DRINFELD ASSOCIATORS, BRAID GROUPS AND EXPLICIT SOLUTIONS OF THE KASHIWARA–VERGNE EQUATIONS

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ABSTRACT

The Kashiwara–Vergne (KV) conjecture states the existence of solutions of a pair of equations related with the Campbell–Baker–Hausdorff series. It was solved by Meinrenken and the first author over $\mathbf{R}$, and in a formal version, by two of the authors over a field of characteristic 0. In this paper, we give a simple and explicit formula for a map from the set of Drinfeld associators to the set of solutions of the formal KV equations. Both sets are torsors under the actions of prounipotent groups, and we show that this map is a morphism of torsors. When specialized to the KZ associator, our construction yields a solution over $\mathbf{R}$ of the original KV conjecture.

Introduction and main results

The Kashiwara–Vergne conjecture. — The desire to understand Duflo's theorem according to which there is an algebra isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{g})^{\mathfrak{g}}$, where $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, led Kashiwara and Vergne to the following conjecture:

Conjecture **1** (See [KV]). — *For* $\mathfrak{g}$ *as above, there exists a pair of Lie series* $A(x, y), B(x, y) \in \hat{\mathfrak{f}}_2^{\mathbf{k}}$, *such that*:

(KV1) $x + y - \log(e^y e^x) = (1 - e^{-\operatorname{ad} x})(A(x, y)) + (e^{\operatorname{ad} y} - 1)(B(x, y))$;
(KV2) A, B *give convergent power series on a neighborhood of* $(0, 0) \in \mathfrak{g}^2$;
(KV3) $\operatorname{tr}_{\mathfrak{g}}((\operatorname{ad} x)\partial_x A + (\operatorname{ad} y)\partial_y B) = \frac{1}{2}\operatorname{tr}_{\mathfrak{g}}(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1)$ (*identity of analytic functions on* $\mathfrak{g}^2$ *near the origin*), *where* $z = \log e^x e^y$ *and for* $(x, y) \in \mathfrak{g}^2$, $(\partial_x A)(x, y) \in \operatorname{End}(\mathfrak{g})$ *is* $a \mapsto \frac{d}{dt}_{|t=0} A(x + ta, y)$, $(\partial_y B)(x, y)(a) = \frac{d}{dt}_{|t=0} B(x, y + ta)$.

Here $\hat{\mathfrak{f}}_2^{\mathbf{k}}$ is the topologically free $\mathbf{k}$-Lie algebra with generators x, y. For $\mathbf{k} = \mathbf{R}$, this conjecture implies an extension of the Duflo isomorphism to germs of invariant distributions on the Lie algebra $\mathfrak{g}$ and on the corresponding Lie group G (the product on distributions being defined by convolution). This extension was first proved in [AST], independently of the KV conjecture.

The KV conjecture triggered the work of several authors (for a review see [T2]). In particular, Kashiwara–Vergne settled it for solvable Lie algebras [KV], Rouvière gave a proof for $\mathfrak{sl}_2$ [R], and Vergne [V] and Alekseev–Meinrenken [AM1] proved it for quadratic Lie algebras; it turns out [AT1] that in the latter case all solutions of equation (KV1) solve equation (KV3). All these constructions lead to explicit formulas for solutions of the KV conjecture, which are both rational and independent of the Lie algebra $\mathfrak{g}$ in the considered class. The general case was settled in the positive by Alekseev–Meinrenken [AM2] using Kontsevich's deformation quantization theory and results in

DOI 10.1007/s10240-010-0029-4

[T1]. The corresponding solution (A, B) is universal, i.e., independent of the Lie algebra $\mathfrak{g}$; the series A, B are defined over $\mathbf{R}$, and expressed as infinite series where coefficients are combinations of Kontsevich integrals on configuration spaces and integrals over simplices. The values of most of these coefficients remain unknown.

An approach based on associators. — In [AT2], two of the authors proposed a new approach to the KV problem, related to the theory of Drinfeld associators [Dr]. Recall first that an associator with coupling constant 1 defined over a $\mathbf{Q}$-ring $\mathbf{k}$ is a series $\Phi(x,y) \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$, such that

$$\log \Phi(x,y) = -\frac{1}{24}[x,y] + \text{terms of degree } \geq 2,$$

$$(\mathbf{1}) \qquad \Phi(y,x) = \Phi(x,y)^{-1}, \qquad \Phi(x,y)e^{x/2}\Phi(-x-y,x)e^{-(x+y)/2}\Phi(y,-x-y)e^{y/2} = 1,$$

$$(\mathbf{2}) \qquad \Phi(t_{23},t_{34})\Phi(t_{12}+t_{13},t_{24}+t_{34})\Phi(t_{12},t_{23}) = \Phi(t_{12},t_{23}+t_{24})\Phi(t_{13}+t_{23},t_{34}),$$

the last relation taking place in the group $\exp(\hat{\mathfrak{t}}_4^{\mathbf{k}})$, where $\mathfrak{t}_4^{\mathbf{k}}$ is the $\mathbf{k}$-Lie algebra with generators t_{ij}, $1 \leq i \neq j \leq 4$ and relations $t_{ji} = t_{ij}$ and $[t_{ij}, t_{ik} + t_{jk}] = [t_{ij}, t_{kl}] = 0$ for i,j,k,l distinct; $\hat{\mathfrak{t}}_4^{\mathbf{k}}$ is its degree completion, where the generators t_{ij} have degree 1; and if $\mathfrak{a}$ is a pronilpotent Lie algebra, the group $\exp(\mathfrak{a})$ is isomorphic to $\mathfrak{a}$, equipped with the Campbell–Baker–Hausdorff product.

We now describe the approach of [AT2]. For any set S, let $\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}}$ be the free $\mathbf{k}$-Lie algebra generated by S and $\hat{\mathfrak{f}}_{\mathrm{S}}^{\mathbf{k}}$ its degree completion (where elements of S have degree 1).

We define a group structure on $\mathrm{Taut}_{\mathrm{S}}(\mathbf{k}) := \exp(\hat{\mathfrak{f}}_{\mathrm{S}}^{\mathbf{k}})^{\mathrm{S}}$ as follows: we have a map $\theta : \mathrm{Taut}_{\mathrm{S}}(\mathbf{k}) \to \mathrm{Aut}(\exp(\hat{\mathfrak{f}}_{\mathrm{S}}^{\mathbf{k}}))$, given by $g = (g_s)_{s\in\mathrm{S}} \mapsto \theta(g) = (e^s \mapsto \mathrm{Ad}_{g_s}(e^s))$. We set $g \circ h = k$, where $k_s := \theta(g)(h_s)g_s$. Then θ is a group morphism.

We define a Lie algebra structure on $\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} := (\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}})^{\mathrm{S}}$ by $[u,v] = w$, where $w_s = d\theta(u)(v_s) - d\theta(v)(u_s) + [u_s, v_s]$, and $d\theta : \mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} \to \mathrm{Der}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}})$ maps $u = (u_s)_{s\in\mathrm{S}}$ to $d\theta(u) : s \mapsto [u_s, s]$. The map $d\theta$ is then a Lie algebra morphism. The degree completion $\widehat{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}}$ of $\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}}$ is the Lie algebra of $\mathrm{Taut}_{\mathrm{S}}(\mathbf{k})$.

The Lie algebra $\mathfrak{t}_{\mathrm{S}}^{\mathbf{k}}$ is presented by generators $t_{ss'}$, $s \neq s' \in \mathrm{S}$, and relations $t_{s's} = t_{ss'}$, $[t_{ss'} + t_{ss''}, t_{s's''}] = 0$, $[t_{ss'}, t_{s''s'''}] = 0$ for $s, \ldots, s'''$ distinct. We then have an injective Lie algebra morphism $\mathfrak{t}_{\mathrm{S}}^{\mathbf{k}} \to \mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}}$, taking $t_{ss'}$ to $t_{ss'} \in \mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}}$ defined by $(t_{ss'})_s = -s'$, $(t_{ss'})_{s'} = -s$, $(t_{ss'})_{s''} = 0$ for $s'' \neq s, s'$.

The assignments $\mathrm{S} \mapsto \mathfrak{f}_{\mathrm{S}}^{\mathbf{k}}, \mathfrak{t}_{\mathrm{S}}^{\mathbf{k}}, \mathrm{Taut}_{\mathrm{S}}(\mathbf{k}), \mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}}$, can be made into contravariant functors from the category $\mathcal{S}$ of sets and partially defined maps, to that of Lie algebras and groups. For $\mathrm{T} \supset \mathrm{D}_\phi \xrightarrow{\phi} \mathrm{S}$ a morphism in $\mathcal{S}$, the corresponding morphisms are (a) $\phi^* : \mathfrak{f}_{\mathrm{S}}^{\mathbf{k}} \to \mathfrak{f}_{\mathrm{T}}^{\mathbf{k}}$, $s \mapsto \sum_{t\in\phi^{-1}(s)} t$; (b) $\phi^* : \mathfrak{t}_{\mathrm{S}}^{\mathbf{k}} \to \mathfrak{t}_{\mathrm{T}}^{\mathbf{k}}$, $t_{ss'} \mapsto \sum_{t\in\phi^{-1}(s), t'\in\phi^{-1}(s')} t_{tt'}$; (c) $\phi^* : \mathrm{Taut}_{\mathrm{S}}(\mathbf{k}) \to \mathrm{Taut}_{\mathrm{T}}(\mathbf{k})$, $g = (g_s)_{s\in\mathrm{S}} \mapsto g^\phi = h = (h_t)_{t\in\mathrm{T}}$, where $h_t = \phi^*(g_{\phi(t)})$. If $\phi(t)$ is undefined, then $g_{\phi(t)} = 1$; (d) $\phi^* : \mathfrak{tder}^{\mathbf{k}}_{\mathrm{S}} \to \mathfrak{tder}^{\mathbf{k}}_{\mathrm{T}}$ is defined in the same way, with $u_{\phi(t)} = 0$ for $\phi(t)$ undefined.

When $\mathrm{S} = [n] = \{1, \dots, n\}$, $\mathrm{Taut}_{\mathrm{S}}(\mathbf{k})$, $\mathfrak{tder}^{\mathbf{k}}_{\mathrm{S}}$, $\mathfrak{f}^{\mathbf{k}}_{\mathrm{S}}$, $\mathfrak{t}^{\mathbf{k}}_{\mathrm{S}}$ are denoted simply $\mathrm{Taut}_n(\mathbf{k})$, $\mathfrak{tder}^{\mathbf{k}}_n$, $\mathfrak{f}^{\mathbf{k}}_n$, $\mathfrak{t}^{\mathbf{k}}_n$, and the generators of $\mathfrak{f}^{\mathbf{k}}_n$ are denoted $x_1, \dots, x_n$. We use the notation $g^{\phi^{-1}(1),\dots,\phi^{-1}(n)}$ for g^ϕ. Thus the maps $\mathrm{Taut}_2(\mathbf{k}) \to \mathrm{Taut}_3(\mathbf{k})$ are $\mu \mapsto \mu^{12,3}, \mu^{2,3}$, etc., where for $\mu = (a_1(x_1, x_2), a_2(x_1, x_2))$, we have $\mu^{12,3} = (a_1(x_1 + x_2, x_3), a_1(x_1 + x_2, x_3), a_2(x_1 + x_2, x_3))$, $\mu^{2,3} = (1, a_1(x_2, x_3), a_2(x_2, x_3))$, etc.

The first result of [AT2] can be formulated as follows:

Theorem **2** ([AT2], Theorem 7.1). — *For every associator* Φ *over* $\mathbf{k}$ *with coupling constant* 1, *there exists* $\mu_\Phi \in \mathrm{Taut}_2(\mathbf{k})$ *such that*

$$\Phi(t_{12}, t_{23}) \circ \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3} \tag{3}$$

holds in $\mathrm{Taut}_3(\mathbf{k})$.

Let ℓ be the 'grading' derivation of $\hat{\mathfrak{f}}^{\mathbf{k}}_2$ defined by $\ell(x_i) = x_i$ for $i = 1, 2$. It is proved in [AT2] that $\theta(\mu_\Phi)^{-1}\ell\theta(\mu_\Phi) - \ell \in \mathrm{Im}(\widehat{\mathfrak{tder}}^{\mathbf{k}}_2 \xrightarrow{d\theta} \mathrm{Der}(\hat{\mathfrak{f}}^{\mathbf{k}}_2))$. Set the identification $(x, y) = (x_1, x_2)$. There is a unique pair $(\mathrm{A}_\Phi, \mathrm{B}_\Phi) \in (\hat{\mathfrak{f}}^{\mathbf{k}}_2)^2$ such that A_Φ (resp., B_Φ) has no constant term in x (resp., y) and $\theta(\mu_\Phi)^{-1}\ell\theta(\mu_\Phi) - \ell = d\theta(\mathrm{A}_\Phi, \mathrm{B}_\Phi)$. We have $d\theta(\mathrm{A}_\Phi, \mathrm{B}_\Phi) = \theta(\mu_\Phi)^{-1}\frac{d}{dt}_{|t=1}\theta(\mu^t_\Phi)$, where for $\mu = (a_1(x, y), a_2(x, y)) \in \mathrm{Taut}_2(\mathbf{k})$, $\mu^t := (a_1(tx, ty), a_2(tx, ty))$.

The next result of [AT2] is:

Theorem **3** ([AT2], Theorems 7.1 and 5.2). — $(\mathrm{A}_\Phi, \mathrm{B}_\Phi)$ *satisfy* (KV1), *and* (KV3) *in which* $\frac{t}{e^t-1}$ *is replaced by a formal power series with even part* $\frac{t}{e^t-1} - 1 - \frac{t}{2}$.

Using the nonemptiness of the set of associators [Dr] and the action of a group $\mathrm{KV}(\mathbf{k})$, the authors of [AT2] then construct joint solutions of (KV1) and (KV3).

The main results. — The automorphism μ_Φ in Theorem 2 is constructed by an inductive procedure. The first result of this paper is a simple formula for μ_Φ:

Theorem **4.** — $\mu_\Phi := (\Phi(x, -x-y), e^{-(x+y)/2}\Phi(y, -x-y)e^{y/2})$ *is a solution of* (3).

The formula for μ_Φ, as well as the proof of the identity $\mu_\Phi(e^xe^y) = e^{x+y}$, which is a consequence of (3), were suggested to us by D. Calaque; a similar formula has been discovered independently by M. Boyarchenko [Bo].

The proof of Theorem 4 sheds some light on the relations between associators and the KV theory. It relies on the following facts:

(a) the geometric/categorical aspect of associators, namely the fact that an associator gives rise to a compatible system of isomorphisms between completions of pure braid groups and explicit prounipotent Lie groups;

(b) the relations between free groups and pure braid groups, more precisely the fact that the free group with $n-1$ generators F_{n-1} is a normal subgroup of the pure braid group with n strands PB_n; the geometric origin of this fact lies in the Fadell–Neuwirth fibration $\mathrm{Cf}_n(\mathbf{C}) \to \mathrm{Cf}_{n-1}(\mathbf{C})$, where $\mathrm{Cf}_n(\mathbf{C}) = \{$injective maps $[n] \to \mathbf{C}\}$ is the configuration space of n points in $\mathbf{C}$.

Let $\Phi_{\mathrm{KZ}} \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{C}})$ be the Knizhnik–Zamolodchikov (KZ) associator (see [Dr]); its normalized version $\tilde{\Phi}_{\mathrm{KZ}}(x,y) = \Phi_{\mathrm{KZ}}(\frac{x}{2\pi\,\mathrm{i}}, \frac{y}{2\pi\,\mathrm{i}})$ is an associator with coupling constant 1, and it may be defined as the holonomy from 0 to 1 of the ordinary differential equation $\mathrm{G}'(t) = \frac{1}{2\pi\,\mathrm{i}}(\frac{x}{t} + \frac{y}{t-1})\mathrm{G}(t)$. Let $(\mathrm{A}_{\mathrm{KZ}}, \mathrm{B}_{\mathrm{KZ}}) := (\mathrm{A}_{\tilde{\Phi}_{\mathrm{KZ}}}, \mathrm{B}_{\tilde{\Phi}_{\mathrm{KZ}}})$ and define $(\mathrm{A}_{\mathbf{R}}, \mathrm{B}_{\mathbf{R}})$ as the real part of $(\mathrm{A}_{\mathrm{KZ}}, \mathrm{B}_{\mathrm{KZ}})$ (with respect to the canonical real structure of $\hat{\mathfrak{f}}_2^{\mathbf{C}}$). Then:

Theorem **5.** — (1) $(\mathrm{A}_{\mathbf{R}}, \mathrm{B}_{\mathbf{R}})$ *satisfies* (KV1), (KV2) *and* (KV3) *for any finite dimensional Lie algebra* $\mathfrak{g}$ *and is therefore a universal solution of the KV conjecture.*

(2) *For any* $t \in \mathbf{R}$, $(\mathrm{A}_t, \mathrm{B}_t) := (\mathrm{A}_{\mathbf{R}} + t(\log(e^x e^y) - x), \mathrm{B}_{\mathbf{R}} + t(\log(e^x e^y) - y))$ *is a universal solution of the KV conjecture.*

(3) *When* $t = -1/4$, *we have* $(\mathrm{A}_t(x,y), \mathrm{B}_t(x,y)) = (\mathrm{B}_t(-y,-x), \mathrm{A}_t(-y,-x))$.

A scheme morphism $\mathrm{M}_1 \to \mathrm{SolKV}$. — A key ingredient of [AT2] is a $\mathbf{Q}$-scheme SolKV. Its definition relies on the notions of non-commutative divergence and Jacobian, which we now recall.

If S is a set and $\mathbf{k}$ is a $\mathbf{Q}$-ring, let $\mathfrak{T}_{\mathrm{S}}^{\mathbf{k}} := \mathrm{U}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}})/[\mathrm{U}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}}), \mathrm{U}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}})]$ be the space spanned by all cyclic words in S; the map $\mathrm{U}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}}) \to \mathfrak{T}_{\mathrm{S}}^{\mathbf{k}}$ is denoted $x \mapsto \langle x \rangle$. The 'non-commutative divergence' map $j : \mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} \to \mathfrak{T}_{\mathrm{S}}^{\mathbf{k}}$ is defined by $j(u) := \langle \sum_{s\in\mathrm{S}} s\partial_s(u_s)\rangle$ for $u = (u_s)_{s\in\mathrm{S}}$, where $\partial_s : \mathrm{U}(f_{\mathrm{S}}^{\mathbf{k}}) \to \mathrm{U}(f_{\mathrm{S}}^{\mathbf{k}})$ is defined by the identity $x = \varepsilon(x)1 + \sum_{s\in\mathrm{S}} \partial_s(x)s$ (where $\varepsilon : \mathrm{U}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}}) \to \mathbf{k}$ is the counit map). The authors of [AT2] then show the existence of a 'non-commutative Jacobian' map $\mathrm{J} : \mathrm{Taut}_{\mathrm{S}}(\mathbf{k}) \to \hat{\mathfrak{T}}_{\mathrm{S}}^{\mathbf{k}}$ (here $\hat{\mathfrak{T}}_{\mathrm{S}}^{\mathbf{k}}$ is the degree completion of $\mathfrak{T}_{\mathrm{S}}^{\mathbf{k}}$, the elements of S being of degree 1), uniquely determined by $\mathrm{J}(1) = 0$ and $\frac{d}{dt}_{|t=0}\mathrm{J}(e^{tx}g) = j(x) + x\cdot\mathrm{J}(g)$ for $g \in \mathrm{Taut}_{\mathrm{S}}(\mathbf{k})$ and $x \in \widehat{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}}$ (the natural action of $\widehat{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}}$ on $\hat{\mathfrak{T}}_{\mathrm{S}}^{\mathbf{k}}$ being understood in the last equation). Then j and J satisfy the cocycle identities

$$j([u,v]) = u\cdot j(v) - v\cdot j(u) \quad \text{and} \quad \mathrm{J}(h\circ g) = \mathrm{J}(h) + h\cdot\mathrm{J}(g).$$

The scheme SolKV is defined by

$$\mathrm{SolKV}(\mathbf{k}) := \Big\{\mu \in \mathrm{Taut}_2(\mathbf{k}) | \theta(\mu)(e^x e^y) = e^{x+y} \\ \text{and } \exists r \in u^2\mathbf{k}[[u]], \mathrm{J}(\mu) = \langle r(x+y) - r(x) - r(y)\rangle\Big\}.$$

As the map $u^2\mathbf{k}[[u]] \to \mathfrak{T}_2$, $r \mapsto \langle r(x+y) - r(x) - r(y)\rangle$ is injective, there is a well-defined map $\mathrm{Duf} : \mathrm{SolKV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$, $\mu \mapsto r$, which we call the Duflo map. It is proved in [AT2] that any $\mu \in \mathrm{SolKV}(\mathbf{k})$ gives rise to a solution (A, B) of both (KV1) and (KV3) in which $\frac{t}{e^t-1}$ is replaced by $t\frac{dr}{dt}(t)$. This solution is given by the formula $d\theta(\mathrm{A},\mathrm{B}) = \mu^{-1}\ell\mu - \ell$.

