rationally connected. Hence F is rationally connected. □

Note that the Bruhat decomposition in particular implies that the variety $G/P \times G/P$ has finitely many G-orbits. In addition all stabilizers of points of $G/P \times G/P$ are connected, for example by [Bor91, Proposition 14.22].

Lemma **15.7.** — *Let* $Z = G/P$, $\mathcal{L}$ *be as in Corollary 15.4 above. Then every geometric fibre of*

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(Z, 1) \to Z$$

is nonempty and rationally connected.

Proof. — Because Z is homogeneous, all fibres are isomorphic and it suffices to prove one fibre is rationally connected. By Lemma 15.6, it suffices to show that $\overline{\mathcal{M}}_{0,1}(Z, 1)$ is rationally connected. Clearly, it suffices to show that $\overline{\mathcal{M}}_{0,0}(Z, 1)$ is rationally connected. Since a line is determined by any pair of points it passes through, this follows from the remarks about the Bruhat decomposition immediately preceding the lemma. □

Lemma **15.8.** — *Let* $Z = G/P$, $\mathcal{L}$ *be as in Corollary 15.4 above. There exists a positive integer* n *and a nonempty open subset* V *of* $Z \times Z$ *such that the geometric fibres of the evaluation* *morphism*

$$\mathrm{ev}_{1,n+1} : \mathrm{FreeChain}_2(Z/k, n) \longrightarrow Z \times Z$$

over V *are nonempty, irreducible and birationally rationally connected. (Notation as in Section 7.)*

Proof. — We will use without mention that every line on Z is free.

We rephrase a few arguments of Campana and Kollár-Miyaoka-Mori in this particularly nice setting. Consider the diagram

$$\overline{\mathcal{M}}_{0,0}(Z, 1) \xleftarrow{\Phi} \overline{\mathcal{M}}_{0,1}(Z, 1) \xrightarrow{\mathrm{ev}} Z$$

For any closed subset $T \subset Z$, the closed set $T' = \mathrm{ev}(\Phi^{-1}(\Phi(\mathrm{ev}^{-1}(T))))$ is the set of points which are connected by a line passing through a point in T. Note that if T is irreducible, then so is T′ since by Corollary 15.4 the morphism ev is smooth with irreducible fibres according to Lemma 15.7. For any point $z \in Z(k)$ consider the increasing sequence of closed subvarieties

$$T_0(z) = \{z\} \subset T_1(z) = T_0(z)' \subset T_2(z) = T_1(z)' \subset \dots$$

The first is the variety of points that lie on a line passing through z. The second is the set of points which lie on a chain of lines of length 2 passing through z, etc. Let $n \geq 0$ be the first integer such that the dimension of $T_n(z)$ is maximal. Then obviously $T_{n+1}(z) = T_n(z)$. Hence $T_{2n} = T_n(z)$. We conclude that for every point $z' \in T_n(z)$ we have $T_n(z') = T_n(z)$. Consider the map $Z \to \mathrm{Hilb}_{Z/k}$, $z \mapsto [T_n(z)]$. By what we just saw the fibres of this map are exactly the varieties $T_n(z)$. Since $\mathrm{Pic}(Z) = \mathbf{Z}$ we conclude that either $T_n(z) = \{z\}$, or $T_n(z) = Z$. The first possibility is clearly excluded and hence $T_n(z) = Z$, in other words every pair of points on Z can be connected by a chain of lines of length n.

Thus the morphism

$$\mathrm{ev} : \mathrm{FreeChain}_2(Z/k, n) \to Z \times_k Z$$

is surjective. This is a G-equivariant morphism for the evident actions of G on the domain and target. There exists an open orbit in $Z \times Z$, and all stabilizers are connected, see the remarks preceding Lemma 15.7. By Lemma 15.6 it suffices to prove that $\mathrm{FreeChain}_2(Z/k, n)$ is rationally connected. For $n = 1$ this was shown in the proof of Lemma 15.7. For larger n it follows by induction on n from Lemma 15.7 in exactly the same way as the proof of the corresponding statement of Lemma 7.11. □

16. Families of projective homogeneous varieties

The main theorem in this section is Theorem 16.6: every split family of homogeneous spaces (see below) over a quasi-projective surface has a rational section. There are several steps in the proof. Lemma 16.1 reduces to the case of Picard number 1. Lemma 16.3 shows it suffices to prove the theorem in characteristic zero. The technique of discriminant avoidence reduces it to the special case where the base is a projective surface, which is proved in Theorem 16.5.