Recall that the scheme M_1 of associators with coupling constant 1 is defined by $\mathrm{M}_1(\mathbf{k}) = \{\Phi \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$ satisfying (1) and (2)$\}$.

Proposition **6.** — *The map* $\Phi \mapsto \mu_\Phi$ *is a morphism of* $\mathbf{Q}$*-schemes* $\mathrm{M}_1 \to \mathrm{SolKV}$.

In order to study the relation of this morphism with the Duflo map, we recall the following result on associators (see [DT, E], and also [Ih]): for any $\Phi(x,y) \in \mathrm{M}_1(\mathbf{k})$, there exists a formal power series $\Gamma_\Phi(u) = e^{\sum_{n\geq 2}(-1)^n\zeta_\Phi(n)u^n/n}$, such that

$$(1 + y\partial_y\Phi(x,y))^{\mathrm{ab}} = \frac{\Gamma_\Phi(\bar{x}+\bar{y})}{\Gamma_\Phi(\bar{x})\Gamma_\Phi(\bar{y})}, \qquad (\mathbf{4})$$

where $\xi \mapsto \xi^{\mathrm{ab}}$ is the abelianization morphism $\mathbf{k}\langle\langle x,y\rangle\rangle \to \mathbf{k}[[\bar{x},\bar{y}]]$. The values of the $\zeta_\Phi(n)$ for n even are independent of Φ; they are expressed in terms of Bernoulli numbers by $\zeta_\Phi(2n) = -\frac{1}{2}\frac{\mathrm{B}_{2n}}{(2n)!}$ for $n \geq 1$, so there is an identity for generating functions $-\frac{1}{2}(\frac{u}{e^u-1} - 1 + \frac{u}{2}) = \sum_{n\geq 1}\zeta_\Phi(2n)u^{2n}$ (we have $\zeta_\Phi(2) = -1/24$, $\zeta_\Phi(4) = 1/1440$, etc.)

Proposition **7.** — $\mathrm{J}(\mu_\Phi) = \langle \log\Gamma_\Phi(x) + \log\Gamma_\Phi(y) - \log\Gamma_\Phi(x+y)\rangle$, *so* $\mathrm{Duf}(\mu_\Phi) = -\log\Gamma_\Phi$. *We therefore have a commutative diagram*

$$\begin{array}{ccc} \mathrm{M}_1(\mathbf{k}) & \stackrel{\Phi\mapsto\mu_\Phi}{\to} & \mathrm{SolKV}(\mathbf{k}) \\ {\scriptstyle \Phi\mapsto\log\Gamma_\Phi}\downarrow & & \downarrow{\scriptstyle \mathrm{Duf}} \\ \{r \in u^2\mathbf{k}[[u]] | r_{ev}(u) = -\frac{u^2}{24} + \frac{u^4}{1440} + \cdots\} & \stackrel{(-1)\times -}{\hookrightarrow} & u^2\mathbf{k}[[u]] \end{array} \qquad (\mathbf{5})$$

where $r_{ev}(u)$ *is the even part of* $r(u)$.

Torsor aspects. — Let us set

$$\mathrm{KV}(\mathbf{k}) := \Big\{\alpha \in \mathrm{Taut}_2(\mathbf{k}) | \theta(\alpha)(e^xe^y) = e^xe^y$$
$$\text{and } \exists\sigma \in u^2\mathbf{k}[[u]], \mathrm{J}(\alpha) = \langle\sigma(\log(e^xe^y)) - \sigma(x) - \sigma(y)\rangle\Big\},$$

and

$$\mathrm{KRV}(\mathbf{k}) := \Big\{a \in \mathrm{Taut}_2(\mathbf{k}) | \theta(a)(e^{x+y}) = e^{x+y}$$
$$\text{and } \exists s \in u^2\mathbf{k}[[u]], \mathrm{J}(a) = \langle s(x+y) - s(x) - s(y)\rangle\Big\};$$

we call $\mathrm{KV}(\mathbf{k})$ the Kashiwara–Vergne group, while $\mathrm{KRV}(\mathbf{k})$ is its graded version. As before, we will denote by $\mathrm{Duf} : \mathrm{KV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$, $\mathrm{KRV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ the maps $\alpha \mapsto \sigma$, $a \mapsto s$.

Proposition **8.** — $\mathrm{KV}(\mathbf{k})$ *and* $\mathrm{KRV}(\mathbf{k})$ *are subgroups of* $\mathrm{Taut}_2(\mathbf{k})$, *and* $\mathrm{Duf}: \mathrm{KV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$, $\mathrm{KRV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ *are group morphisms.* $\mathrm{SolKV}(\mathbf{k})$ *is a torsor under the commuting left action of* $\mathrm{KV}(\mathbf{k})$ *and right action of* $\mathrm{KRV}(\mathbf{k})$ *given by* $(\alpha, \mu) \mapsto \mu \circ \alpha^{-1}$ *and* $(\mu, a) \mapsto a^{-1} \circ \mu$, *and* $\mathrm{Duf}: \mathrm{SolKV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ *is a morphism of torsors.*

In particular, every element of $\mathrm{SolKV}(\mathbf{k})$ gives rise to an isomorphism $\mathfrak{kv} \to \mathfrak{krv}$ between the Lie algebras of these groups, whose associated graded morphism is the canonical identification $\mathrm{gr}(\mathfrak{kv}) \simeq \mathfrak{krv}$.

The prounipotent radical of the Grothendieck–Teichmüller group is

$$\begin{aligned}\mathrm{GT}_1(\mathbf{k}) = \big\{ & f \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}}) | f(y,x) = f(x,y)^{-1}, \\ & f(x,y) f(\log e^{-y}e^{-x}, x) f(y, \log e^{-y}e^{-x}) = 1, \\ & f(\xi_{23}, \xi_{34}) f(\log e^{\xi_{12}}e^{\xi_{13}}, \log e^{\xi_{24}}e^{\xi_{34}}) f(\xi_{12}, \xi_{23}) \\ & = f(\xi_{12}, \log e^{\xi_{23}}e^{\xi_{24}}) f(\log e^{\xi_{13}}e^{\xi_{23}}, \xi_{34}) \big\},\end{aligned}$$

where the last equation holds in the prounipotent completion $\mathrm{PB}_4(\mathbf{k})$ of the pure braid group in four strands PB_4; $x_{ij} = (\sigma_{j-2} \cdots \sigma_i)^{-1} \sigma_{j-1}^2 (\sigma_{j-2} \cdots \sigma_i)$ where $\sigma_1, \sigma_2, \sigma_3$ are the Artin generators of the braid group in four strands B_4 and $\xi_{ij} = \log x_{ij}$ (here x_{ij} is identified with its image under the canonical morphism $\mathrm{PB}_4 \to \mathrm{PB}_4(\mathbf{k})$ and $\log : \mathrm{PB}_4(\mathbf{k}) \to \mathrm{Lie}\,\mathrm{PB}_4(\mathbf{k})$ is the logarithm map, which is a bijection between a prounipotent Lie group and its Lie algebra). It is equipped with the product $(f_1 * f_2)(x,y) = f_1(\mathrm{Ad}_{f_2(x,y)}(x), y) f_2(x,y)$. Its graded version is

$$\begin{aligned}\mathrm{GRT}_1(\mathbf{k}) = \big\{ & g(x,y) \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}}) | g(y,x) g(x,y) = 1, \\ & \mathrm{Ad}_{g(x,-x-y)}(x) + \mathrm{Ad}_{g(y,-x-y)}(y) = x + y, \\ & g(x,y) g(-x-y, x) g(y, -x-y) = 1, \text{ and } g \text{ satisfies (2)} \big\}\end{aligned}$$

with product $(g_1 * g_2)(x,y) = g_1(\mathrm{Ad}_{g_2(x,y)}(x), y) g_2(x,y)$.

It is proved in [Dr] that $\mathrm{M}_1(\mathbf{k})$ is a torsor under the commuting left action of $\mathrm{GT}_1(\mathbf{k})$ and right action of $\mathrm{GRT}_1(\mathbf{k})$ by $(f, \Phi) \mapsto (f * \Phi)(x,y) := f(\mathrm{Ad}_{\Phi(x,y)}(x), y) \Phi(x,y)$ and $(\Phi, g) \mapsto (\Phi * g)(x,y) := \Phi(\mathrm{Ad}_{g(x,y)}(x), y) g(x,y)$.

The following Theorem 9 and Proposition 10 express the torsor properties of the map $\Phi \mapsto \mu_\Phi$.

Theorem **9.** — *There are unique group morphisms* $\mathrm{GT}_1(\mathbf{k}) \to \mathrm{KV}(\mathbf{k})$, $f \mapsto \alpha_f^{-1}$, *where*

$$\alpha_f = (f(x, \log e^{-y}e^{-x}), f(y, \log e^{-y}e^{-x})),$$

and $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{KRV}_1(\mathbf{k})$, $g \mapsto a_g^{-1}$, *where*

$$a_g(x) = (g(x, -x-y), g(y, -x-y)).$$

These group morphisms are compatible with the map $\mathrm{M}_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$, *which is therefore a morphism of torsors.*

Proposition **10.** — *The diagram* (5) *is a diagram of torsors, where the sets in the lower line are viewed as affine spaces.*

In Appendix A, we show that α_f satisfies the cocycle identity

$$f(\log x_{12}, \log x_{23}) \circ \alpha_f^{\widetilde{12,3}} \circ \alpha_f^{1,2} = \alpha_f^{\widetilde{1,23}} \circ \alpha_f^{2,3} \tag{6}$$

in $\mathrm{Taut}_3(\mathbf{k})$, where for $\alpha = (\alpha_1(x_1, x_2), \alpha_2(x_1, x_2)) \in \mathrm{Taut}_2(\mathbf{k})$, we set

$$\alpha^{\widetilde{12,3}} := (\alpha_1(\log e^{x_1}e^{x_2}, x_3), \alpha_1(\log e^{x_1}e^{x_2}, x_3), \alpha_2(\log e^{x_1}e^{x_2}, x_3))$$

and $\alpha^{\widetilde{1,23}} := (\alpha_1(x_1, \log e^{x_2}e^{x_3}), \alpha_2(x_1, \log e^{x_2}e^{x_3}), \alpha_2(x_1, \log e^{x_2}e^{x_3}))$, and $x_{12}, x_{23} \in \mathrm{Taut}_3(\mathbf{k})$ are the images of $x_{12} = \sigma_1^2$, $x_{23} = \sigma_2^2$ under the natural morphism $\mathrm{PB}_3 \to \mathrm{Taut}_3(\mathbf{k})$ (see Proposition 19), given by $x_{12} = (e^{-x_2}, e^{-x_2}e^{-x_1}, 1)$ and $x_{23} = (1, e^{-x_3}, e^{-x_3}e^{-x_2})$.

The group $\mathrm{GT}_1(\mathbf{k})$ admits profinite and pro-l versions, where l is a prime number. The morphism $f \mapsto \alpha_f^{-1}$ admits variants in these setups, which satisfy analogues of (6) (in the profinite setup, identity (6) was independently obtained by P. Lochak and L. Schneps [LS]).

Appendix B is devoted to the study of the following problem: Theorem 4 is proved by studying restrictions μ_{O} to free groups of morphisms $\tilde{\mu}_{\mathrm{O}}$ between braid groups and their infinitesimal analogues, where O is a parenthesized word with n identical letters. We express μ_{O} and its Jacobian using $\mu_{\bullet(\bullet\bullet)} = \mu_\Phi$ and Γ_Φ.

Finally, Appendix C is devoted to the computation of centralizers in infinitesimal analogues of pure braid groups, which are used in the proof of Theorem 4.

Organization. — In Section 1, we recall the relations between associators and 1-formality isomorphisms for braid groups. In Section 2, we study the relation between these isomorphisms. In Section 3, we recall the relations between braid and free groups. In Section 4, we show that these isomorphisms give rise to the tangential automorphism μ_Φ; using the results of Section 2, we show a key relation satisfied by μ_Φ. This enables us to prove Theorem 4 and Propositions 6 and 7 in Section 5. In Section 6, we prove Proposition 8, Theorem 9 and Proposition 10 on the group and torsor aspects of our work. In Section 7, we study the analytic aspects of our construction, which enables us to prove Theorem 5.

1. Associators and 1-formality of braid groups

In [Dr], Drinfeld showed that associators give rise to 1-formality isomorphisms for braid groups. This statement was reformulated by Bar-Natan in the framework of braided monoidal categories [B]. This section is devoted to an exposition of this material.

1.1. *(Braided) (strict) monoidal categories.* — Recall that a monoidal category is a category $\mathcal{C}$, equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbf{1}$ and a natural constraint $a_{X,Y,Z} \in \mathrm{Iso}_{\mathcal{C}}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$ such that

$$a_{X,Y,Z\otimes T} a_{X\otimes Y,Z,T} = (\mathrm{id}_X \otimes a_{Y,Z,T}) a_{X,Y\otimes Z,T} (a_{X,Y,Z} \otimes \mathrm{id}_T).$$

A braiding is then a natural constraint $\beta_{X,Y} \in \mathrm{Iso}_{\mathcal{C}}(X \otimes Y, Y \otimes X)$, such that

$$(\mathrm{id}_Y \otimes \beta^{\pm}_{X,Z}) a_{Y,X,Z} (\beta^{\pm}_{X,Y} \otimes \mathrm{id}_Z) = a_{Y,Z,X} \beta^{\pm}_{X,Y\otimes Z} a_{X,Y,Z},$$

where $\beta^{+}_{X,Y} = \beta_{X,Y}$ while $\beta^{-}_{X,Y} = \beta^{-1}_{Y,X}$. It is called strict if the trifunctors $\otimes \circ (\otimes \times \mathrm{id})$ and $\otimes \circ (\mathrm{id} \times \otimes) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ coincide and $a_{X,Y,Z} = \mathrm{id}_{X\otimes Y\otimes Z}$.

Let B_n be the braid group in n strands. We recall its Artin presentation: the generators are $\sigma_1, \dots, \sigma_{n-1}$ and the relations are $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$. Recall that the symmetric group $\mathfrak{S}_n$ has generators $s_1, \dots, s_{n-1}$ and the same relations, with the additional $s_i^2 = 1$ for $i = 1, \dots, n-1$. We therefore have a morphism $B_n \to \mathfrak{S}_n$, $\sigma_i \mapsto s_i$. The pure braid group in n strands is $PB_n := \mathrm{Ker}(B_n \to \mathfrak{S}_n)$; it is the smallest normal subgroup of B_n containing σ_i^2 for $i = 1, \dots, n-1$.

A braided monoidal category (b.m.c.) $\mathcal{C}$ then gives rise to morphisms $B_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})$, where $X^{\otimes n}$ is defined inductively by $X^{\otimes 0} = \mathbf{1}$, $X^{\otimes n} = X \otimes X^{\otimes n-1}$, given by $\sigma_i \mapsto a_i^{-1} (\mathrm{id}_{X^{\otimes i-1}} \otimes \beta_{X,X} \otimes \mathrm{id}_{\otimes X^{\otimes n-1-i}}) a_i$, where $a_i : X^{\otimes n} \to X^{\otimes i-1} \otimes X^{\otimes 2} \otimes X^{\otimes n-1-i}$ is the morphism constructed from the associativity constraints (this morphism is unique by McLane's coherence theorem). A b.m.c. also gives rise to morphisms $PB_n \to \mathrm{Aut}_{\mathcal{C}}(X_1 \otimes \dots \otimes X_n)$.

1.2. *The categories* **PaB**, **PaCD**. — In [JS], Section 2, Joyal and Street introduced the free braided monoidal category $F_b(\mathcal{A})$ generated by a small category $\mathcal{A}$. For S a set, let $\mathcal{A}_S$ be the category with $\mathrm{Ob}(\mathcal{A}_S) = S$, and

$$\mathrm{Hom}_{\mathcal{A}_S}(s, t) := \begin{cases} \{\mathrm{id}_s\} & \text{if } t = s, \\ \emptyset & \text{otherwise.} \end{cases}$$

We set $\mathbf{PaB}_S := F_b(\mathcal{A}_S)$ and for $S = \{\bullet\}$, $\mathbf{PaB} := \mathbf{PaB}_{\{\bullet\}}$. These are the free b.m.c.'s generated by S (resp., by one object $\bullet$).

The category **PaB** coincides with Bar-Natan's category of parenthesized braids [B], which can be described explicitly as follows. Its set of objects is $\mathbf{Par} = \bigsqcup_{n\geq 0} \mathbf{Par}_n$, where $\mathbf{Par}_n$ is the set of parenthesizations of the word $\bullet \cdots \bullet$ (n letters); alternatively, the set of planar binary trees with n leaves (we will set $|O| = n$ for $O \in \mathbf{Par}_n$). The object with $n = 0$ is denoted $\mathbf{1}$. Morphisms are defined by[1]

$$\mathbf{PaB}(O, O') := \begin{cases} B_n & \text{if } |O| = |O'| = n, \\ \emptyset & \text{if } |O| \neq |O'|; \end{cases}$$

[1] If $\mathcal{C}$ is a category and $X \in \mathrm{Ob}\,\mathcal{C}$, we set $\mathcal{C}(X, X') := \mathrm{Hom}_{\mathcal{C}}(X, X')$.

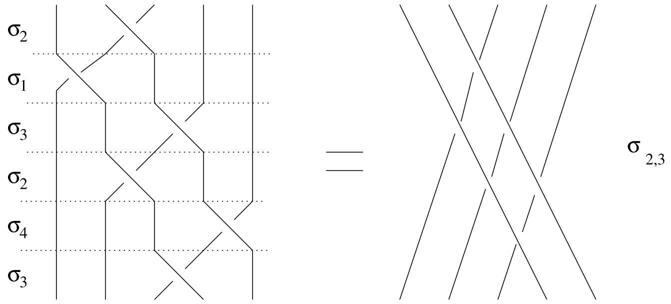

FIG. 1. — Braiding in **PaB**

the composition is then defined using the product in B_n.

PaB is a braided monoidal category (see e.g. [JS]), where the tensor product of objects is $(n, P) \otimes (n', P') := (n + n', P * P')$ (where $P * P'$ is the concatenation of parenthesized words, e.g. for $P = \bullet\bullet$ and $P' = (\bullet\bullet)\bullet$, $P * P' = (\bullet\bullet)((\bullet\bullet)\bullet)$). The tensor product of morphisms $\mathbf{PaB}(O_1, O'_1) \times \mathbf{PaB}(O_2, O'_2) \to \mathbf{PaB}(O_1 \otimes O_2, O'_1 \otimes O'_2)$ is induced by the juxtaposition of braids $B_{|O_1|} \times B_{|O_2|} \to B_{|O_1|+|O_2|}$ (the group morphism $(\sigma_i, e) \mapsto \sigma_i$, $(e, \sigma_j) \mapsto \sigma_{j+|O_1|}$). The braiding $\beta_{O,O'} \in \mathbf{PaB}(O \otimes O', O' \otimes O)$ is the braid $\sigma_{n,n'} \in B_{n+n'}$ where the n first strands are globally exchanged with the n' last strands (see Figure 1); we have $\sigma_{n,n'} = (\sigma_n \cdots \sigma_1)(\sigma_{n+1} \cdots \sigma_2) \cdots (\sigma_{n+n'-1} \cdots \sigma_{n'})$ (where $n = |O|$, $n' = |O'|$). Finally, the associativity constraint $a_{O,O',O''} \in \mathbf{PaB}((O \otimes O') \otimes O'', O \otimes (O' \otimes O''))$ corresponds to the trivial braid $e \in B_{|O|+|O'|+|O''|}$.

Moreover, the pair $(\mathbf{PaB}, \bullet)$ is universal for pairs $(\mathcal{C}, M)$ of a braided monoidal category and an object, i.e., for each such a pair, there exists a unique tensor functor $\mathbf{PaB} \to \mathcal{C}$ taking $\bullet$ to M.

Bar-Natan introduced another category **PaCD** of 'parenthesized chord diagrams'. It is constructed using the family of Lie algebras $\mathfrak{t}_S^{\mathbf{k}}$ defined in the Introduction. Note that the permutation group $\mathfrak{S}_S$ of S acts on $\mathfrak{t}_S^{\mathbf{k}}$ by $\sigma \cdot t_{ss'} = t_{\sigma(s)\sigma(s')}$. Then the Lie algebra $\mathfrak{t}_S^{\mathbf{k}}$ is graded, where $t_{ss'}$ has degree 1, and we denote by $\hat{\mathfrak{t}}_S$ its degree completion. When $S = [n]$, we denote by $\mathfrak{t}_S^{\mathbf{k}}, \hat{\mathfrak{t}}_S^{\mathbf{k}}$ by $\mathfrak{t}_n^{\mathbf{k}}, \hat{\mathfrak{t}}_n^{\mathbf{k}}$; we have $\mathfrak{S}_S = \mathfrak{S}_n$.

The category **PaCD** can then be described as follows. Its set of objects is **Par**, and

$$\mathbf{PaCD}(O, O') := \begin{cases} \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \mathfrak{S}_n & \text{if } |O| = |O'| = n, \\ \emptyset & \text{if } |O| \neq |O'|. \end{cases}$$

We define the tensor product as above at the level of objects, and by the juxtaposition map $(\exp \hat{\mathfrak{t}}_n^{\mathbf{k}} \rtimes \mathfrak{S}_n) \times (\exp \hat{\mathfrak{t}}_{n'}^{\mathbf{k}} \rtimes \mathfrak{S}_{n'}) \to \exp \hat{\mathfrak{t}}_{n+n'}^{\mathbf{k}} \rtimes \mathfrak{S}_{n+n'}$, $((\exp x, s), (\exp x', s')) \mapsto (\exp(x * x'), s * s')$, where $\hat{\mathfrak{t}}_n^{\mathbf{k}} \times \hat{\mathfrak{t}}_{n'}^{\mathbf{k}} \to \hat{\mathfrak{t}}_{n+n'}^{\mathbf{k}}$, $(x, x') \mapsto x * x'$ is the Lie algebra morphism such that $t_{ij} * 0 = t_{ij}$ and $0 * t_{i'j'} = t_{n+i',n+j'}$, and $\mathfrak{S}_n \times \mathfrak{S}_{n'} \to \mathfrak{S}_{n+n'}$, $(s, s') \mapsto s * s'$ is the group morphism such that $s_i * 1 = s_i$, $1 * s_{i'} = s_{n+i'}$.

Every $\Phi \in M_1(\mathbf{k})$ gives rise to a structure of braided monoidal category $\mathbf{PaCD}_\Phi$ on **PaCD**, as follows: $\beta_{O,O'} = (e^{t_{12}/2})^{[n],n+[n']} s_{n,n'}$, where $n = |O|, n' = |O'|$, and $s_{n,n'} \in$

$\mathfrak{S}_{n+n'}$ is given by $s_{n,n'}(i) = n' + i$ for $i \in [n]$, $s_{n,n'}(n+i) = i$ for $i \in [n']$, and $a_{\mathrm{O},\mathrm{O}',\mathrm{O}''} = \Phi(t_{12}, t_{23})^{[n],n+[n'],n+n'+[n'']}$ for $n = |\mathrm{O}|$, $n' = |\mathrm{O}'|$, $n'' = |\mathrm{O}''|$. By the universal property of **PaB**, there is a unique tensor functor $\mathbf{PaB} \to \mathbf{PaCD}_\Phi$, which is the identity at the level of objects.

1.3. *Morphisms* $\mathrm{B}_n \to \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \mathfrak{S}_n$, $\mathrm{PB}_n \to \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}})$. — Fix $\Phi \in \mathrm{M}_1(\mathbf{k})$. By the universal property of **PaB**, there is a unique tensor functor $\mathrm{F}_\Phi : \mathbf{PaB} \to \mathbf{PaCD}_\Phi$, inducing the identity at the level of objects. So for any $n \geq 1$ and any $\mathrm{O} \in \mathrm{Ob}(\mathbf{PaB})$, $|\mathrm{O}| = n$, we get a group morphism[2]

$$\mathrm{F}_\Phi(\mathrm{O}) = \tilde{\mu}_\mathrm{O} : \mathrm{B}_n \simeq \mathbf{PaB}(\mathrm{O}) \to \mathbf{PaCD}_\Phi(\mathrm{O}) = \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \mathfrak{S}_n,$$

such that

$$\begin{array}{ccc} \mathrm{B}_n & \xrightarrow{\tilde{\mu}_\mathrm{O}} & \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \mathfrak{S}_n \\ & \searrow \quad \swarrow & \\ & \mathfrak{S}_n & \end{array}$$

commutes. It follows that $\tilde{\mu}_\mathrm{O}$ restricts to a morphism

$$\tilde{\mu}_\mathrm{O} : \mathrm{PB}_n \to \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}). \tag{7}$$

Let us show that the various $\tilde{\mu}_\mathrm{O}$ are all conjugated to each other. Let $\mathrm{can}_{\mathrm{O},\mathrm{O}'} \in \mathbf{PaB}(\mathrm{O}, \mathrm{O}')$ correspond to $e \in \mathrm{B}_n$. Then $\mathrm{can}_{\mathrm{O}',\mathrm{O}''} \circ \mathrm{can}_{\mathrm{O},\mathrm{O}'} = \mathrm{can}_{\mathrm{O},\mathrm{O}''}$. Moreover, if we denote by $\sigma_\mathrm{O} : \mathrm{B}_n \to \mathbf{PaB}(\mathrm{O})$ the canonical identification, then $\sigma_{\mathrm{O}'}(b) = \mathrm{can}_{\mathrm{O},\mathrm{O}'}\, \sigma_\mathrm{O}(b) \times \mathrm{can}_{\mathrm{O},\mathrm{O}'}^{-1}$. Let us set $\Phi_{\mathrm{O},\mathrm{O}'} := \mathrm{F}_\Phi(\mathrm{can}_{\mathrm{O},\mathrm{O}'})$. Then:

(1) $\Phi_{\mathrm{O},\mathrm{O}'} \in \exp(\hat{\mathfrak{t}}_n)$, $\Phi_{\mathrm{O}',\mathrm{O}''}\Phi_{\mathrm{O},\mathrm{O}'} = \Phi_{\mathrm{O},\mathrm{O}''}$;
(2) $\tilde{\mu}_{\mathrm{O}'}(b) = \Phi_{\mathrm{O},\mathrm{O}'}\tilde{\mu}_\mathrm{O}(b)\Phi_{\mathrm{O},\mathrm{O}'}^{-1}$.

If $\mathrm{O} = \bullet(\ldots(\bullet\bullet))$ is the 'right parenthesization', the explicit formula for $\tilde{\mu}_\mathrm{O}$ is

$$\tilde{\mu}_\mathrm{O}(\sigma_i) = \Phi^{i,i+1,i+2\ldots n} e^{t_{i,i+1}/2} s_i (\Phi^{i,i+1,i+2\ldots n})^{-1}, \quad i = 0, \ldots, n-1.$$

1.4. *Prounipotent completions.* — Recall that a group scheme over $\mathbf{Q}$ is a functor $\{\mathbf{Q}\text{-rings}\} \to \{\text{groups}\}$, $\mathrm{G}(-) = (\mathbf{k} \mapsto \mathrm{G}(\mathbf{k}))$. Such a group scheme is called prounipotent if there exists a pronilpotent $\mathbf{Q}$-Lie algebra $\mathfrak{g}$, such that $\mathrm{G}(\mathbf{k}) \simeq \exp(\mathfrak{g}^{\mathbf{k}})$, where $\mathfrak{g}^{\mathbf{k}} = \lim_{\leftarrow}(\mathfrak{g}/\mathrm{D}^n(\mathfrak{g})) \otimes \mathbf{k}$, and $\mathrm{D}^1(\mathfrak{g}) = \mathfrak{g}$, $\mathrm{D}^{n+1}(\mathfrak{g}) = [\mathfrak{g}, \mathrm{D}^n(\mathfrak{g})]$. To each finitely generated group Γ, one may attach a prounipotent group scheme $\Gamma(-)$, equipped with a morphism $\Gamma \to \Gamma(\mathbf{Q})$, with the following universal property: any unipotent $\mathbf{Q}$-group scheme $\mathrm{U}(-)$ and any group morphism $\Gamma \to \mathrm{U}(\mathbf{Q})$ give rise to a morphism $\Gamma(-) \to \mathrm{U}(-)$ of

[2] It $\mathcal{C}$ is a category and $\mathrm{X} \in \mathrm{Ob}\,\mathcal{C}$, we write $\mathcal{C}(\mathrm{X}) := \mathcal{C}(\mathrm{X}, \mathrm{X}) = \mathrm{End}_\mathcal{C}(\mathrm{X})$.

prounipotent group schemes, such that the composite map $\Gamma \to \Gamma(\mathbf{Q}) \to \mathrm{U}(\mathbf{Q})$ coincides with $\Gamma \to \mathrm{U}(\mathbf{Q})$. The scheme $\Gamma(-)$ is called the prounipotent (or Malcev) completion of Γ.

If S is a finite set, let F_S be the free group generated by S and $\hat{\mathfrak{f}}_\mathrm{S}^{\mathbf{Q}}$ be the topologically free Lie algebra generated by symbols $\log s$, $s \in \mathrm{S}$. Then we have an injective morphism $\mathrm{F}_\mathrm{S} \stackrel{\mathrm{can}}{\to} \exp(\hat{\mathfrak{f}}_\mathrm{S}^{\mathbf{Q}})$, $s \mapsto \exp(\log s)$. If Γ is presented as $\langle \mathrm{S} | f(t), t \in \mathrm{T}\rangle$ for some map $\mathrm{T} \stackrel{f}{\to} \mathrm{F}_\mathrm{S}$, then $\mathrm{Lie}\,\Gamma(-)$ may be presented as the quotient of $\hat{\mathfrak{f}}_\mathrm{S}^{\mathbf{Q}}$ by the topological ideal generated by all $\log \mathrm{can} f(t)$, $t \in \mathrm{T}$. In particular, we have a canonical identification $\mathrm{Lie}\,\mathrm{F}_\mathrm{S}(-) \simeq \hat{\mathfrak{f}}_\mathrm{S}^{\mathbf{Q}}$.

1.5. *1-formality isomorphisms for braid groups.* — We show how the morphisms $\tilde{\mu}_\mathrm{O}$ (see (7)) extend to isomorphisms between prounipotent completions. The prounipotent completion of B_n relative to $\mathrm{B}_n \to \mathfrak{S}_n$ will be denoted $\mathrm{B}_n(\mathbf{k}, \mathfrak{S}_n)$; it may be constructed as follows: B_n acts by automorphisms of PB_n, hence of $\mathrm{PB}_n(\mathbf{k})$; $\mathrm{B}_n(\mathbf{k}, \mathfrak{S}_n)$ is defined as the quotient of the semidirect product $\mathrm{PB}_n(\mathbf{k}) \rtimes \mathrm{B}_n$ by the image of the morphism $\mathrm{PB}_n \to \mathrm{PB}_n(\mathbf{k}) \rtimes \mathrm{B}_n$, $g \mapsto (g^{-1}, g)$ (which is a normal subgroup). Then $\mathrm{B}_n(\mathbf{k}, \mathfrak{S}_n)$ fits into an exact sequence $1 \to \mathrm{PB}_n(\mathbf{k}) \to \mathrm{B}_n(\mathbf{k}, \mathfrak{S}_n) \to \mathfrak{S}_n \to 1$.

The morphisms $\tilde{\mu}_\mathrm{O}$ then give rise to isomorphisms

$$\begin{array}{ccc} \mathrm{PB}_n(\mathbf{k}) & \stackrel{\sim}{\to} & \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \\ \downarrow & & \downarrow \\ \mathrm{B}_n(\mathbf{k}, \mathfrak{S}_n) & \stackrel{\sim}{\to} & \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \mathfrak{S}_n \end{array} \tag{8}$$

also denoted $\tilde{\mu}_\mathrm{O}$. When Φ is the KZ associator with coupling constant $2\pi i$ (see the Introduction), these isomorphisms are given by Sullivan's theory of minimal models applied to the configuration space of n points in the complex plane. This theory computes all the rational homotopy groups of a simply-connected Kähler manifold, but only the prounipotent completion of its fundamental group in the non-simply-connected case, whence the name '1-formality' [Su].

2. Operadic properties of 1-formality isomorphisms of braid groups

In this section, we establish operadic properties of the morphisms $\tilde{\mu}_\mathrm{O}$ introduced in Section 1.3. The operadic structure of the collection of braid groups is described by the cabling morphisms, which we review in the following section.

2.1. *Cabling morphisms.* — Let $n \geq 1$, $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbf{N}^n$ and $m := |\mathbf{m}| = m_1 + \cdots + m_n$. For $s \in \mathfrak{S}_n$, define $s_\mathbf{m} \in \mathfrak{S}_m$ by $s_\mathbf{m}(m_1 + \cdots + m_{i-1} + j) = m_{s^{-1}(1)} + \cdots +$

$m_{s^{-1}(s(i)-1)} + j$ for any $i \in [n]$ and any $j \in [m_i]$. Then the diagram

$$\begin{array}{ccc} [m] & \stackrel{s_{\mathbf{m}}}{\to} & [m] \\ \phi_{\mathbf{m}} \downarrow & & \downarrow \phi_{\mathbf{m}\circ s^{-1}} \\ [n] & \stackrel{s}{\to} & [n] \end{array}$$

commutes, where $\phi_{\mathbf{m}} : [m] \to [n]$ is defined by $\phi_{\mathbf{m}}(m_1 + \cdots + m_{i-1} + j) = i$ for any $i \in [n]$ and any $j \in [m_i]$. One checks:

Lemma **11.** — *For $s, t \in \mathfrak{S}_n$, $(ts)_{\mathbf{m}} = t_{\mathbf{m}\circ s^{-1}} s_{\mathbf{m}}$ (where we view $\mathbf{m}$ as a map $[n] \to \mathbf{N}$).*

Recall that for $a, b \geq 0$, $\sigma_{a,b} = (\sigma_b \cdots \sigma_1) \cdots (\sigma_{a+b-1} \cdots \sigma_a) \in \mathrm{B}_{a+b}$, and that $(\sigma, \tau) \mapsto \sigma * \tau$ is the group morphism $\mathrm{B}_a \times \mathrm{B}_b \to \mathrm{B}_{a+b}$, such that $\sigma_i * 1 = \sigma_i$ for $i \in [a-1]$ and $1 * \sigma_j = \sigma_{a+j}$ for $j \in [b-1]$.

Proposition **12.** — *There exists a unique collection of maps $\mathrm{B}_n \to \mathrm{B}_m$, $\sigma \mapsto \sigma_{\mathbf{m}}$, such that:*

(a) $(\sigma_i)_{\mathbf{m}} = 1_{m_1+\cdots+m_{i-1}} * \sigma_{m_i, m_{i+1}} * 1_{m_{i+2}+\cdots+m_n}$ *for any $i \in [n]$ (where $1_n \in \mathfrak{S}_n$ is the identity permutation);*

(b) *for any $\sigma, \tau \in \mathrm{B}_n$, $(\tau\sigma)_{\mathbf{m}} = \tau_{\mathbf{m}\circ s^{-1}} \sigma_{\mathbf{m}}$, where $s = \mathrm{im}(\sigma \in \mathrm{B}_n \to \mathfrak{S}_n)$.*

For any $\mathbf{m}$, the diagram

$$(\mathbf{9}) \qquad \begin{array}{ccc} \mathrm{B}_n & \stackrel{\sigma \mapsto \sigma_{\mathbf{m}}}{\to} & \mathrm{B}_m \\ \downarrow & & \downarrow \\ \mathfrak{S}_n & \stackrel{s \mapsto s_{\mathbf{m}}}{\to} & \mathfrak{S}_m \end{array}$$

commutes, and the map $\sigma \mapsto \sigma_{\mathbf{m}}$ restricts to a group morphism $\mathrm{PB}_n \to \mathrm{PB}_m$.

Remark **13.** — The morphisms $f_{\mathbf{m}} : \mathrm{PB}_n \to \mathrm{PB}_m$ can be interpreted topologically as follows. Recall the isomorphisms $\mathrm{PB}_n \simeq \pi_1(\mathrm{Cf}_n, \mathrm{P}_n)$ where $\mathrm{Cf}_n = \{f : [n] \to \mathbf{C} | f$ is injective$\}$, and $\mathrm{P}_n = \{f : [n] \to \mathbf{R} | f(1) < \cdots < f(n)\}$ (this is well-defined as P_n is contractible). For $\varepsilon > 0$, define $\mathrm{Cf}_n^{\varepsilon} \subset \mathrm{Cf}_n$ as $\mathrm{Cf}_n^{\varepsilon} = \{f | \forall i \neq j, |f(i) - f(j)| > \varepsilon\}$ and let $g_{\mathbf{m}} : \mathrm{Cf}_n^{\varepsilon} \to \mathrm{Cf}_n$ be the map $f \mapsto g$, where $g(m_1 + \cdots + m_{i-1} + j) = f(i) + \frac{j}{m_i}\varepsilon$. Then $g_{\mathbf{m}}(\mathrm{P}_n \cap \mathrm{Cf}_n^{\varepsilon}) \subset \mathrm{P}_m$, and $\mathrm{Cf}_n^{\varepsilon} \subset \mathrm{Cf}_n$ is a homotopy equivalence, so the diagram of maps $\mathrm{Cf}_n \supset \mathrm{Cf}_n^{\varepsilon} \to \mathrm{Cf}_m$ induces a group morphism $\mathrm{PB}_n \to \mathrm{PB}_m$, which coincides with $f_{\mathbf{m}}$. The maps $f_{\mathbf{m}} : \mathrm{B}_n \to \mathrm{B}_m$ can be defined in a similar fashion (see Figure 2). □

Proof of Proposition 12. — This proposition could be proved topologically, following Remark 13; however, we give an algebraic proof as it involves techniques which will be used in Proposition 14.

Condition (b) imposes $(1_n)_{\mathbf{m}} = 1_m$, therefore (a) and (b) imply $(\sigma_i^{-1})_{\mathbf{m}} = 1_{m_1+\cdots+m_{i-1}} * \sigma^{-1}_{m_{i+1}, m_i} * 1_{m_{i+2}+\cdots+m_n}$. As $\sigma_i^{\pm 1}$ generate B_n, this equality and conditions (a) and (b) deter-

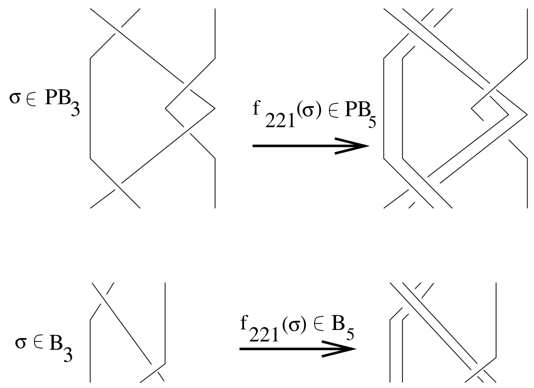

Fig. 2. — Cabling morphisms

mine the value of $\sigma_{\mathbf{m}}$ for each $\sigma \in \mathrm{B}_n$. This proves the uniqueness of the collection of maps $\sigma \mapsto \sigma_{\mathbf{m}}$.