In this section k is an algebraically closed field of any characteristic. A *projective homogeneous space* over k or an algebraically closed extension of k is a variety isomorphic to G/P where G is a connected, semi-simple, simply connected linear algebraic group and $P \subset G$ is a parabolic subgroup. (We do not allow exotic, nonreduced, parabolic subgroup schemes.) Let S be a variety over k, and let $\overline{\eta} \to S$ be a geometric generic point. A *split family of homogeneous spaces* over S will be a proper smooth morphism $X \to S$ such that $X_{\overline{\eta}}$ is a projective homogeneous space and that $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{\eta}})$ is surjective.

Lemma **16.1.** — *Let k be an algebraically closed field of any characteristic. Let $X \to S$ be a split family of homogeneous spaces over S. Then, after possibly shrinking S, there exists a factorization*

$$X \to Y \to S$$

such that $X \to Y$ is a split family of homogeneous spaces over Y, such that $Y \to S$ is a split family of homogeneous spaces over S and such that $\mathrm{Pic}(Y_{\overline{\eta}}) = \mathbf{Z}$.

Proof. — Choose an isomorphism $X_{\overline{\eta}} = G/P$ for some G, P over $\overline{\eta}$. Choose a maximal parabolic $P \subset P'$ containing P. Let $\mathcal{L}_0$ be an ample generator of $\mathrm{Pic}(G/P')$, as in Corollary 15.4, its proof and Remark 15.5. Let $\mathcal{L}$ be an invertible sheaf on X whose restriction to $X_{\overline{\eta}}$ is isomorphic to the pullback of $\mathcal{L}_0$ to $X_{\overline{\eta}} = G/P$ via the morphism $\pi : G/P \to G/P'$. Since $\mathcal{L}_0$ is ample, some power $\mathcal{L}_0^N$ is very ample and globally generated. Also, $H^0(G/P, \pi^*\mathcal{L}_0^N) = H^0(G/P', \mathcal{L}_0^N)$ as the fibres of π are connected projective varieties. We conclude, by cohomology and base change, that after shrinking S we may assume that $\mathcal{L}^N$ is globally generated on X.

Choose a finite collection of sections $s_0, \dots, s_M$ which generate the $\kappa(\overline{\eta})$-vector space $H^0(G/P, \pi^*\mathcal{L}_0^N) = H^0(G/P', \mathcal{L}_0^N)$. Consider the morphism $X \to \mathbf{P}^M_S$ over S. Note that on the geometric generic fibre the image is the projective embedding of G/P' via the sections $s_0, \dots, s_M$. Hence, after possibly shrinking S and using Stein factorization, we get a factorization $X \to Y \to S$ such that $Y_{\overline{\eta}} \cong G/P'$ with $X_{\overline{\eta}} \to Y_{\overline{\eta}}$ equal to π. The family $Y \to S$ is split because the invertible sheaf $\mathcal{L}$ descends to Y due to the fact that it does so on the geometric generic fibre; really, just take the pushforward of $\mathcal{L}$ under $X \to Y$.

The geometric generic fibre of $X \to Y$ is isomorphic to P/P' which is a projective homogeneous variety as well (for a possibly different group). To show that $X \to Y$ is split it suffices to show that $\mathrm{Pic}(G/P) \to \mathrm{Pic}(P'/P)$ is surjective, as $X \to S$ is split. This surjectivity can be proved using the fact that the derived group of the Levi group of P' is simply connected (since G is simply connected), combined with for example the results of [Pop74]. We prefer to prove it by considering the Leray spectral sequence for $f : G/P \to G/P'$ and the sheaf $\mathbf{G}_m$. Namely, $R^1 f_* \mathbf{G}_m$ is a constant sheaf with value