Let us now prove its existence. We first recall from [JS] the construction of the free strict braided monoidal category $\mathrm{F}_s(\mathcal{A}_{\mathrm{S}}) = \mathbf{B}_{\mathrm{S}}$, where S is a set (the category $\mathcal{A}_{\mathrm{S}}$ is defined in Section 1.2).

$\mathbf{B}_{\mathrm{S}}$ is small and its set of objects is $\mathrm{Ob}(\mathbf{B}_{\mathrm{S}}) = \bigsqcup_{k\geq 0} \mathrm{S}^k$; it identifies with the semigroup $\langle \mathrm{S} \rangle$ freely generated by S. For $w \in \mathrm{Ob}(\mathbf{B}_{\mathrm{S}})$, we denote by $|w|$ the index k such that $w \in \mathrm{S}^k$ (k is the length of w). Then for $w, w' \in \mathrm{Ob}(\mathbf{B}_{\mathrm{S}})$, we set $\mathbf{B}_{\mathrm{S}}(w, w') = \emptyset$ if $|w| \neq |w'|$, and $\mathbf{B}_{\mathrm{S}}(w, w') = \mathrm{B}_k \times_{\mathfrak{S}_k} \mathfrak{S}_{w,w'}$ if $|w| = |w'| = k$; here $\mathfrak{S}_{w,w'} = \{\sigma \in \mathfrak{S}_k | w \circ \sigma^{-1} = w'\}$ (we view w, w' as maps $[k] \to \mathrm{S}$).

The tensor product is defined at the level of objects using the semigroup law, so $w \otimes w'$ is defined by $|w \otimes w'| = |w| + |w'|$, $(w \otimes w')(i) = w(i)$ for $i \in [|w|]$, $(w \otimes w')(|w| + i) = w'(i)$ for $i \in [|w'|]$. It is defined at the level of morphisms by restricting the map $\mathrm{B}_{|w|} \times \mathrm{B}_{|w'|} \to \mathrm{B}_{|w|+|w'|}$, $(\sigma, \sigma') \mapsto \sigma * \sigma'$. The braiding is $\beta_{w,w'} := \sigma_{|w|,|w'|} \in \mathbf{B}_{\mathrm{S}}(w \otimes w', w' \otimes w)$.

When $\mathrm{S} = \{\bullet\}$, $\mathbf{B}_{\mathrm{S}}$ is simply denoted $\mathbf{B}$; then $\mathrm{Ob}(\mathbf{B}) = \mathbf{N}$, $\mathbf{B}(k, k') = \emptyset$ if $k \neq k'$, $\mathbf{B}(k) = \mathrm{B}_k$, $k \otimes k' = k + k'$, and the tensor product coincides with $*$ at the level of morphisms.

Let now $n \geq 1$ and $\mathbf{m} \in \mathbf{N}^n$. By the universal properties of $\mathbf{B}_{[n]}$, there exists a unique tensor functor $\mathrm{F}_{\mathbf{m}} : \mathbf{B}_{[n]} \to \mathbf{B}$, such that $\mathrm{F}_{\mathbf{m}}(i) = m_i$ for each $i \in [n]$. For $s \in \mathfrak{S}_n$, we set $\mathrm{B}_n^s := \mathrm{B}_n \times_{\mathfrak{S}_n} \{s\}$. Then $\mathrm{B}_n = \bigsqcup_{s \in \mathfrak{S}_n} \mathrm{B}_n^s$. Define the map $f_{\mathbf{m}} : \mathrm{B}_n \to \mathrm{B}_m$ by the condition that for any $s \in \mathfrak{S}_n$, the diagram

$$
\begin{array}{ccc}
\mathbf{B}_{[n]}(1 \otimes \cdots \otimes n, s^{-1}(1) \otimes \cdots \otimes s^{-1}(n)) & \xrightarrow{\mathrm{F}_{\mathbf{m}}} & \mathbf{B}(m_1 \otimes \cdots \otimes m_n, m_{s^{-1}(1)} \otimes \cdots \otimes s_{s^{-1}(s_n)}) \\
\| & & \| \\
\mathrm{B}_n^s & \xrightarrow{f_{\mathbf{m}}} & \mathrm{B}_m
\end{array} \qquad (\mathbf{10})
$$

commutes. We now prove that the maps $f_{\mathbf{m}}$ satisfy conditions (a) and (b). For $s, t \in \mathfrak{S}_n$,

$$
\begin{array}{ccccccccc}
\mathrm{B}_n^s \times \mathrm{B}_n^t & = & \begin{array}{c} \mathbf{B}_{[n]}(1 \otimes \cdots \otimes n, \\ s^{-1}1 \otimes \cdots \otimes s^{-1}n) \\ \times \mathbf{B}_{[n]}(1 \otimes \cdots \otimes n, \\ t^{-1}1 \otimes \cdots \otimes t^{-1}n) \end{array} & \simeq & \begin{array}{c} \mathbf{B}_{[n]}(1 \otimes \cdots \otimes n, \\ s^{-1}1 \otimes \cdots \otimes s^{-1}n) \\ \times \mathbf{B}_{[n]}(s^{-1}1 \otimes \cdots \otimes s^{-1}n, \\ s^{-1}t^{-1}1 \otimes \cdots \otimes s^{-1}t^{-1}n) \end{array} & \stackrel{(f,g)\mapsto g\circ f}{\to} & \begin{array}{c} \mathbf{B}_{[n]}(1 \otimes \cdots \otimes n, \\ (ts)^{-1}1 \otimes \cdots \otimes (ts)^{-1}n) \end{array} & = & \mathrm{B}_n^{ts} \\
f_{\mathbf{m}} \times f_{\mathbf{m}\circ s^{-1}} \downarrow & & & & \downarrow \mathrm{F}_{\mathbf{m}} \times \mathrm{F}_{\mathbf{m}\circ s^{-1}} & & \downarrow \mathrm{F}_{\mathbf{m}} & & \downarrow f_{\mathbf{m}} \\
\mathrm{B}_m \times \mathrm{B}_m & & \simeq & & \begin{array}{c} \mathbf{B}(m_1 \otimes \cdots \otimes m_n, \\ m_{s^{-1}1} \otimes \cdots \otimes m_{s^{-1}m}) \\ \times \mathbf{B}(m_{s^{-1}1} \otimes \cdots \otimes m_{s^{-1}n}, \\ m_{s^{-1}t^{-1}1} \otimes \cdots \otimes \\ m_{s^{-1}t^{-1}n}) \end{array} & \stackrel{(f,g)\mapsto g\circ f}{\to} & \begin{array}{c} \mathbf{B}_{[m]}(m_1 \otimes \cdots \otimes m_n, \\ m_{s^{-1}t^{-1}1} \otimes \cdots \otimes \\ m_{s^{-1}t^{-1}n}) \end{array} & = & \mathrm{B}_m
\end{array}
$$

commutes, so the diagram

$$
\begin{array}{ccc}
\mathrm{B}_n^s \times \mathrm{B}_n^t & \stackrel{(\sigma,\tau)\mapsto\tau\sigma}{\to} & \mathrm{B}_n \\
f_{\mathbf{m}} \times f_{\mathbf{m}\circ\sigma^{-1}} \downarrow & & \downarrow f_{\mathbf{m}} \\
\mathrm{B}_m \times \mathrm{B}_m & \stackrel{(\sigma,\tau)\mapsto\tau\sigma}{\to} & \mathrm{B}_m
\end{array}
$$

commutes. So the family of maps $(f_{\mathbf{m}})_{|\mathbf{m}|=m}$ satisfies condition (a). The value of $f_{\mathbf{m}}(\sigma_i)$ is obtained by a direct computation, which shows that $(f_{\mathbf{m}})_{|\mathbf{m}|=m}$ satisfies condition (b).

Note that the tensor functor $\mathrm{F}_{\mathbf{m}} : \mathbf{B}_{[n]} \to \mathbf{B}$ factors as $\mathrm{F}_{\mathbf{m}} = p \circ \mathrm{G}_{\mathbf{m}}$, where $\mathrm{G}_{\mathbf{m}} : \mathbf{B}_{[n]} \to \mathbf{B}_{[m]}$ is defined by $\mathrm{G}_{\mathbf{m}}(i) := \bigotimes_{j \in m_1+\cdots+m_{i-1}+[m_i]} j$ for any $i \in [n]$ and $p : \mathbf{B}_{[m]} \to \mathbf{B}$ is defined by $p(j) = \bullet$ for any $j \in [m]$. It follows that we have a factorization of (10) as

$$
(\mathbf{11}) \qquad
\begin{array}{ccccc}
\mathbf{B}_{[n]}(1 \otimes \cdots \otimes n, s^{-1}(1) \otimes \cdots \otimes s^{-1}(n)) & \stackrel{\mathrm{G}_{\mathbf{m}}}{\to} & \begin{array}{c} \mathbf{B}_{[m]}(\mathrm{G}_{\mathbf{m}}(1) \otimes \cdots \otimes \mathrm{G}_{\mathbf{m}}(n), \\ \mathrm{G}_{\mathbf{m}}(s^{-1}(1)) \otimes \cdots \otimes \mathrm{G}_{\mathbf{m}}(s^{-1}(n))) \end{array} & \stackrel{p}{\to} & \mathbf{B}(m) \\
|| & & & & || \\
\mathrm{B}_n^s & \to & \mathrm{B}_m^{s_{\mathbf{m}}} & \to & \mathrm{B}_m
\end{array}
$$

which implies that $f_{\mathbf{m}}(\mathrm{B}_n^s) \subset \mathrm{B}_m^{s_{\mathbf{m}}}$, as wanted. The commutativity of (9) and identity (b) then imply that $f_{\mathbf{m}}$ restricts to a group morphism $\mathrm{PB}_n \to \mathrm{PB}_m$. $\square$

Identities (a) and (b) immediately imply that if $m_1 = 1$, then

$$f_{\mathbf{m}}(\mathrm{X}_i) = \mathrm{X}_{m_1+\cdots+m_{i-1}+2} \cdots \mathrm{X}_{m_1+\cdots+m_i+1}$$

for $i = 2, \ldots, n$, where $\mathrm{X}_i \in \mathrm{PB}_n$ is given by $\mathrm{X}_i = \sigma_{i-1} \cdots \sigma_2 \sigma_1^2 \sigma_2 \cdots \sigma_{i-1}$.

2.2. *A commutative diagram.* — Let $\mathrm{O} \in \mathbf{Par}_n$, and let $\mathrm{O}_1, \ldots, \mathrm{O}_n \in \mathbf{Par}$. Let $\mathrm{O}(\mathrm{O}_1, \ldots, \mathrm{O}_n) \in \mathbf{Par}$ be obtained by replacing the object $\bullet$ occurring n times in O successively by $\mathrm{O}_1, \ldots, \mathrm{O}_n$. (For example, for $\mathrm{O} = \bullet(\bullet\bullet)$, $\mathrm{O}_1 = \bullet\bullet$, $\mathrm{O}_2 = \bullet(\bullet\bullet)$, $\mathrm{O}_3 = \bullet$, $\mathrm{O}(\mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3) = (\bullet\bullet)((\bullet(\bullet\bullet))\bullet)$.)

Proposition **14.** — *Fix* $\Phi \in \mathrm{M}_1(\mathbf{k})$. *The diagram*

$$\begin{array}{ccc} \mathrm{B}_n & \stackrel{\tilde\mu_{\mathrm{O}}}{\to} & \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \mathfrak{S}_n \\ f_{\mathbf{m}} \downarrow & & \downarrow g_{\mathbf{m}} \\ \mathrm{B}_m & \stackrel{\tilde\mu_{\mathrm{O}(\mathrm{O}_1,\dots,\mathrm{O}_n)}}{\to} & \exp(\hat{\mathfrak{t}}_m^{\mathbf{k}}) \rtimes \mathfrak{S}_m \end{array}$$

commutes, where $g_{\mathbf{m}}(e^x, s) = (e^y, s_{\mathbf{m}})$ *with* $y = x^{[m_1],m_1+[m_2],\dots,m_1+\dots+m_{n-1}+[m_n]}$. *In particular, we have a commutative diagram of group morphisms*

$$(\mathbf{12}) \qquad \begin{array}{ccc} \mathrm{PB}_n & \stackrel{\tilde\mu_{\mathrm{O}}}{\to} & \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \\ f_{\mathbf{m}} \downarrow & & \downarrow g_{\mathbf{m}} \\ \mathrm{PB}_m & \stackrel{\tilde\mu_{\mathrm{O}(\mathrm{O}_1,\dots,\mathrm{O}_n)}}{\to} & \exp(\hat{\mathfrak{t}}_m^{\mathbf{k}}) \end{array}$$

Proof. — We first recall the construction of the free b.m.c. $\mathbf{PaB}_{\mathrm{S}} = \mathrm{F}_b(\mathcal{A}_{\mathrm{S}})$ (see Section 1.2). Its set of objects is $\mathrm{Ob}(\mathbf{PaB}_{\mathrm{S}}) := \mathrm{Ob}(\mathbf{B}_{\mathrm{S}}) \times_{\mathbf{N}} \mathbf{Par} = \{(w,p) | |w| = |p|\}$; it may be viewed as the free magma generated by S (recall that a magma is a set equipped with a binary law and a neutral element). The morphisms are then $\mathbf{PaB}_{\mathrm{S}}((w,p),(w',p')) := \mathbf{B}_{\mathrm{S}}(w,w')$. The tensor product is defined at the level of objects by $(w,p) \otimes (w',p') := (w \otimes w', p \otimes p')$, and may be identified with the magma product. At the level of morphisms, the tensor product law $\mathbf{PaB}_{\mathrm{S}}((w_1,p_1),(w_2,p_2)) \times \mathbf{PaB}_{\mathrm{S}}((w_3,p_3),(w_4,p_4)) \to \mathbf{PaB}_{\mathrm{S}}((w_1,p_1) \otimes (w_3,p_3),(w_2,p_2) \otimes (w_4,p_4))$ is defined as the tensor product law $\mathbf{B}_{\mathrm{S}}(w_1,w_2) \times \mathbf{B}_{\mathrm{S}}(w_3,w_4) \to \mathbf{B}_{\mathrm{S}}(w_1 \otimes w_3, w_2 \otimes w_4)$ of $\mathbf{B}_{\mathrm{S}}$. The braiding constraint for $\mathbf{PaB}_{\mathrm{S}}$ is $\beta_{(w,p),(w',p')} := \beta_{w,w'} \in \mathbf{B}_{\mathrm{S}}(w \otimes w', w' \otimes w) = \mathbf{PaB}_{\mathrm{S}}((w,p) \otimes (w',p'), (w',p') \otimes (w,p))$, and the associativity constraint is $a_{(w,p),(w',p'),(w'',p'')} := \mathrm{id}_{w \otimes w' \otimes w''} \in \mathbf{B}_{\mathrm{S}}(w \otimes w' \otimes w'') = \mathbf{PaB}_{\mathrm{S}}(((w,p) \otimes (w',p')) \otimes (w'',p''), (w,p) \otimes ((w',p') \otimes (w'',p'')))$.

For $\Phi \in \mathrm{M}_1(\mathbf{k})$, we then construct a b.m.c. $\mathbf{PaCD}_{\mathrm{S}}^{\Phi}$ as follows. Its set of objects is defined by $\mathrm{Ob}(\mathbf{PaCD}_{\mathrm{S}}^{\Phi}) := \mathrm{Ob}(\mathbf{PaB}_{\mathrm{S}})$. The morphisms are defined by $\mathbf{PaCD}_{\mathrm{S}}^{\Phi}((w,p),(w',p')) := \exp(\hat{\mathfrak{t}}_k^{\mathbf{k}}) \rtimes \mathfrak{S}_{w,w'}$ if $|w| = |w'| = k$, and $\mathbf{PaCD}_{\mathrm{S}}^{\Phi}(w,w') = \emptyset$ otherwise. There exists a unique b.m.c. structure on $\mathbf{PaCD}_{\mathrm{S}}^{\Phi}$, such that the tensor product is the same as that of $\mathbf{PaB}_{\mathrm{S}}$ at the level of objects, and the functor $\mathbf{PaCD}_{\mathrm{S}}^{\Phi} \to \mathbf{PaCD}_{\Phi}$, defined at the level of objects by $(w,p) \mapsto p$ and at the level of morphisms by the canonical inclusion $\mathbf{PaCD}_{\mathrm{S}}^{\Phi}((w,p),(w',p')) \subset \mathbf{PaCD}_{\Phi}(p,p')$, is a tensor functor.

We associate a tensor functor $\mathrm{G}_{\mathbf{m},\mathbf{p}} : \mathbf{PaCD}_{[n]}^{\Phi} \to \mathbf{PaCD}_{[m]}^{\Phi}$ to the following data:

(1) a map $\mathbf{m} : [n] \to \mathbf{N}$, such that $m = \sum_{i=1}^n m_i$;
(2) a collection $\mathbf{p} = (p_i)_{i \in [n]}$, where for each i, $p_i \in \mathbf{Par}_{m_i}$.

$\mathrm{G}_{\mathbf{m},\mathbf{p}}$ is constructed as follows. At the level of objects, it induces the unique tensor map $g_{\mathbf{m},\mathbf{p}} : \mathrm{Ob}(\mathbf{PaCD}_{[n]}^{\Phi}) \to \mathrm{Ob}(\mathbf{PaCD}_{[m]}^{\Phi})$ taking $i \in [n]$ to $((m_1 + \dots + m_{i-1} + 1, \dots, m_1 +$

$\cdots + m_i), p_i) \in [m]^{m_i} \times \mathbf{Par}_{m_i}$. Note that the diagram

$$\begin{array}{ccc} \mathrm{Ob}(\mathbf{PaCD}^{\Phi}_{[n]}) & \stackrel{g_{\mathbf{m},\mathbf{p}}}{\to} & \mathrm{Ob}(\mathbf{PaCD}^{\Phi}_{[m]}) \\ \downarrow & & \downarrow \\ \langle [n] \rangle & \stackrel{g_{\mathbf{m}}}{\to} & \langle [m] \rangle \end{array}$$

commutes, where we recall that $\langle S \rangle$ is the semigroup generated by S and $g_{\mathbf{m}}$ is the semigroup morphism defined by $g_{\mathbf{m}}(i) = (m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_i)$.

Let $w = (w_1, \ldots, w_k)$, $w' = (w'_1, \ldots, w'_k) \in [n]^k$ and $p, p' \in \mathbf{Par}_k$; the map

$$\mathbf{PaCD}^{\Phi}_{[n]}((w,p),(w',p')) \to \mathbf{PaCD}^{\Phi}_{[m]}(g_{\mathbf{m},\mathbf{p}}(w,p), g_{\mathbf{m},\mathbf{p}}(w',p'))$$

induced by $G_{\mathbf{m},\mathbf{p}}$ on morphisms is determined by the condition that

$$\begin{array}{ccc} \mathbf{PaCD}^{\Phi}_{[n]}((w,p),(w',p')) & \to & \mathbf{PaCD}^{\Phi}_{[m]}(g_{\mathbf{m},\mathbf{p}}(w,p), g_{\mathbf{m},\mathbf{p}}(w',p')) \\ || & & || \\ \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_k) \rtimes \mathfrak{S}_{w,w'} & \stackrel{g^k_{\mathbf{m}}}{\to} & \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{k'}) \rtimes \mathfrak{S}_{g_{\mathbf{m}}(w), g_{\mathbf{m}}(w')} \end{array}$$

commutes, where $k' = \sum_{i=1}^k m(w_i)$, and $g^k_{\mathbf{m}}(\exp x, s) = (\exp y, s_{m(w_1),\ldots,m(w_k)})$, where $y = x^{[m(w_1)],\ldots,m(w_1)+\cdots+m(w_{k-1})+[m(w_k)]}$. One checks that $G_{\mathbf{m},\mathbf{p}}$ is a tensor functor.

There are tensor functors $\mathbf{PaB}_{[n]} \to \mathbf{PaCD}^{\Phi}_{[n]}$ and $\mathbf{PaB}_{[m]} \to \mathbf{PaCD}^{\Phi}_{[m]}$, uniquely determined by the condition that they induce the identity at the level of objects. We also have a tensor functor $\mathbf{PaB}_{[n]} \to \mathbf{PaB}_{[m]}$, uniquely determined by the condition that it induces the map $g_{\mathbf{m},\mathbf{p}}$ at the level of objects. Then the diagram of tensor functors

$$\begin{array}{ccc} \mathbf{PaB}_{[n]} & \to & \mathbf{PaB}_{[m]} \\ \downarrow & & \downarrow \\ \mathbf{PaCD}^{\Phi}_{[n]} & \to & \mathbf{PaCD}^{\Phi}_{[m]} \end{array}$$

commutes; to prove this, one checks that two tensor functors $\mathbf{PaB}_{[n]} \to \mathbf{PaCD}^{\Phi}_{[m]}$ are equal by considering the images of objects.

We also have a commutative diagram of tensor functors

$$\begin{array}{ccc} \mathbf{PaB}_{[m]} & \to & \mathbf{PaB} \\ \downarrow & & \downarrow \\ \mathbf{PaCD}^{\Phi}_{[m]} & \to & \mathbf{PaCD}_{\Phi} \end{array}$$

where $\mathbf{PaB}_{[m]} \to \mathbf{PaB}$ is induced by the unique map $[m] \to \{\bullet\}$, $\mathbf{PaCD}^{\Phi}_{[m]} \to \mathbf{PaCD}_{\Phi}$ is similarly defined on objects and by the natural inclusions of the sets of morphisms.

Composing these commutative diagrams, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{PaB}_{[n]} & \to & \mathbf{PaB} \\ \downarrow & & \downarrow \\ \mathbf{PaCD}^{\Phi}_{[n]} & \to & \mathbf{PaCD}_{\Phi} \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc} \mathbf{PaB}_{[n]}(1\otimes\cdots\otimes n, s^{-1}(1)\otimes\cdots\otimes s^{-1}(n)) & \to & \mathbf{PaB}(m) \\ \downarrow & & \downarrow \\ \mathbf{PaCD}^{\Phi}_{[n]}(1\otimes\cdots\otimes n, s^{-1}(1)\otimes\cdots\otimes s^{-1}(n)) & \to & \mathbf{PaCD}_{\Phi}(m) \end{array}$$

for any $s\in\mathfrak{S}_n$. The latter induces the desired commutative diagram

$$\begin{array}{ccc} \mathrm{B}^s_m & \stackrel{\sigma\mapsto\sigma_{\mathbf{m}}}{\to} & \mathrm{B}_m \\ \downarrow & & \downarrow \\ \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_n)\rtimes\mathfrak{S}_n & \to & \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_m)\rtimes\mathfrak{S}_m \end{array}$$

□

3. Braid groups and free groups

In this section, we recall the relations between the free and (pure) braid groups, as well as between their infinitesimal analogues. We also recall material from [AT2] about the non-commutative Jacobian and complexes of spaces of cyclic words.

3.1. *Action of braid groups on free Lie algebras.* — For S a finite totally ordered set, define the braid group B_S by $\mathrm{B}_\mathrm{S} := \mathrm{B}_{|\mathrm{S}|}$. The images of the Artin generators of $\mathrm{B}_{|\mathrm{S}|}$ are then σ_s, $s\in\mathrm{S}$ non-maximal.

Set $\mathrm{B}_{1,n} := \mathrm{B}_{\{0,\ldots,n\}}\times_{\mathfrak{S}_{n+1}}\mathfrak{S}_n$, where $\mathfrak{S}_n\hookrightarrow\mathfrak{S}_{n+1}=\mathfrak{S}_{\{0,\ldots,n\}}$ is $\sigma\mapsto\tilde\sigma$, where $\tilde\sigma$ extends σ by $\tilde\sigma(0)=0$. Then $\mathrm{B}_{1,n}$ is a braid group of type B. If we set $\tau:=\sigma_0^2$, its presentation is as follows:

$$\text{generators: } \tau,\sigma_1,\ldots,\sigma_{n-1},$$
$$\text{relations: } (\tau\sigma_1)^2=(\sigma_1\tau)^2,\quad \tau\sigma_i=\sigma_i\tau \text{ for } i\geq 2,$$
$$\text{Artin relations between } \sigma_1,\ldots,\sigma_{n-1}. \tag{13}$$

Define elements of $\mathrm{B}_{1,n}$ as follows:

$$\mathrm{X}_1:=\tau,\qquad \mathrm{X}_2:=\sigma_1\tau\sigma_1^{-1},\quad\ldots,\quad \mathrm{X}_n:=(\sigma_{n-1}\cdots\sigma_1)\tau(\sigma_{n-1}\cdots\sigma_1)^{-1}.$$

We have then:

$$\begin{aligned} &\sigma_i\mathrm{X}_i\sigma_i^{-1}=\mathrm{X}_{i+1},\qquad \sigma_i\mathrm{X}_{i+1}\sigma_i^{-1}=\mathrm{X}_{i+1}^{-1}\mathrm{X}_i\mathrm{X}_{i+1},\\ &\sigma_i\mathrm{X}_j\sigma_i^{-1}=\mathrm{X}_j\quad \text{if } j\neq i,i+1\end{aligned}\tag{14}$$

for $i=1,\ldots,n-1$, $j=1,\ldots,n$. One checks that $\mathrm{B}_{1,n}$ may be presented as follows:

$$\text{generators: } \mathrm{X}_1,\ldots,\mathrm{X}_n,\sigma_1,\ldots,\sigma_{n-1},$$
$$\text{relations: relations (14)},\quad \text{Artin relations between the } \sigma_i. \tag{15}$$

More precisely, one shows directly that the presentations (13) and (15) are equivalent.

Proposition **15.** — (1) *There is a unique group morphism* $B_n \to \mathrm{Aut}(F_n)$*, taking* σ_i $(i = 1, \dots, n-1)$ *to the automorphism* $X_i \mapsto X_{i+1}$, $X_{i+1} \mapsto X_{i+1}^{-1} X_i X_{i+1}$, $X_j \mapsto X_j$ *for* $j \neq i, i+1$.

(2) *We have an isomorphism* $B_{1,n} \simeq F_n \rtimes B_n$*, where the semidirect product is with respect to the above action. Its inverse is* $(X_i, 1) \mapsto X_i$, $(1, \sigma_i) \mapsto \sigma_i$.

Proof. — (1) is well-known (see e.g. [Mag]). As mentioned in the Introduction, this group morphism admits an interpretation in terms of the Fadell–Neuwirth fibration. (2) follows from the fact that the presentation (15) is that of a semidirect product. □

Note that we have a commutative diagram

$$\begin{array}{ccc} B_{1,n} & \xrightarrow{\sim} & F_n \rtimes B_n \\ & \searrow & \downarrow \\ & & \mathfrak{S}_n \end{array}$$

Taking kernels, we obtain:

Corollary **16.** — *We have an isomorphism* $PB_{n+1} \simeq F_n \rtimes PB_n$*, where the semidirect product is with respect to the restriction* $PB_n \to \mathrm{Aut}(F_n)$ *of the action of Proposition* 15.

These statements have prounipotent counterparts:

Proposition **17.** — (1) *The morphism* $B_n \to \mathrm{Aut}(F_n)$ *in Proposition* 15 *extends to a morphism* $B_n(\mathbf{k}, \mathfrak{S}_n) \to \mathrm{Aut}(F_n(\mathbf{k}))$.

(2) *We have an isomorphism* $PB_{n+1}(\mathbf{k}) \simeq F_n(\mathbf{k}) \rtimes PB_n(\mathbf{k})$.

Proof. — Immediate. □

Remark **18.** — The results of this subsection can be reformulated 'invariantly' as follows. If S is a finite totally ordered set, set $S^+ := \{o\} \sqcup S$, where $o < s$ for any $s \in S$. We then set $B_{1,S} := B_{S^+} \times_{\mathfrak{S}_{S^+}} \mathfrak{S}_S$, and $X_s := (\prod_{t<s}^{-} \sigma_t)\sigma_{\min S}^2 (\prod_{t<s}^{-} \sigma_t)^{-1}$, where $\prod^-$ means the product in decreasing order. Then we have injective group morphisms $F_S \hookrightarrow B_{1,S}$, $s \mapsto X_s$ and $B_S \hookrightarrow B_{1,S}$, which lead to an isomorphism $B_{1,S} \simeq F_S \rtimes B_S$. It restricts to an isomorphism $PB_{S^+} \simeq F_S \rtimes PB_S$.

3.2. *Lie algebraic analogues.* — One checks that there is a unique Lie algebra morphism $\mathfrak{t}_S \to \mathrm{Der}(\mathfrak{f}_S)$, given by $t_{ss'} \mapsto (s \mapsto [s, s'], s' \mapsto [s', s], t \mapsto 0$ for $t \neq s, s')$. It follows from the presentations of $\mathfrak{t}_S$ and $\mathfrak{t}_{S^+}$ that we have an isomorphism

$$\mathfrak{t}_{S^+} \simeq \mathfrak{f}_S \rtimes \mathfrak{t}_S,$$

given by $t_{os} \mapsto (s, 0)$, $t_{ss'} \mapsto (0, t_{ss'})$.

3.3. *Tangential derivations and tangential automorphisms.* — If S is a set, define $\mathrm{Eauts}_{\mathrm{S}} := \mathfrak{S}_{\mathrm{S}} \times (\mathrm{F}_{\mathrm{S}})^{\mathrm{S}}$; it is equipped with the semigroup law $(\sigma, g)(\tau, h) := (\sigma\tau, k)$, where $k_s = \theta(\sigma, g)(h_s) g_{\tau(s)}$ and $\theta : \mathrm{Eauts}_{\mathrm{S}} \to \mathrm{End}(\mathrm{F}_{\mathrm{S}})$ is given by $\theta(\sigma, g)(s) = \mathrm{Ad}_{g_s}(\sigma(s))$. Then θ is a semigroup morphism. We have an isomorphism $\mathrm{Ker}\,\theta \simeq \mathbf{Z}^{\mathrm{S}}$, with inverse given by $(n_s)_{s\in\mathrm{S}} \mapsto (g_s)_{s\in\mathrm{S}}$, where $g_s = s^{n_s}$ for any $s \in \mathrm{S}$. We set $\underline{\mathrm{Eaut}}_{\mathrm{S}} := \mathrm{Im}\,\theta = \mathrm{Eauts}_{\mathrm{S}} / \mathbf{Z}^{\mathrm{S}}$ and call its elements extended tangential endomorphisms of the free group.

A section of $\mathrm{Eauts}_{\mathrm{S}} \to \underline{\mathrm{Eaut}}_{\mathrm{S}}$ may be defined by $(\sigma, (g_s)_{s\in\mathrm{S}}) \mapsto (\sigma, (g'_s)_{s\in\mathrm{S}})$, where $g'_s = g_s s^{-\text{degree of } g_s \text{ in } s}$. We then have

$$\mathrm{Eauts}_{\mathrm{S}} = \mathbf{Z}^{\mathrm{S}} \rtimes \underline{\mathrm{Eaut}}_{\mathrm{S}},$$

where the action of $\underline{\mathrm{Eaut}}_{\mathrm{S}}$ on $\mathbf{Z}^{\mathrm{S}}$ is via $\underline{\mathrm{Eaut}}_{\mathrm{S}} \to \mathfrak{S}_{\mathrm{S}}$, $(\sigma, (g_s)_{s\in\mathrm{S}}) \mapsto \sigma$.

Set $\mathrm{Tauts}_{\mathrm{S}} := \mathrm{Ker}(\mathrm{Eauts}_{\mathrm{S}} \to \mathfrak{S}_{\mathrm{S}})$, $\underline{\mathrm{Taut}}_{\mathrm{S}} := \mathrm{Ker}(\underline{\mathrm{Eaut}}_{\mathrm{S}} \to \mathfrak{S}_{\mathrm{S}})$. Then $\mathrm{Ker}\,\theta \subset \mathrm{Tauts}_{\mathrm{S}}$ is central, and the above section of θ restricts to a morphism $\underline{\mathrm{Taut}}_{\mathrm{S}} \to \mathrm{Tauts}_{\mathrm{S}}$, therefore

$$\mathrm{Tauts}_{\mathrm{S}} = \mathbf{Z}^{\mathrm{S}} \oplus \underline{\mathrm{Taut}}_{\mathrm{S}}.$$

The semigroup morphism $\underline{\mathrm{Eaut}}_{\mathrm{S}} \to \mathfrak{S}_{\mathrm{S}}$, $(\sigma, g) \mapsto \sigma$ admits a section $\sigma \mapsto (\sigma, 1)$. We then have isomorphisms

$$\mathrm{Eauts}_{\mathrm{S}} = \mathrm{Tauts}_{\mathrm{S}} \rtimes \mathfrak{S}_{\mathrm{S}}, \qquad \underline{\mathrm{Eaut}}_{\mathrm{S}} = \underline{\mathrm{Taut}}_{\mathrm{S}} \rtimes \mathfrak{S}_{\mathrm{S}}$$

compatible with the above decompositions.

These semigroups admit prounipotent versions. We set $\mathrm{Eauts}_{\mathrm{S}}(\mathbf{k}) := \mathfrak{S}_{\mathrm{S}} \times \mathrm{F}_{\mathrm{S}}(\mathbf{k})^{\mathrm{S}}$, $\underline{\mathrm{Eaut}}_{\mathrm{S}}(\mathbf{k}) := \mathrm{Im}(\mathrm{Eauts}_{\mathrm{S}}(\mathbf{k}) \to \mathrm{Aut}(\mathrm{F}_{\mathrm{S}}(\mathbf{k})))$; we recall that $\mathrm{F}_{\mathrm{S}}(\mathbf{k}) \simeq \exp(\hat{\mathfrak{f}}_{\mathrm{S}}^{\mathbf{k}})$. A section of $\mathrm{Eauts}_{\mathrm{S}}(\mathbf{k}) \to \underline{\mathrm{Eaut}}_{\mathrm{S}}(\mathbf{k})$ is defined as above, with $g'_s := g_s e^{-(\text{coefficient of } \log s \text{ in } \log g_s) \log s}$. Then as above,

$$\mathrm{Eauts}_{\mathrm{S}}(\mathbf{k}) = \mathbf{k}^{\mathrm{S}} \rtimes \underline{\mathrm{Eaut}}_{\mathrm{S}}(\mathbf{k}), \qquad \mathrm{Tauts}_{\mathrm{S}}(\mathbf{k}) = \mathbf{k}^{\mathrm{S}} \oplus \underline{\mathrm{Taut}}_{\mathrm{S}}(\mathbf{k}),$$

$$\mathrm{Eauts}_{\mathrm{S}}(\mathbf{k}) = \mathrm{Tauts}_{\mathrm{S}}(\mathbf{k}) \rtimes \mathfrak{S}_{\mathrm{S}}, \qquad \underline{\mathrm{Eaut}}_{\mathrm{S}}(\mathbf{k}) = \underline{\mathrm{Taut}}_{\mathrm{S}}(\mathbf{k}) \rtimes \mathfrak{S}_{\mathrm{S}}.$$

Set $\underline{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}} := \mathrm{Im}(\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} \to \mathrm{Der}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}}))$. Then the projection $\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} \to \underline{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}}$ admits a section $(u_s)_{s\in\mathrm{S}} \mapsto (u'_s)_{s\in\mathrm{S}}$, where $u'_s = u_s - (\text{coefficient of } u_s \text{ in } s)s$. Then we have an isomorphism

$$\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} = \mathbf{k}^{\mathrm{S}} \oplus \underline{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}},$$

which is equivariant under $\mathfrak{S}_{\mathrm{S}}$ and is the Lie algebraic version of the above decompositions.

Proposition **19.** — (1) *There is a unique semigroup morphism* $\mathrm{B}_{1,n} \xrightarrow{\mathrm{Ad}} \underline{\mathrm{Eaut}}_n$, *given by* $\sigma_i \mapsto (s_i, g_i)$, *where* $(g_i)_{i+1} = \mathrm{X}_{i+1}^{-1}$, *and* $(g_i)_j = 1$ *for* $j \neq i+1$, *and* $\mathrm{X}_i \mapsto (1, h_i)$, *where* $(h_i)_j = \mathrm{X}_i$ *for* $j = 1, \dots, n$. *The composite map* $\mathrm{B}_{1,n} \to \underline{\mathrm{Eaut}}_n \to \mathrm{Aut}(\mathrm{F}_n)$ *is the adjoint action of* $\mathrm{B}_{1,n}$ *on its normal subgroup* F_n.

(2) *This morphism restricts to a morphism* $\mathrm{PB}_{n+1} \to \underline{\mathrm{Taut}}_n$; *the latter morphism extends to a morphism* $\mathrm{PB}_{n+1}(\mathbf{k}) \to \underline{\mathrm{Taut}}_n(\mathbf{k})$. *The composite maps* $\mathrm{PB}_{n+1} \to \underline{\mathrm{Taut}}_n \to \mathrm{Aut}(\mathrm{F}_n)$ *and* $\mathrm{PB}_{n+1}(\mathbf{k}) \to \underline{\mathrm{Taut}}_n(\mathbf{k}) \to \mathrm{Aut}(\mathrm{F}_n(\mathbf{k}))$ *are the adjoint actions of* PB_{n+1} *on* F_n (*resp., of* $\mathrm{PB}_{n+1}(\mathbf{k})$ *on* $\mathrm{F}_n(\mathbf{k})$).

The proof is straightforward.

As $\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} = \mathrm{Lie}\,\mathrm{Taut}_{\mathrm{S}}(\mathbf{k})$, $\underline{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}} = \mathrm{Lie}\,\underline{\mathrm{Taut}}_{\mathrm{S}}(\mathbf{k})$, the Lie algebraic version of the sequence of morphisms $\mathrm{Taut}_{\mathrm{S}}(\mathbf{k}) \to \underline{\mathrm{Taut}}_{\mathrm{S}}(\mathbf{k}) \hookrightarrow \mathrm{Aut}(\mathrm{F}_{\mathrm{S}}(\mathbf{k}))$ is $\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} \twoheadrightarrow \underline{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}} \hookrightarrow \mathrm{Der}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}})$, where $\underline{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}} = \mathrm{Im}(\mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}} \to \mathrm{Der}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}})) = \mathfrak{tder}_{\mathrm{S}}^{\mathbf{k}}/\mathbf{k}^{\mathrm{S}}$.

Proposition **20.** — *There exists a unique morphism* $\mathfrak{t}_{n+1} \to \underline{\mathfrak{tder}}_n$, *given by* $t_{ij} \mapsto (i \mapsto -x_j, j \mapsto -x_i, k \mapsto 0$ *for* $k \neq i, j)$ *and* $t_{0i} \mapsto (j \mapsto x_i)$. *The composite map* $\mathfrak{t}_n \to \underline{\mathfrak{tder}}_n \to \mathrm{Der}(\mathfrak{f}_n)$ *coincides with the adjoint action of* $\mathfrak{t}_{n+1}$ *on its ideal* $\mathfrak{f}_n$.

This follows from the Section 3.2.

3.4. *Contravariant functors from the category* $\mathcal{S}_{ord}$. — We define $\mathcal{S}_{ord}$ as the category where objects are totally ordered finite sets and morphisms are partially defined non-decreasing maps.