$\mathrm{Pic}(P/P')$ (due to the fact that G/P' has trivial fundamental group). Hence the obstruction to surjectivity is an element in the Brauer group $\mathrm{B}r(G/P')$. Since k is algebraically closed and G/P' is rational (by Bruhat decomposition) we see $\mathrm{B}r(G/P') = 0$ and we win. □

Let k be an algebraically closed field of positive characteristic. Let R be a Cohen ring for k (for example the Witt ring of k). Let G be a connected, semi-simple, simply connected linear algebraic group over k, and let $P \subset G$ be a parabolic subgroup. According to Chevalley ([Che95], see also [Dem65] and [Spr98]) there exists a reductive group scheme G_R over R and a parabolic subgroup scheme $P_R \subset G_R$ whose fibre over k recovers the pair (G, P). Denote $Z_R = G_R/P_R$ the smooth projective G_R-scheme over R whose special fibre is $Z = G/P$. Since $\mathrm{Pic}(Z_R)$ is isomorphic to the character group of P_R, and since by construction the character group of P_R equals that of P, we see that $\mathrm{Pic}(Z_R) = \mathrm{Pic}(Z)$. Choose an ample invertible sheaf $\mathcal{O}(1)$ on Z_R.

Lemma **16.2.** — *The automorphism scheme* $Aut_R((Z_R, \mathcal{O}(1)))$ *is smooth over* R.

Proof. — The scheme of automorphisms $\mathrm{Aut}_R(Z_R)$ is smooth over R by [Dem77, Proposition 4]. Thus it suffices to show that the morphism $\mathrm{Aut}_R((Z_R, \mathcal{O}(1))) \to \mathrm{Aut}_R(Z_R)$ is smooth. By deformation theory it suffices to show that $H^1(Z, \mathcal{O}_Z) = 0$. This follows from [Kem76, Section 6 Theorem 1]. □

The following lemma follows from a general "lifting" argument that was explained to us by Ofer Gabber, Jean-Louis Colliot-Thélène and Max Lieblich. For a brief explanation, see Remark 16.4 below.

Lemma **16.3.** — *Let* k, R, G, P, G_R, P_R *as above. Assume that* P *is maximal parabolic in* G. *Let* $\Omega = \overline{K}$ *be an algebraic closure of the field of fractions of* R. *Suppose that every split family of homogeneous spaces over a surface over* Ω *whose general fibre is isomorphic to* G_Ω/P_Ω *has a rational section. Then the same is true over* k.

Proof. — Let $\mathcal{O}(1)$ on Z_R be an ample generator of $\mathrm{Pic}(Z_R) = \mathrm{Pic}(Z) = \mathbf{Z}$. Denote $H_R = \mathrm{Aut}_R((Z_R, \mathcal{O}(1)))$. By Lemma 16.2 we see that H_R is flat over R. At this point [dJS05a, Corollary 2.3.3] applies with our H_R playing the role of the group named G_R in ibid., and with our Z_R playing the role of the scheme called V_R in ibid. □

Remark **16.4.** — What is involved in the proof of [dJS05a, Corollary 2.3.3]? Consider a split family $X \to S/k$ of homogeneous spaces whose general fibre is isomorphic to $Z = G/P$. Let $\mathcal{L}$ be an invertible sheaf on X restricting to a generator of the Picard group of a general fibre. Then $\mathrm{Isom}((X, \mathcal{L}), (Z, \mathcal{O}_Z(1)))$ is a H_k-torsor over a nonempty affine open $U \subset S$ of the surface S. After shrinking U we may assume U is smooth over k. After this one lifts the smooth morphism $U \to \mathrm{Spec}(k)$ to a smooth morphism $W \to \mathrm{Spec}(R)$ using [Elk73]. The "lifting" result mentioned above is the statement that a H_k-torsor over

U is the restriction of a H_R-torsor over W after possibly replacing W by $W' \to W$ étale and trivial over a nonempty open U. This torsor in turn defines a split family of homogeneous spaces $\mathcal{X} \to W$ restricting to $X_U \to U$ over U. It is split because the geometric generic fibre has Picard group $\mathbf{Z}$ and the ample generator exists by virtue of the fact that H_R acts on $\mathcal{O}(1)$ on Z_R. At this point, a standard specialization argument implies the result. For details see [dJS05a].