The functor $\mathrm{S} \mapsto \hat{\mathfrak{f}}_{\mathrm{S}}^{\mathbf{k}}$ is then a contravariant functor $\mathcal{S}_{ord} \to \{\text{Lie algebras}\}$, where to the morphism $\mathrm{T} \supset \mathrm{D}_\phi \xrightarrow{\phi} \mathrm{S}$ is assigned $\tilde{\phi}^* : \hat{\mathfrak{f}}_{\mathrm{S}}^{\mathbf{k}} \to \hat{\mathfrak{f}}_{\mathrm{T}}^{\mathbf{k}}$, $s \mapsto \mathrm{cbh}(t, t \in \phi^{-1}(s))$, and cbh is the Campbell–Baker–Hausdorff product (according to the order in $\phi^{-1}(s)$).

Likewise, the functor $\mathrm{S} \mapsto \mathrm{Taut}_{\mathrm{S}}(\mathbf{k})$ is a contravariant functor $\mathcal{S}_{ord} \to \{\text{groups}\}$, where to ϕ is assigned $\tilde{\phi}^* : g = (g_s)_{s\in\mathrm{S}} \mapsto g^{\tilde{\phi}} = h = (h_t)_{t\in\mathrm{T}}$, where $h_t = \tilde{\phi}^*(g_{\phi(t)})$; we use the convention $g_{\phi(t)} = 1$ for $\phi(t)$ undefined. The corresponding contravariant functor $\mathcal{S}_{ord} \to \{\text{Lie algebras}\}$ is $\mathrm{S} \mapsto \widehat{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}}$ (the hat denotes the degree completion); the maps $\widehat{\mathfrak{tder}}_{\mathrm{S}}^{\mathbf{k}} \to \widehat{\mathfrak{tder}}_{\mathrm{T}}^{\mathbf{k}}$ are defined in the same way, with the convention $u_{\phi(t)} = 0$ if $\phi(t)$ is undefined.

The contravariant functor structure of $\mathrm{S} \mapsto \hat{\mathfrak{f}}_{\mathrm{S}}^{\mathbf{k}}$ induces structures of contravariant functors $\mathcal{S}_{ord} \to \{\text{algebras}\}$ and $\mathcal{S}_{ord} \to \{\text{vector spaces}\}$ on $\mathrm{S} \mapsto \widehat{\mathrm{U}(\mathfrak{f}_{\mathrm{S}}^{\mathbf{k}})}$ and $\mathrm{S} \mapsto \hat{\mathfrak{T}}_{\mathrm{S}}^{\mathbf{k}}$ (where the hats again denote the degree completions).

We use the notation $\tilde{\phi}^*(g) = g^{\tilde{\phi}} = g^{\widetilde{\mathrm{I}_1,\dots,\mathrm{I}_n}}$ for $\mathrm{S} = [n]$, where $\mathrm{I}_i = \phi^{-1}(i)$.

Remark **21.** — The simplicial category Δ has the same objects as $\mathcal{S}_{ord}$, and its morphisms are the (everywhere defined) non-decreasing maps. We thus have a functor $\Delta \to \mathcal{S}_{ord}$.

3.5. *Properties of the non-commutative Jacobian map.* —

Proposition **22.** — *The composite map* $\exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{n+1}) \to \mathrm{Taut}_n(\mathbf{k}) \xrightarrow{\mathrm{J}} \hat{\mathfrak{T}}^{\mathbf{k}}_n$ *is zero.*

Proof. — This follows from the relations between J and j and the fact that $(\mathfrak{t}^{\mathbf{k}}_{n+1} \xrightarrow{\mathrm{ad}} \mathfrak{tder}^{\mathbf{k}}_n \xrightarrow{j} \mathfrak{T}^{\mathbf{k}}_n) = 0$, which follows from the cocycle identity for j and $j(\mathrm{ad}\, t_{ii'}) = 0$ for any i, i'. □

Proposition **23.** — *The composite map* $\mathrm{PB}_{n+1}(\mathbf{k}) \to \mathrm{Taut}_n(\mathbf{k}) \xrightarrow{\mathrm{J}} \hat{\mathfrak{T}}^{\mathbf{k}}_n$ *is zero.*

Proof. — The map J admits an extension to a cocycle $\underline{\mathrm{Eaut}_n}(\mathbf{k}) \xrightarrow{\mathrm{J}} \hat{\mathfrak{T}}^{\mathbf{k}}_n$, uniquely defined by the condition that $\mathrm{J}(\sigma) = 0$ for $\sigma \in \mathfrak{S}_{\mathrm{S}}$. One checks that $\mathrm{J}(\mathrm{Ad}\,\sigma_i) = 0$ for $i = 1, \ldots, n-1$ and $\mathrm{J}(\mathrm{Ad}\,\mathrm{X}_i) = 0$ for $i = 1, \ldots, n$, which implies the statement. □

We now study the compatibility of j and J with the simplicial structure. Any partially defined map $[m] \supset \mathrm{D}_\phi \xrightarrow{\phi} [n]$ gives rise to a Lie algebra morphism $\phi^* : \mathfrak{f}_n \to \mathfrak{f}_m$, $x \mapsto x^\phi$. These morphisms give rise to linear maps $\mathfrak{T}^{\mathbf{k}}_n \to \mathfrak{T}^{\mathbf{k}}_m$. Then one can show:

Proposition **24.** — $j(u^\phi) = j(u)^\phi$, $\mathrm{J}(g^\phi) = \mathrm{J}(g)^\phi$ *for* $u \in \mathfrak{tder}^{\mathbf{k}}_n$, $g \in \mathrm{Taut}_n(\mathbf{k})$.

J and j are also compatible with the ordered simplicial structure. One can show:

Proposition **25.** — *For any non-decreasing partially defined map* $[m] \supset \mathrm{D}_\phi \to [n]$, *we have* $j(u^{\tilde{\phi}}) = j(u)^{\tilde{\phi}}$, $\mathrm{J}(g^{\tilde{\phi}}) = \mathrm{J}(g)^{\tilde{\phi}}$ *for* $u \in \widehat{\mathfrak{tder}}^{\mathbf{k}}_n$, $g \in \mathrm{Taut}_n(\mathbf{k})$.

3.6. *Complexes.* — We define a complex $\mathfrak{T}_1 \xrightarrow{\delta} \mathfrak{T}_2 \xrightarrow{\delta} \mathfrak{T}_3 \ldots$ as follows: the map $\mathfrak{T}_n \xrightarrow{\delta} \mathfrak{T}_{n+1}$ is

$$f \mapsto \sum_{k=1}^{n} (-1)^{k+1} f^{1,\ldots,kk+1,\ldots,n+1} - f^{2,3,\ldots,n+1} + (-1)^n f^{1,2,\ldots,n},$$

so the first maps are $f(x_1) \mapsto f(x_1 + x_2) - f(x_1) - f(x_2) = f^{12} - f^1 - f^2$, $f(x_1, x_2) \mapsto f(x_1 + x_2, x_3) - f(x_1, x_2 + x_3) - f(x_2, x_3) + f(x_1, x_2) = f^{12,3} - f^{1,23} - f^{2,3} + f^{1,2}$, etc.

Proposition **26** (See [AT2]). — *This complex is acyclic in degree* 2 (*the degree of* $\mathfrak{T}_i$ *is* i). *The kernel of* $\mathfrak{T}_1 \xrightarrow{\delta} \mathfrak{T}_2$ *is 1-dimensional, spanned by the class of* $x_1 \in \mathrm{U}(\mathfrak{f}^{\mathbf{k}}_1) \simeq \mathfrak{T}_1$.

We similarly define a complex $\hat{\mathfrak{T}}_1 \xrightarrow{\tilde{\delta}} \hat{\mathfrak{T}}_2 \xrightarrow{\tilde{\delta}} \hat{\mathfrak{T}}_3 \ldots$ by requiring that $\hat{\mathfrak{T}}_n \xrightarrow{\tilde{\delta}} \hat{\mathfrak{T}}_{n+1}$ is

$$f \mapsto \sum_{k=1}^{n} (-1)^{k+1} f^{1,\ldots,\widetilde{kk+1},\ldots,n+1} - f^{2,3,\ldots,n+1} + (-1)^n f^{1,2,\ldots,n},$$

so the first maps are $f(x_1) \mapsto f(\log(e^{x_1}e^{x_2})) - f(x_1) - f(x_2) = f^{\widetilde{12}} - f^1 - f^2$, $f(x_1, x_2) \mapsto f(\log(e^{x_1}e^{x_2}), x_3) - f(x_1, \log(e^{x_2}e^{x_3})) - f(x_2, x_3) + f(x_1, x_2)$.

Proposition **27.** — *This complex is acyclic in degree* 2, *and* $\mathrm{Ker}(\hat{\mathfrak{T}}_1 \xrightarrow{\tilde\delta} \hat{\mathfrak{T}}_2)$ *is 1-dimensional, spanned by the class of* $x_1 \in \widehat{\mathrm{U}(\mathfrak{f}_1^{\mathbf{k}})} \simeq \hat{\mathfrak{T}}_1$.

Proof. — The complex $\hat{\mathfrak{T}}_1 \xrightarrow{\tilde\delta} \cdots$ has a decreasing filtration by the degree, and its associated graded is the complex $\mathfrak{T}_1 \xrightarrow{\delta} \cdots$, which is acyclic by Proprosition 26; so the complex $\hat{\mathfrak{T}}_1 \xrightarrow{\tilde\delta} \cdots$ is again acyclic in degree 2. The second statement follows from the fact that $\log(e^{x_1}e^{x_2}) - x_1 - x_2$ is a sum of brackets. □

4. Automorphisms of free groups

Fix $\Phi \in \mathrm{M}_1(\mathbf{k})$. In this section, we first show that for any $\mathrm{O} \in \mathbf{Par}_{n+1}$, the isomorphism $\tilde\mu_{\mathrm{O}} : \mathrm{PB}_{n+1}(\mathbf{k}) \to \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{n+1})$ (see (8)) restricts to an isomorphism $\mathrm{F}_n(\mathbf{k}) \to \exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n)$ (in the case of the left parenthesization, this was proved in [HM]). We then set $\mu_{\mathrm{O}} := \tilde\mu_{\mathrm{O}|\mathrm{F}_n(\mathbf{k})} \circ \mathrm{can} \in \mathrm{Aut}(\hat{\mathfrak{f}}^{\mathbf{k}}_n)$, where $\mathrm{can} : \exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n) \to \mathrm{F}_n(\mathbf{k})$ is induced by $e^{x_i} \mapsto \mathrm{X}_i$, and we show that $\mu_{\mathrm{O}} \in \underline{\mathrm{Taut}}_n(\mathbf{k}) \subset \mathrm{Aut}(\exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n))$. We then show how the μ_{O} are related for various $\mathrm{O} \in \mathbf{Par}$. If $\mathrm{O} \in \mathbf{Par}_n$ has letters successively indexed by $0, \ldots, n-1$, and if $i \in [n]$, we denote by $\mathrm{O}^{(i)}$ the element of $\mathbf{Par}_{n+1}$ obtained from O by replacing the letter $\bullet$ with index i by $(\bullet\bullet)$. Our main result (Theorem 30) is the identity

$$\mu_{\mathrm{O}^{(i)}} = \mu_{\mathrm{O}}^{1,2,\ldots,ii+1,\ldots,n} \circ \mu^{i,i+1}_{\bullet(\bullet\bullet)},$$

where we view $\mu_{\mathrm{O}^{(i)}}, \mu_{\mathrm{O}}, \mu_{\bullet(\bullet\bullet)}$ as elements of $\mathrm{Taut}_k(\mathbf{k})$ for $k = n+1, n, 2$, by virtue of the inclusion $\underline{\mathrm{Taut}}_k(\mathbf{k}) \subset \underline{\mathrm{Taut}}_k(\mathbf{k}) \oplus \mathbf{k}^k = \mathrm{Taut}_n(\mathbf{k})$.

4.1. *Restriction of formality isomorphisms to free groups.* — Let $\mathrm{S} := [n]$. We identify $\mathrm{S}^+ = \{0, \ldots, n\}$. Then the inclusions of normal subgroups $\mathrm{F}_{\mathrm{S}} \subset \mathrm{PB}_{\mathrm{S}^+}$ and $\exp(\hat{\mathfrak{f}}^{\mathbf{k}}_{\mathrm{S}}) \subset \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{\mathrm{S}^+})$ identify with $\mathrm{F}_n \subset \mathrm{PB}_{n+1}$ and $\exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n) \subset \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{n+1})$. Recall that the generators of $\mathrm{B}_{n+1} \supset \mathrm{PB}_{n+1}$ are $\sigma_0, \ldots, \sigma_{n-1}$, the generators of $\mathrm{F}_n \subset \mathrm{PB}_{n+1}$ are $\mathrm{X}_1, \ldots, \mathrm{X}_n$ with $\mathrm{X}_1 = \sigma_0^2, \ldots, \mathrm{X}_n = (\sigma_{n-1} \cdots \sigma_1)\sigma_0^2(\sigma_{n-1} \cdots \sigma_1)^{-1}$, the generators of $\mathfrak{t}^{\mathbf{k}}_{n+1}$ are t_{ij} with $i \neq j \in \{0, \ldots, n\}$, and the generators of $\mathfrak{f}^{\mathbf{k}}_n$ are $x_1, \ldots, x_n$ with $x_i = t_{0i}$.

The generators of B_{n+1} and of $\mathrm{F}_n \lhd \mathrm{PB}_{n+1} \subset \mathrm{B}_{n+1}$ are depicted in Figure 3.

Proposition **28.** — *For any* $\mathrm{O} \in \mathbf{Par}_{n+1}$, *the morphism* $\tilde\mu_{\mathrm{O}}$ *restricts to an isomorphism* $\mathrm{F}_n(\mathbf{k}) \to \exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n)$. *The composition of* $\tilde\mu_{\mathrm{O}|\mathrm{F}_n(\mathbf{k})}$ *with the isomorphism* $\exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n) \xrightarrow{\mathrm{can}} \mathrm{F}_n(\mathbf{k})$, $\exp(x_i) \mapsto \mathrm{X}_i$ *belongs to* $\underline{\mathrm{Taut}}_n(\mathbf{k}) \subset \mathrm{Aut}(\exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n))$. *We denote it* μ_{O}.

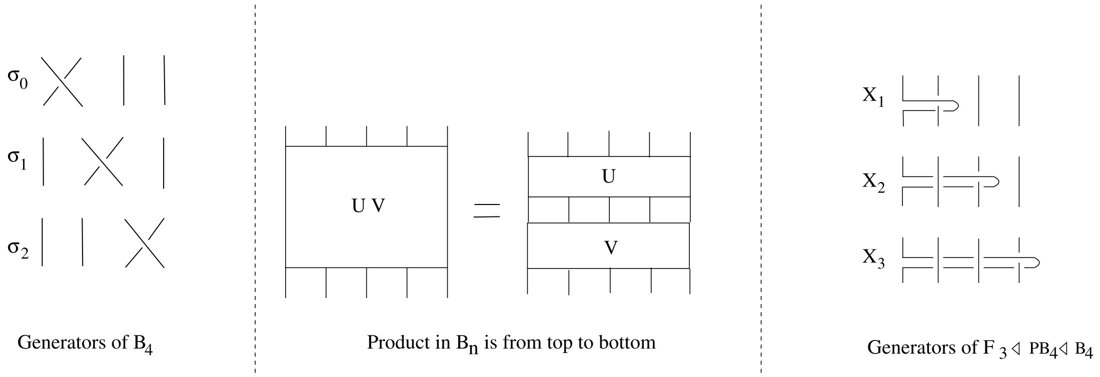

FIG. 3. — Generators of B_4 and F_3

Proof. — Let us first treat the case of $O := \bullet(\ldots(\bullet\bullet))$. As $X_i = \sigma_{i-1}\cdots\sigma_1 \times \sigma_0^2(\sigma_{i-1}\cdots\sigma_1)^{-1}$, we have $\tilde\mu_O(X_i) = \mathrm{Ad}_{\tilde\mu_O(\sigma_{i-1}\cdots\sigma_1)\Phi^{0,1,2\ldots n}}(e^{t_{01}})$.

Now we have $\tilde\mu_O(\sigma_{i-1}\cdots\sigma_1)\Phi^{0,1,2\ldots n} = e^{y_i}s_{i-1}\cdots s_0$ for some $y_i \in \hat{\mathfrak{t}}^{\mathbf{k}}_{n+1}$, so $\tilde\mu_O(X_i) = \mathrm{Ad}_{e^{y_i}}(e^{t_{0i}})$. As $\hat{\mathfrak{t}}^{\mathbf{k}}_{n+1}$ acts on $\hat{\mathfrak{f}}^{\mathbf{k}}_n$ by tangential automorphisms, we have $\mathrm{Ad}_{e^{y_i}}(e^{t_{0i}}) = \mathrm{Ad}_{e^{z_i}}(e^{t_{0i}}) = \mathrm{Ad}_{e^{z_i}}(e^{x_i})$ for some $z_i \in \hat{\mathfrak{f}}^{\mathbf{k}}_n$. So $\tilde\mu_{O|F_n(\mathbf{k})} \circ \mathrm{can} \in \underline{\mathrm{Taut}}_n(\mathbf{k})$. The general case follows from the identity $\tilde\mu_{O'} = \mathrm{Ad}_{\Phi_{O,O'}} \circ \tilde\mu_O$ and the fact that for any $\Psi \in \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{n+1})$, $\mathrm{Ad}_\Psi \in \underline{\mathrm{Taut}}_n(\mathbf{k})$. □

Proposition **29.** — *If moreover* $O = \bullet \otimes \bar{O}$ *from some* $\bar{O} \in \mathbf{Par}_n$, *then*

$$\tilde\mu_{O|F_n(\mathbf{k})}(X_1\cdots X_n) = e^{x_1+\cdots+x_n}.$$

Proof. — We have $X_1\cdots X_n = \sigma_1\cdots\sigma_{n-1}^2\cdots\sigma_1$. Now $\sigma_{n-1}\cdots\sigma_1 = \beta_{\bullet,\bar{O}} \in \mathbf{PaB}(\bullet\otimes\bar{O}, \bar{O}\otimes\bullet)$ while $\sigma_1\cdots\sigma_{n-1} = \beta_{\bar{O},\bullet} \in \mathbf{PaB}(\bar{O}\otimes\bullet, \bullet\otimes\bar{O})$. So $\mu_O(X_1\cdots X_n) = \tilde\mu_O(\beta_{\bar{O},\bullet}\beta_{\bullet,\bar{O}}) = \beta^{\mathbf{PaCD}_\Phi}_{\bar{O},\bullet}\beta^{\mathbf{PaCD}_\Phi}_{\bullet,\bar{O}} = e^{t_{01}+\cdots+t_{0n}} = e^{x_1+\cdots+x_n}$, as announced. □

Whereas the isomorphisms $\tilde\mu_O$ are related by inner automorphisms, the various isomorphisms $\tilde\mu_{O|F_n(\mathbf{k})}$ are related by the identities

$$\tilde\mu_{O'|F_n(\mathbf{k})} = \mathrm{Ad}(\Phi_{O,O'}) \circ \tilde\mu_{O|F_n(\mathbf{k})}, \tag{16}$$

where the automorphisms $\mathrm{Ad}(\Phi_{O,O'})$ of $\exp(\hat{\mathfrak{f}}^{\mathbf{k}}_n)$ are no longer necessarily inner.

4.2. *Relation between* μ_O *and* $\mu_{O^{(i)}}$. — Let $O \in \mathbf{Par}_n$. We index letters in O by $0,\ldots,n-1$, fix an index $i \neq 0$ and construct $O^{(i)}$ by doubling inside O the letter $\bullet$ with index i.

O gives rise to a morphism $\tilde\mu_O : B_n(\mathbf{k}) \to \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_n) \rtimes \mathfrak{S}_n$, which induces $\mu_O \in \underline{\mathrm{Taut}}_{n-1}(\mathbf{k}) \subset \mathrm{Taut}_{n-1}(\mathbf{k})$. Similarly, $\tilde\mu_{O^{(i)}} : B_{n+1}(\mathbf{k}) \to \exp(\hat{\mathfrak{t}}^{\mathbf{k}}_{n+1}) \rtimes \mathfrak{S}_{n+1}$ induces $\mu_{O^{(i)}} \in \underline{\mathrm{Taut}}_n(\mathbf{k}) \subset \mathrm{Taut}_n(\mathbf{k})$.