Theorem **16.5.** — *Let k be an algebraically closed field of characteristic* 0*. Let* G *be a linear algebraic group over k and let* $P \subset G$ *be a maximal parabolic subgroup. Let* S *be a nonsingular projective surface over k. Let* $X \to S$ *be a smooth projective morphism all of whose geometric fibres are isomorphic to* G/P*. Assume there exists an invertible sheaf* $\mathcal{L}$ *on* X *which restricts to an ample generator of Pic*(G/P) *on a fibre. Then* $X \to S$ *has a rational section.*

Proof. — The varieties G/P are rationally simply connected by chains of free lines and contain very twisting surfaces, see Corollary 15.4 and Lemmas 15.7 and 15.8. Therefore using Corollary 13.2 we immediately obtain the theorem. □

Theorem **16.6.** — *Let k be an algebraically closed field of any characteristic. Let* S *be a quasi-projective surface over k. Let* $X \to S$ *be a split family of projective homogeneous spaces. Then* $X \to S$ *has a rational section.*

Proof. — By Lemma 16.1 it suffices to prove the theorem in case the general fibre has Picard number $\rho = 1$, i.e., the parabolic group is maximal. By Lemma 16.3 it suffices to prove the result in the $\rho = 1$ case in characteristic zero. The case $\rho = 1$, S is a *projective* smooth surface and $X \to S$ is smooth and projective is Theorem 16.5.

Thus it remains to deduce the general $\rho = 1$ case in characteristic 0 from the case where the base is projective. This is done by the method of *discriminant avoidance*. Precise references are [dJS05b, Theorem 1.3], or [dJS05a, Theorem 1.0.1] (either reference contains a complete argument; the first giving a more down to earth approach than the second). Please see Remark 16.7 below for a brief synopsis of the argument. □

Remark **16.7.** — Let k be an algebraically closed field (of any characteristic). As before let $Z = G/P$ with P maximal parabolic and let $\mathcal{O}(1)$ be an ample generator of Pic(Z). Also, as before let $H = \mathrm{Aut}((Z, \mathcal{O}(1)))$ be the automorphism scheme of the pair. According to the proof of Lemma 16.2 and the references therein, the group scheme H is reductive. Let S be a quasi-projective surface over k. Suppose that $X \to S$ is a split family of projective homogeneous spaces whose general fibre is isomorphic to Z. We will explain the method of [dJS05a] reducing the problem of finding a rational section of $X \to S$ to the problem in a case where the base surface is projective.

Choose an invertible sheaf $\mathcal{L}$ on X whose restriction to a general fibre is an ample generator. As in Remark 16.4 there exists an open U and an H-torsor $\mathcal{T}$ over U such that

$(X_U, \mathcal{L}_U)$ is isomorphic over U to $(\mathcal{T} \times_H Z, \mathcal{O} \boxtimes_H \mathcal{O}(1))$. The main result of [dJS05a] implies that there exists a flat morphism $W \to \mathrm{Spec}(k[[t]])$, an open immersion $j : W_k \to U$, a H-torsor $\mathcal{T}'$ over W such that (1) $\mathcal{T}'_k \cong j^*\mathcal{T}$ and (2) the generic fibre $W_{k((t))}$ is projective. We may use the torsor $\mathcal{T}'$ to obtain an extension of the pullback family j^*X to a split family $\mathcal{X} \to W$ of projective homogeneous spaces over all of W. At this point a standard specialization argument implies that the existence of a rational section of $\mathcal{X}_{\overline{k((t))}} \to W_{\overline{k((t))}}$ implies the existence of a rational section of the original $X \to S$.

The main result of [dJS05a] mentioned above is the following: If H is a reductive algebraic group, and $c \geq 0$ an integer then the algebraic stack $B(H) = [\mathrm{Spec}(k)/H]$ allows a morphism $X \to B(H)$ surjective on field points such that X has a projective compactification $X \subset \overline{X}$ whose boundary has codimension $\geq c$ in $\overline{X}$. This result easily implies the existence of the family of torsors as above. For details see the references above.