We now prove:

Theorem **30.**

$$\mu_{\mathrm{O}^{(i)}} = \mu_{\mathrm{O}}^{1,2,\dots,ii+1,\dots,n} \circ \mu_{\bullet(\bullet\bullet)}^{i,i+1}. \tag{17}$$

Proof. — We first show that there are uniquely determined elements $g_1, \dots, g_{n-1} \in \exp(\hat{\mathfrak{f}}_{n-1}^{\mathbf{k}})$ and $g, h \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$ such that:

(a) $\mu_{\mathrm{O}} = \theta(g_1, \dots, g_{n-1})$, $\log g_i = -\frac{1}{2}(x_1 + \cdots + x_{i-1}) + \mathrm{O}(x^2)$, and[3]
(b) $\mu_{\bullet(\bullet\bullet)} = \theta(g, h)$, $\log g = \mathrm{O}(x^2)$, $\log h = -\frac{1}{2}x_1 + \mathrm{O}(x^2)$.

Let us prove the first statement (it actually contains the second statement as a particular case). The elements $g_i = g_i(x_1, \dots, x_{n-1})$ are uniquely determined by the equality $\mu_{\mathrm{O}} = \theta(g_1, \dots, g_{n-1})$, together with the condition that the coefficient of x_i in the expansion of $\log g_i$ vanishes. We should then prove that $\log g_i = -\frac{1}{2}(x_1 + \cdots + x_{i-1}) + \mathrm{O}(x^2)$. We have

$$\tilde{\mu}_{\mathrm{O}}(\sigma_j) = e^{a_j} \cdot e^{t_{j-1,j}/2} s_j \cdot e^{-a_j},$$

where $a_j \in \hat{\mathfrak{t}}_n^{\mathbf{k}}$ has valuation ≥ 2 (we write this as $a_j \in \mathrm{O}(t^2)$), and

$$\tilde{\mu}_{\mathrm{O}}(\mathrm{X}_i) = \tilde{\mu}_{\mathrm{O}}(\sigma_1)^{-1} \cdots \tilde{\mu}_{\mathrm{O}}(\sigma_{i-1})^{-1} \tilde{\mu}_{\mathrm{O}}(\sigma_i)^2 \tilde{\mu}_{\mathrm{O}}(\sigma_{i-1}) \cdots \tilde{\mu}_{\mathrm{O}}(\sigma_1).$$

Now

$$\tilde{\mu}_{\mathrm{O}}(\sigma_{i-1}) \cdots \tilde{\mu}_{\mathrm{O}}(\sigma_1) = s_{i-1} \cdots s_1 e^{\frac{1}{2}(x_1 + \cdots + x_{i-1}) + \mathrm{O}(t^2)}$$

and $\tilde{\mu}_{\mathrm{O}}(\sigma_i^2) = e^{a_i} e^{t_{i-1,i}} e^{-a_i}$. It follows that

$$\tilde{\mu}_{\mathrm{O}}(\mathrm{X}_i) = \mathrm{Ad}_{e^{-\frac{1}{2}(x_1 + \cdots + x_{i-1}) + \mathrm{O}(t^2)} e^{\tilde{a}_i}}(e^{x_i}),$$

where $\tilde{a}_i = s_1 \cdots s_{i-1} \cdot a_i \cdot s_{i-1} \cdots s_1 \in \mathrm{O}(t^2)$, so $\tilde{\mu}_{\mathrm{O}}(\mathrm{X}_i) = \mathrm{Ad}_{e^{-\frac{1}{2}(x_1 + \cdots + x_{i-1}) + \mathrm{O}(t^2)}}(e^{x_i})$, which implies that g_i has the announced form.

To prove (17), we need to prove the equality

$$\begin{aligned} \mu_{\mathrm{O}^{(i)}} = \theta\big(&g_1(x_1, \dots, x_i + x_{i+1}, \dots, x_n), \dots, g_i(x_1, \dots, x_i + x_{i+1}, \dots, x_n) g(x_i, x_{i+1}), \\ &g_i(x_1, \dots, x_i + x_{i+1}, \dots, x_n) h(x_i, x_{i+1}), \dots, g_{n-1}(x_1, \dots, x_i + x_{i+1}, \dots, x_n)\big). \end{aligned} \tag{18}$$

(12) implies that the diagram

$$\begin{array}{ccc} \mathrm{F}_{n-1} & \to & \mathrm{F}_n \\ {\scriptstyle \tilde{\mu}_{\mathrm{O}|\mathrm{F}_{n-1}}} \downarrow & & \downarrow {\scriptstyle \tilde{\mu}_{\mathrm{O}^{(i)}|\mathrm{F}_n}} \\ \exp(\hat{\mathfrak{f}}_{n-1}^{\mathbf{k}}) & \to & \exp(\hat{\mathfrak{f}}_n^{\mathbf{k}}) \end{array}$$

[3] $\mathrm{O}(x^2)$ means an element of $\hat{\mathfrak{f}}_{n-1}^{\mathbf{k}}$ of valuation ≥ 2.

commutes, where the upper morphism takes X_j ($j \in [n-1]$) to: X_j if $j < i$, X_iX_{i+1} if $j = i$, X_{j+1} if $j > i+1$; and where the lower morphism is similarly defined (replacing products by sums and X_k's by x_k's). Specializing to the generators X_j ($j \neq i$) of F_{n-1}, this gives

$$\tilde{\mu}_{O^{(i)}}(X_j) = \mathrm{Ad}_{g_j^{0,1,\dots,ii+1,\dots,n}}(e^{x_j})$$

for $j < i$ and

$$\tilde{\mu}_{O^{(i)}}(X_j) = \mathrm{Ad}_{g_{j-1}^{0,1,\dots,ii+1,\dots,n}}(e^{x_j})$$

for $j > i+1$, which implies that (18) holds when applied to the $e^{x_j}, j \neq i, i+1$.

We now prove that (18) also holds when applied to e^{x_i} and $e^{x_{i+1}}$.

The morphism $X_i \in B_n = \mathbf{PaB}(O, O)$ can be decomposed as

$$O \overset{(\sigma_{i-2}\cdots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2 \overset{\sigma_{i-1}^2}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2 \overset{\sigma_{i-2}\cdots\sigma_0}{\to} O.$$

Here the braid group elements indicate the morphisms. Let $\gamma \in \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}}) \rtimes \mathfrak{S}_n$ be the image of the morphism $O \overset{(\sigma_{i-2}\cdots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2$ under $\mathbf{PaB} \to \mathbf{PaCD}_\Phi$; its image in $\mathfrak{S}_n$ is the permutation $s_0 \cdots s_{i-2}$, i.e., $(0, \dots, n-1) \mapsto (i-1, 0, 1, \dots, i-2, i, i+1, \dots, n-1)$. The image of $(O_1 \otimes (\bullet\bullet)) \otimes O_2 \overset{\sigma_{i-1}^2}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2$ is $e^{t_{i-1,i}}$, therefore the image of X_i is

$$\mu_O(X_i) = \gamma e^{t_{i-1,i}} \gamma^{-1}.$$

We have $\gamma = \gamma_0 s_0 \cdots s_{i-2}$, where $\gamma_0 \in \exp(\hat{\mathfrak{t}}_n^{\mathbf{k}})$. As $s_0 \cdots s_{i-2} \cdot t_{i-1,i} = x_i$, we have

$$\mu_O(X_i) = \gamma_0 e^{x_i} \gamma_0^{-1}.$$

As this image is also $\mathrm{Ad}_{g_i(x_1,\dots,x_{n-1})}(e^{x_i})$, we derive from this that $g_i^{-1}\gamma_0$ commutes with x_i, hence by Proposition 51 has the form $e^{\lambda x_i}\alpha^{0i,1,2,\dots,i-1,i+1,\dots,n-1}$, where $\alpha \in \exp(\hat{\mathfrak{t}}_{n-1}^{\mathbf{k}})$.

Since $\mu_O(\sigma_j) = s_j e^{t_{j,j+1}/2}$, we get $\log \gamma_0 = -\frac{1}{2}(x_1 + \cdots + x_{i-1}) + O(x^2)$. Comparing linear terms in x_i, we get $\lambda = 0$.

Let us now compute $\mu_{O^{(i)}}(X_i)$. The morphism $X_i \in B_{n+1} = \mathbf{PaB}(O^{(i)}, O^{(i)})$ can be decomposed as

$$O^{(i)} \overset{(\sigma_{i-2}\cdots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2 \overset{\sigma_{i-1}^2}{\to} (O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2 \overset{\sigma_{i-2}\cdots\sigma_0}{\to} O^{(i)}$$

(here σ_{i-1}^2 involves the two first $\bullet$ of $\bullet(\bullet\bullet)$). The morphism $O^{(i)} \overset{(\sigma_{i-2}\cdots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2$ is obtained from $O^{(i)} \overset{(\sigma_{i-2}\cdots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2$ by the operation of doubling of the ith strand, so its image is $\gamma^{0,1,2,\dots,ii+1,\dots,n} = \gamma_0^{0,1,2,\dots,ii+1,\dots,n}(s_0 \cdots s_{i-2})$. The image of $\bullet(\bullet\bullet) \xrightarrow{\sigma_1^2} \bullet(\bullet\bullet)$ is $\mathrm{Ad}_{g(x_1,x_2)}(e^{x_1})$, so the image of

$$(\mathrm{O}_1 \otimes (\bullet(\bullet\bullet))) \otimes \mathrm{O}_2 \xrightarrow{\sigma_{i-1}^2} (\mathrm{O}_1 \otimes (\bullet(\bullet\bullet))) \otimes \mathrm{O}_2$$

is $\mathrm{Ad}_{g(t_{i-1,i},t_{i-1,i+1})}(e^{t_{i-1,i}})$. It follows that

$$\tilde{\mu}_{\mathrm{O}^{(i)}}(\mathrm{X}_i) = \mathrm{Ad}_{\gamma^{0,1,2,\dots,ii+1,\dots,n} g(t_{i-1,i},t_{i-1,i+1})}(e^{t_{i-1,i}}) = \mathrm{Ad}_{\gamma_0^{0,1,2,\dots,ii+1,\dots,n} g(x_i,x_{i+1})}(e^{x_i}).$$

Now we claim that

$$\mathrm{Ad}_{\gamma_0^{0,1,2,\dots,ii+1,\dots,n} g(x_i,x_{i+1})}(e^{x_i}) = \mathrm{Ad}_{g_i^{0,1,2,\dots,ii+1,\dots,n} g(x_i,x_{i+1})}(e^{x_i}).$$

Indeed,

$$\begin{aligned}
&\mathrm{Ad}_{(g_i^{-1}\gamma_0)^{0,1,2,\dots,ii+1,\dots,n} g(x_i,x_{i+1})}(e^{x_i}) \\
&\quad = \mathrm{Ad}_{(\alpha^{0i,1,2,\dots,i-1,i+1,\dots,n-1})^{0,1,2,\dots,ii+1,\dots,n} g(x_i,x_{i+1})}(e^{x_i}) \\
&\quad = \mathrm{Ad}_{\alpha^{0ii+1,2,3,\dots,i-1,i+2,\dots,n} g(x_i,x_{i+1})}(e^{x_i}).
\end{aligned}$$

Now x_i and x_{i+1} commute with any $\alpha^{0ii+1,\cdots}$, so this is $\mathrm{Ad}_{g(x_i,x_{i+1})}(e^{x_i})$.

So we get

$$\tilde{\mu}_{\mathrm{O}^{(i)}}(\mathrm{X}_i) = \mathrm{Ad}_{g_i^{0,1,2,\dots,ii+1,\dots,n} g(x_i,x_{i+1})}(e^{x_i}).$$

The same argument shows that

$$\tilde{\mu}_{\mathrm{O}^{(i)}}(\mathrm{X}_{i+1}) = \mathrm{Ad}_{g_i^{0,1,2,\dots,ii+1,\dots,n} h(x_i,x_{i+1})}(e^{x_{i+1}}),$$

as wanted. □

5. Proof of Theorem 4 and Propositions 6 and 7

5.1. *Proof of Theorem 4.* — We first recall the formulation of Theorem 4:

Theorem **31.** — *Let* $\Phi \in \mathrm{M}_1(\mathbf{k})$. *Then* $\mu_\Phi := (\Phi(x_1, -x_1 - x_2), e^{-(x_1+x_2)/2}\Phi(x_2, -x_1 - x_2)e^{x_2/2}) \in \underline{\mathrm{Taut}}_2(\mathbf{k})$ *satisfies* $\Phi(t_{12}, t_{23}) \circ \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3}$.

Proof. — We first prove that $\mu_{\bullet(\bullet\bullet)} = \mu_\Phi$. $\mathrm{X}_1 \in \mathrm{B}_3 = \mathbf{PaB}(\bullet(\bullet\bullet))$ corresponds to $a_{\bullet,\bullet,\bullet} \circ (\beta^2_{\bullet,\bullet} \otimes \mathrm{id}_\bullet) \circ a^{-1}_{\bullet,\bullet,\bullet}$. Then $\mu_{\bullet(\bullet\bullet)}(e^{x_1}) = \tilde{\mu}_{\bullet(\bullet\bullet)}(\mathrm{X}_1) = \Phi(t_{01}, t_{12})e^{t_{01}}\Phi(t_{01}, t_{12})^{-1}$. Since $t_{01} + t_{12} + t_{02}$ is central in $\mathfrak{t}_3$ and since Φ is group-like, this is $\Phi(t_{01}, -t_{01} - t_{02})e^{t_{01}}\Phi(t_{01}, -t_{01} - t_{02})^{-1} = \Phi(x_1, -x_1 - x_2)e^{x_1}\Phi(x_1, -x_1 - x_2)^{-1} = \mu_\Phi(e^{x_1})$.

Similarly, X_2 corresponds to $(\mathrm{id}_\bullet \otimes \beta_{\bullet,\bullet}) \circ a_{\bullet,\bullet,\bullet} \circ (\beta^2_{\bullet,\bullet} \otimes \mathrm{id}_\bullet) \circ a^{-1}_{\bullet,\bullet,\bullet} \circ (\mathrm{id}_\bullet \otimes \beta^{-1}_{\bullet,\bullet})$. Then

$$\begin{aligned}
\mu_{\bullet(\bullet\bullet)}(e^{x_2}) &= \tilde{\mu}_{\bullet(\bullet\bullet)}(\mathrm{X}_2) \\
&= e^{t_{12}/2}(12)\Phi(t_{01}, t_{12})e^{t_{01}}\Phi(t_{01}, t_{12})^{-1}(12)e^{-t_{12}/2} \\
&= e^{t_{12}/2}\Phi(t_{02}, t_{12})e^{t_{02}}\Phi(t_{02}, t_{12})^{-1}e^{-t_{12}/2} \\
&= e^{-(t_{01}+t_{02})/2}\Phi(t_{02}, -t_{01} - t_{02})e^{t_{02}}\Phi(t_{02}, -t_{01} - t_{02})^{-1}e^{(t_{01}+t_{02})/2} \\
&= e^{-(x_1+x_2)/2}\Phi(x_2, -x_1 - x_2)e^{x_2}\Phi(x_2, -x_1 - x_2)^{-1}e^{(x_1+x_2)/2} \\
&= \mu_\Phi(e^{x_2}).
\end{aligned}$$

So $\mu_{\bullet(\bullet\bullet)} = \mu_\Phi$.

Set now $\mathrm{O} := \bullet((\bullet\bullet)\bullet)$, $\mathrm{O}' := \bullet(\bullet(\bullet\bullet))$. Then $\mathrm{can}_{\mathrm{O},\mathrm{O}'} = \mathrm{id}_\bullet \otimes a_{\bullet,\bullet,\bullet} \in \mathbf{PaB}(\mathrm{O}, \mathrm{O}')$, whose image in $\mathbf{PaCD}_\Phi(\mathrm{O}, \mathrm{O}') = \exp(\hat{\mathfrak{t}}_4) \rtimes \mathfrak{S}_4$ is $\Phi(t_{12}, t_{23}) = \Phi_{\mathrm{O},\mathrm{O}'}$. It follows that $\Phi_{\bullet((\bullet\bullet)\bullet),\bullet(\bullet(\bullet\bullet))} = \Phi(t_{12}, t_{23})$.

Theorem 30 implies that $\mu_{\bullet((\bullet\bullet)\bullet)} = \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2}$ and $\mu_{\bullet(\bullet(\bullet\bullet))} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3}$ and (16) implies that $\mu_{\bullet(\bullet(\bullet\bullet))} = \mathrm{Ad}\,\Phi(t_{12}, t_{23}) \circ \mu_{\bullet((\bullet\bullet)\bullet)}$. All this implies Theorem 31. □

5.2. *Proof of Proposition 6.* — We recall the formulation of Proposition 6. The scheme SolKV is defined by

$$\begin{aligned}
\mathrm{SolKV}(\mathbf{k}) := \big\{\mu \in \mathrm{Taut}_2(\mathbf{k}) | \theta(\mu)(e^{x_1}e^{x_2}) = e^{x_1+x_2} \\
\text{and } \exists r \in u^2\mathbf{k}[[u]], \mathrm{J}(\mu) = \langle r(x_1 + x_2) - r(x_1) - r(x_2)\rangle\big\},
\end{aligned}$$

for any $\mathbf{Q}$-ring $\mathbf{k}$, and Proposition 6 says:

Proposition **32.** — *The map* $\Phi \mapsto \mu_\Phi$ *is a morphism of* $\mathbf{Q}$*-schemes* $\mathrm{M}_1 \to \mathrm{SolKV}$.

Proof. — Let $\Phi \in \mathrm{M}_1(\mathbf{k})$. We first should prove that $\theta(\mu_\Phi)(e^{x_1}e^{x_2}) = e^{x_1+x_2}$. We will give three proofs of this fact:

First proof. We have

$$\begin{aligned}
\theta(\mu_\Phi)(e^{x_1}x^{x_2}) &= \theta(\mu_\Phi)(e^{x_1})\theta(\mu_\Phi)(e^{x_2}) \\
&= \Phi(x_1, -x_1 - x_2)e^{x_1}\Phi(-x_1 - x_2, x_1)e^{-(x_1+x_2)/2} \\
&\quad \times \Phi(x_2, -x_1 - x_2)e^{x_2}\Phi(-x_1 - x_2, x_1)e^{(x_1+x_2)/2} \\
&= \Phi(x_1, -x_1 - x_2)e^{x_1/2}\Phi(x_2, x_1)e^{x_2/2}\Phi(-x_1 - x_2, x_1)e^{(x_1+x_2)/2} \\
&= e^{x_1+x_2},
\end{aligned}$$

where the second equality follows from the duality identity and the third and fourth equalities both follow from the hexagon identity.

Second proof. Let us set $\nu := \mu_\Phi^{-1}$. Since μ_Φ satisfies (3), we have

$$\nu^{2,3} \circ \nu^{1,23} = \nu^{1,2} \circ \nu^{12,3} \circ \mathrm{Ad}(\Phi(t_{12}, t_{23})). \tag{19}$$

Let us set $\mathrm{C}(x_1, x_2) := \theta(\nu)(x_1 + x_2)$, and apply (19) to $x_1 + x_2 + x_3$ to obtain $\mathrm{C}(x_1, \mathrm{C}(x_2, x_3)) = \mathrm{C}(\mathrm{C}(x_1, x_2), x_3)$. According to [AT2], this implies $\mathrm{C}(x_1, x_2) = s^{-1}\log(e^{sx_1}e^{sx_2})$ for some $s \in \mathbf{k}^\times$. Checking degree 1 and 2 terms in ν, we get $s = 1$.