17. Application to families of complete intersections

In this section we sketch a proof of Tsen's theorem in characteristic 0 for families of *hypersurfaces* using the main results of this paper. As explained in the introduction this is silly, but hopefully provides a useful illustration of our methods.

Let $K \supset k$ be the function field of a surface over an algebraically closed field k. Tsen's theorem states that any complete intersection of type $(d_1, \ldots, d_c)$ in $\mathbf{P}^n_K$ has a K-rational point if $\sum d_i^2 \leq n$. See [Lan52]. When the inequality is violated then there are function fields K and complete intersections without rational points.

First of all we may replace k by an uncountable algebraically closed overfield of characteristic zero (by standard limit techniques). Second we may assume $d_i \geq 2$ for all i. The moduli space M of all complete intersections X of type $(d_1, \ldots, d_c)$ in $\mathbf{P}^n$ is an open subset of a product of projective spaces $M \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_c}$. Note that we do not require complete intersections to be nonsingular; it is then easy to see that the complement of M in $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_c}$ has codimension ≥ 2. It is also straightforward to see that the total space of the universal family $\mathcal{X} \to M$ is nonsingular. It follows by standard specialization techniques that to prove Tsen's theorem for all K and all complete intersections, it suffices to prove it for those X/K such that K corresponds to the function field of a surface $S \subset M$ with the following properties:

(1) S passes through a very general point of M,
(2) the closure $\overline{S} \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_c}$ differs from S by finitely many points, and
(3) the restriction of $\mathcal{X}$ to S has smooth total space.

At this point Theorem 1.1 says that it is enough to show that a very general complete intersection Y of type $(d_1, \ldots, d_c)$ in $\mathbf{P}^n$ is rationally simply connected by chains of free lines, and contains a very twisting surface.

To check that Y is rationally simply connected by chains of free lines, i.e., that $Y \to \mathrm{Spec}(k)$ satisfies Hypothesis 7.8 we point out the following:

(1) The space of lines through a general point $y \in Y(k)$ is a nonsingular complete intersection of type $(1, 2, \ldots, d_1, 1, 2, \ldots, d_2, \ldots, 1, 2, \ldots, d_c)$ in a $\mathbf{P}^{n-1}$. Because $1 + 2 + \cdots + d_1 + 1 + 2 + \cdots + d_2 + \cdots + 1 + 2 + \cdots + d_c = \sum d_i(d_i + 1)/2 \leq \sum(d_i^2 - 1) \leq n - 1$ it is a smooth projective Fano and hence rationally connected.

(2) For a general pair of points $x, y \in Y(k)$ the space of pairs of lines connecting x and y is parametrized by a nonsingular complete intersection of type $(1, 2, \ldots, d_1 - 1, 1, 2, \ldots, d_1 - 1, d_1, \ldots, 1, 2, \ldots, d_c - 1, 1, 2, \ldots, d_c - 1, d_c)$ in $\mathbf{P}^n$. (To see this consider the equations for the locus of the intersection points of the two lines.) The sum of the degrees is equal to $\sum d_i^2 \leq n$ and again we see that this is a smooth projective Fano variety and hence rationally connected.

To prove this argue as in [Kol96, Chapter V Section 4], especially the exercises. See also [dJS06]. This proves Hypothesis 7.8 for Y/k.

As usual the hardest thing to check is the existence of a very twisting scroll. A simple parameter count shows that a very twisting scroll should exist on a general complete intersection with $\sum d_i^2 \leq n$. We do not know how to prove the existence unless $c = 1$. Namely, in [HS05] it was shown that very twisting scrolls exist if $d_1^2 + d_1 + 1 \leq n$. In the preprint [Sta04] it is shown that very twisting scrolls exist in a general hypersurface if $d_1^2 \leq n$, thereby concluding our sketch of the proof of Tsen's theorem for families of hypersurfaces.

Final remark: In the preprint [dJS06] very twisting surfaces are constructed in any smooth complete intersection in $\mathbf{P}^n$ under the assumption that $n + 1 \geq \sum(2d_i^2 - d_i)$. But note that these surfaces are not ruled by lines in every case; sometimes they are ruled by conics—to which the methods of this paper do not apply.

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