Third proof. Set $\mathrm{O} := \bullet(\bullet\bullet)$, then $\mu_{\mathrm{O}} = \mu_\Phi$. Proposition 29 implies that $\tilde\mu_{\mathrm{O}}(\mathrm{X}_1\mathrm{X}_2) = e^{x_1+x_2}$. Then $\theta(\mu_{\mathrm{O}})(e^{x_1}e^{x_2}) = \tilde\mu_{\mathrm{O}|\mathrm{F}_2(\mathbf{k})} \circ \mathrm{can}(e^{x_1}e^{x_2}) = \tilde\mu_{\mathrm{O}}(\mathrm{X}_1\mathrm{X}_2) = e^{x_1+x_2}$.

We now prove that $\mathrm{J}(\mu_\Phi)$ has the desired form. It follows from Proposition 22 that

$$\mathrm{J}(\mathrm{Ad}\,\Phi(t_{12}, t_{23})) = 0.$$

Proposition 24 implies that $\mathrm{J}(\mu_\Phi^{12,3}) = \mathrm{J}(\mu_\Phi)^{12,3}$, etc., and we get by applying J to (3),

$$\begin{aligned}&\Phi(t_{12}, t_{23}) \cdot \mathrm{J}(\mu_\Phi)^{12,3} + \Phi(t_{12}, t_{23}) \circ \mu_\Phi^{12,3} \cdot \mathrm{J}(\mu_\Phi)^{1,2}\\ &\quad = \mathrm{J}(\mu_\Phi)^{1,23} + \mu_\Phi^{1,23} \cdot \mathrm{J}(\mu_\Phi)^{2,3}.\end{aligned}$$

Applying the inverse of (3), we get

$$\begin{aligned}&(\mu_\Phi^{1,2})^{-1} \circ (\mu_\Phi^{12,3})^{-1} \cdot \mathrm{J}(\mu_\Phi)^{12,3} + (\mu_\Phi^{1,2})^{-1} \cdot \mathrm{J}(\mu_\Phi)^{1,2}\\ &\quad = (\mu_\Phi^{2,3})^{-1} \circ (\mu_\Phi^{1,23})^{-1} \cdot \mathrm{J}(\mu_\Phi)^{1,23} + (\mu_\Phi^{2,3})^{-1} \cdot \mathrm{J}(\mu_\Phi)^{2,3},\end{aligned}$$

and since $a^{12,3} \cdot t^{12,3} = (a \cdot t)^{12,3}$, etc.,

$$\begin{aligned}&(\mu_\Phi^{1,2})^{-1} \cdot (\mu_\Phi^{-1} \cdot \mathrm{J}(\mu_\Phi))^{12,3} + (\mu_\Phi^{-1} \cdot \mathrm{J}(\mu_\Phi))^{1,2}\\ &\quad = (\mu_\Phi^{2,3})^{-1} \cdot (\mu_\Phi^{-1} \cdot \mathrm{J}(\mu_\Phi))^{1,23} + (\mu_\Phi^{-1} \cdot \mathrm{J}(\mu_\Phi))^{2,3}.\end{aligned}$$

Now $\theta(\mu_\Phi)^{-1}(x_1 + x_2) = \log(e^{x_1}e^{x_2})$ implies that $(\mu_\Phi^{1,2})^{-1} \cdot t^{12,3} = t^{\widetilde{12,3}}$ and $(\mu_\Phi^{2,3})^{-1} \cdot t^{1,23} = t^{\widetilde{1,23}}$, so $\tilde\delta(\mu_\Phi^{-1} \cdot \mathrm{J}(\mu_\Phi)) = 0$. According to Proposition 27, there exists $r \in \hat{\mathfrak{T}}_1$ with valuation ≥ 2 such that $\mu_\Phi^{-1} \cdot \mathrm{J}(\mu_\Phi) = \tilde\delta(r)$. Now $\mu_\Phi \cdot r^{\widetilde{12}} = r^{12}$, and $\mu_\Phi \cdot r^1 = r^1$, $\mu_\Phi \cdot r^2 = r^2$ as $\mu_\Phi(x_i)$ is conjugated to x_i for $i = 1, 2$ in $\exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$. Therefore $\mu_\Phi \cdot \tilde\delta(r) = \delta(r)$. So $\mathrm{J}(\mu_\Phi) = \delta(r) = \langle r(x_1 + x_2) - r(x_1) - r(x_2)\rangle$, where $r \in u^2\mathbf{k}[[u]]$. $\square$

5.3. *Proof of Proposition 7.* — For $\Phi \in \mathrm{M}_1(\mathbf{k})$, recall [DT, E] that there exists a unique formal series $\Gamma_\Phi(u) \in 1 + u^2\mathbf{k}[[u]]$, such that

$$(1 + y\partial_y\Phi(x, y))^{\mathrm{ab}} = \frac{\Gamma_\Phi(\bar x + \bar y)}{\Gamma_\Phi(\bar x)\Gamma_\Phi(\bar y)}.$$

Proposition 7 then says:

Proposition **33.** — $\mathrm{J}(\mu_\Phi) = \langle \log\Gamma_\Phi(x + y) - \log\Gamma_\Phi(x) - \log\Gamma_\Phi(y)\rangle$.

Proof. — For $\mathrm{A}(x,y) \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$ such that $\log \mathrm{A}(x,y)$ has vanishing linear term in y, let $\mathrm{U} := (1, \mathrm{A}(x,y)) \in \underline{\mathrm{Taut}}_2(\mathbf{k})$. Let

$$\log \mathrm{A}(x,y) = \sum_{k\geq 1} \alpha_k (\mathrm{ad}\, x)^k(y) + \mathrm{O}(y^2)$$

be the expansion of $\log \mathrm{A}(x,y)$; here and later $\mathrm{O}(y^2)$ means a series of elements with y-degree ≥ 2. Then

$$\log \mathrm{U} = \left(0, \sum_{k\geq 1} \alpha_k (\mathrm{ad}\, x)^k(y) + \mathrm{O}(y^2)\right) \in \widehat{\mathfrak{tder}}_2^{\mathbf{k}},$$

and $\mathrm{J}(\mathrm{U}) = j(u) + \mathrm{O}(y^2)$. Now $j(\log \mathrm{U}) = \langle \sum_{k\geq 1} \alpha_k y(-x)^k + \mathrm{O}(y^2)\rangle$. So

$$\mathrm{J}(\mathrm{U}) = \left\langle \sum_{k\geq 1} \alpha_k (-x)^k y + \mathrm{O}(y^2)\right\rangle.$$

On the other hand, the hexagon identity implies that $\mu_\Phi = \mathrm{Inn}(\Phi(x, -x-y)e^{-x/2}) \circ \dot{\mu}_\Phi$, where $\dot{\mu}_\Phi = (1, \Phi(x,y)^{-1})$ and $\mathrm{Inn}(a) = (ae^{-a_x x}, ae^{-a_y y})$ for $a \in \exp(\hat{\mathfrak{f}}_2^{\mathbf{k}})$ with $\log a = a_x x + a_y y +$ (terms of degree ≥ 2), and we then have $\mathrm{J}(\dot{\mu}_\Phi) = \mathrm{J}(\mu_\Phi)$.

If we set $\log \Gamma_\Phi(u) = \sum_{n\geq 2} (-1)^n \zeta_\Phi(n) u^n/n$, then we have

$$\log \Phi(x,y) = -\sum_{k\geq 1} \zeta_\Phi(k+1)(\mathrm{ad}\, x)^k(y) + \mathrm{O}(y^2),$$

therefore

$$\mathrm{J}(\mu_\Phi) = \mathrm{J}(\dot{\mu}_\Phi) = \left\langle \sum_{k\geq 1} (-1)^k \zeta_\Phi(k+1) x^k y\right\rangle + \mathrm{O}(y^2).$$

As we have $\mathrm{J}(\mu_\Phi) = \langle f(x) + f(y) - f(x+y)\rangle$ for some series $f(x)$, we get

$$\begin{aligned}\mathrm{J}(\mu_\Phi) &= \left\langle (-1)^k \frac{\zeta_\Phi(k+1)}{k+1}((x+y)^{k+1} - x^{k+1} - y^{k+1})\right\rangle \\ &= \langle \log \Gamma_\Phi(x) + \log \Gamma_\Phi(y) - \log \Gamma_\Phi(x+y)\rangle.\end{aligned} \tag{20}$$

This proves Proposition 7. □

6. Group and torsor aspects

This section is devoted to the proof of Proposition 8, Theorem 9 and Proposition 10, which describe the torsor structure of $\mathrm{SolKV}(\mathbf{k})$ and show that the map $\Phi \mapsto \mu_\Phi$ is a morphism of torsors.

6.1. *Group structures of* $\mathrm{KV}(\mathbf{k})$ *and* $\mathrm{KRV}(\mathbf{k})$. — Recall that

$$\mathrm{KV}(\mathbf{k}) := \big\{\alpha \in \mathrm{Taut}_2(\mathbf{k}) | \theta(\alpha)(e^x e^y) = e^x e^y \\ \text{and} \quad \exists \sigma \in u^2\mathbf{k}[[u]], \mathrm{J}(\alpha) = \langle \sigma(\log(e^x e^y)) - \sigma(x) - \sigma(y)\rangle \big\},$$

and

$$\mathrm{KRV}(\mathbf{k}) := \big\{a \in \mathrm{Taut}_2(\mathbf{k}) | \theta(a)(e^{x+y}) = e^{x+y} \\ \text{and} \quad \exists s \in u^2\mathbf{k}[[u]], \mathrm{J}(a) = \langle s(x+y) - s(x) - s(y)\rangle \big\}.$$

By Proposition 26, σ and s as above are unique, and we set $s := \mathrm{Duf}(\alpha)$, $s := \mathrm{Duf}(a)$. The first part of Proposition 8 states:

Proposition **34.** — $\mathrm{KV}(\mathbf{k})$ *and* $\mathrm{KRV}(\mathbf{k})$ *are subgroups of* $\mathrm{Taut}_2(\mathbf{k})$*, and* $\mathrm{Duf}: \mathrm{KV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$*,* $\mathrm{Duf}: \mathrm{KRV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ *are group morphisms.*

Proof. — The statements on $\mathrm{KRV}(\mathbf{k})$ are proved in [AT2].

Let us prove that $\mathrm{KV}(\mathbf{k})$ is a group. For $\alpha \in \mathrm{KV}(\mathbf{k})$, let $\sigma_\alpha := \mathrm{Duf}(\alpha)$, so $\sigma_\alpha \in u^2\mathbf{k}[[u]]$, and $\mathrm{J}(\alpha) = \tilde{\delta}(\sigma_\alpha)$. If $\alpha, \alpha' \in \mathrm{KV}(\mathbf{k})$, we have $\theta(\alpha' \circ \alpha)(e^x e^y) = e^x e^y$. Moreover, $\alpha'(e^x), \alpha'(e^y)$ are conjugate to e^x, e^y, and $\alpha'(e^x e^y) = e^x e^y$, which implies

$$\forall t \in \hat{\mathfrak{T}}_1, \quad \alpha' \cdot \tilde{\delta}(t) = \tilde{\delta}(t). \tag{21}$$

Then $\mathrm{J}(\alpha' \circ \alpha) = \mathrm{J}(\alpha') + \alpha' \cdot \mathrm{J}(\alpha) = \tilde{\delta}(\sigma_{\alpha'}) + \alpha' \cdot \tilde{\delta}(\sigma_\alpha) = \tilde{\delta}(\sigma_\alpha + \sigma_{\alpha'})$, where the last equality follows from (21). It follows that $\alpha' \circ \alpha \in \mathrm{KV}(\mathbf{k})$, and that $\sigma_{\alpha' \circ \alpha} = \sigma_\alpha + \sigma_{\alpha'}$. One proves similarly that $\alpha^{-1} \in \mathrm{KV}(\mathbf{k})$. □

6.2. *The torsor structure of* $\mathrm{SolKV}(\mathbf{k})$. — The second part of Proposition 8 states:

Proposition **35.** — $\mathrm{SolKV}(\mathbf{k})$ *is a torsor under the commuting left action of* $\mathrm{KV}(\mathbf{k})$ *and right action of* $\mathrm{KRV}(\mathbf{k})$*, and* $\mathrm{Duf}: \mathrm{SolKV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ *is a morphism of torsors.*

Proof. — It is proved in [AT2] that $\mathrm{KRV}(\mathbf{k})$ acts freely and transitively on $\mathrm{SolKV}(\mathbf{k})$.

Let us prove that $\mathrm{KV}(\mathbf{k})$ acts on $\mathrm{SolKV}(\mathbf{k})$. For $\mu \in \mathrm{SolKV}(\mathbf{k})$, $\alpha \in \mathrm{KV}(\mathbf{k})$, we have $\theta(\mu \circ \alpha)(e^x e^y) = \theta(\mu)(e^x e^y) = e^{x+y}$.

Since $\theta(\mu)(e^x), \theta(\mu)(e^y)$ are conjugate to e^x, e^y, and since $\theta(\mu)(e^x e^y) = e^{x+y}$, we have

$$\forall t \in \hat{\mathfrak{T}}_2, \quad \delta(t) = \mu \cdot \tilde{\delta}(t).$$

Let now $r_\mu := \mathrm{Duf}(\mu)$, so $\mathrm{J}(\mu) = \delta(r_\mu)$. Then $\mathrm{J}(\mu \circ \alpha) = \mathrm{J}(\mu) + \mu \cdot \mathrm{J}(\alpha) = \delta(r_\mu) + \mu \cdot \tilde{\delta}(\sigma_\alpha) = \delta(r_\mu + \sigma_\alpha)$, where the last equality uses the above identity. So

$\mu \circ \alpha \in \mathrm{SolKV}(\mathbf{k})$, and $r_{\mu\circ\alpha} = r_\mu + \sigma_\alpha$. It follows that $\mathrm{KV}(\mathbf{k})$ acts on $\mathrm{SolKV}(\mathbf{k})$, and that the map $\mathrm{Duf}: \mathrm{SolKV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ is compatible with $\mathrm{Duf}: \mathrm{KV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$.

Let us now prove that the action of $\mathrm{KV}(\mathbf{k})$ on $\mathrm{SolKV}(\mathbf{k})$ is free and transitive. For $\mu, \mu' \in \mathrm{SolKV}(\mathbf{k})$, set $\alpha := \mu^{-1} \circ \mu'$; then $\theta(\alpha)(e^x e^y) = \theta(\mu)^{-1}(e^{x+y}) = e^x e^y$, and $\mathrm{J}(\alpha) = \mathrm{J}(\mu^{-1}) + \mu^{-1} \cdot \mathrm{J}(\mu') = \mu^{-1} \cdot (\mathrm{J}(\mu') - \mathrm{J}(\mu))$ as $\mathrm{J}(\mu^{-1}) = -\mu^{-1} \cdot \mathrm{J}(\mu)$. Then $\mathrm{J}(\alpha) = \mu^{-1} \cdot (\delta(r_{\mu'} - r_\mu)) = \tilde{\delta}(r_{\mu'} - r_\mu)$, where the last equality uses $\mu^{-1} \cdot \delta(t) = \tilde{\delta}(t)$ for $t \in \hat{\mathfrak{T}}_1$. So $\alpha \in \mathrm{KV}(\mathbf{k})$.

6.3. *Compatibilities of morphisms with group structures and actions (proof of Theorem 9).* — We now show that: (a) $f \mapsto \alpha_f^{-1}$ is a group morphism $\mathrm{GT}_1(\mathbf{k}) \to \mathrm{KV}(\mathbf{k})$, (b) $g \mapsto a_g^{-1}$ is a group morphism $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{KRV}(\mathbf{k})$, (c) the map $\Phi \mapsto \mu_\Phi$ is compatible with the actions of these groups.

For this, we will show that

$$\mu_{f*\Phi} = \mu_\Phi \circ \alpha_f, \qquad \mu_{\Phi*g} = a_g \circ \mu_\Phi. \tag{22}$$

Since these are identities in $\underline{\mathrm{Taut}}_2(\mathbf{k}) \subset \mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbf{k}})$, it suffices to check them on the generators x, y of $\hat{\mathfrak{f}}_2^{\mathbf{k}}$. We give the proofs in the case of x, the proofs in the case of y being similar.

The proofs go as follows:

$$\begin{aligned}
\theta(\mu_{f*\Phi})(x) &= \mathrm{Ad}_{(f*\Phi)(x,-x-y)}(x) \\
&= \mathrm{Ad}_{f(\Phi(x,-x-y)e^x\Phi(x,-x-y)^{-1}, e^{-x-y})\Phi(x,-x-y)}(x) \\
&= \mathrm{Ad}_{f(\mu_\Phi(e^x), \mu_\Phi(e^{-y}e^{-x}))}(\mu_\Phi(x)) \\
&= \mathrm{Ad}_{\mu_\Phi(f(e^x, e^{-y}e^{-x}))}(x) = \theta(\mu_\Phi \circ \alpha_f)(x)
\end{aligned}$$

and

$$\begin{aligned}
\theta(\mu_{\Phi*g})(x) &= \mathrm{Ad}_{(\Phi*g)(x,-x-y)}(x) \\
&= \mathrm{Ad}_{\Phi(g(x,-x-y)xg(x,-x-y)^{-1}, -x-y)g(x,-x-y)}(x) \\
&= \mathrm{Ad}_{\Phi(a_g(x), a_g(-x-y))}(a_g(x)) \\
&= a_g(\Phi(x, -x-y)x\Phi(x, -x-y)^{-1}) = \theta(a_g \circ \mu_\Phi)(x).
\end{aligned}$$

The first part of (22) implies the following: (a) if $f \in \mathrm{GT}_1(\mathbf{k})$, then $\alpha_f \in \mathrm{KV}(\mathbf{k})$; (b) $\alpha_{f_1 * f_2} = \alpha_{f_2} \circ \alpha_{f_1}$; (c) $\mathrm{M}_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$ is compatible with the group morphism $f \mapsto \alpha_f^{-1}$.

Indeed, using the nonemptiness of $\mathrm{M}_1(\mathbf{k})$ (see [Dr]) we get $\alpha_f = \mu_\Phi^{-1} \circ \mu_{f*\Phi}$, which implies $\alpha_f \in \mathrm{KV}(\mathbf{k})$ according to Section 6.2, i.e., (a). Again using the nonemptiness of $\mathrm{M}_1(\mathbf{k})$, we get $\alpha_{f_1*f_2} = \mu_\Phi^{-1} \circ \mu_{(f_1*f_2)*\Phi} = (\mu_\Phi^{-1} \circ \mu_{f_2*\Phi}) \circ (\mu_{f_2*\Phi}^{-1} \circ \mu_{f_1*(f_2*\Phi)}) = \alpha_{f_2} \circ \alpha_{f_1}$ (where we used $(f_1 * f_2) * \Phi = f_1 * (f_2 * \Phi)$), which proves (b). (c) is then tautological.

Similarly, the second part of (22) implies: (a) if $g \in \mathrm{GRT}_1(\mathbf{k})$, then $a_g \in \mathrm{KRV}(\mathbf{k})$; (b) $a_{g_1 * g_2} = a_{g_2} \circ a_{g_1}$; (c) $\mathrm{M}_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$ is compatible with the group morphism $g \mapsto a_g^{-1}$. All this proves Theorem 9.

It is easy to prove the identities $\alpha_{f_1 * f_2} = \alpha_{f_2} \circ \alpha_{f_1}$, $a_{g_1 * g_2} = a_{g_2} \circ a_{g_1}$ directly (i.e., not using the nonemptiness of $\mathrm{M}_1(\mathbf{k})$). Indeed, these are identities in $\underline{\mathrm{Taut}}_2(\mathbf{k}) \subset \mathrm{Aut}(\hat{\mathfrak{f}}_2^{\mathbf{k}})$, which can be checked on x, y. The verification in the case of x goes as follows:

$$\begin{aligned}\theta(\alpha_{f_1 * f_2})(x) &= \mathrm{Ad}_{(f_1 * f_2)(e^x, e^{-y}e^{-x})}(x)\\ &= \mathrm{Ad}_{f_1(f_2(e^x, e^{-y}e^{-x})e^x f_2(e^x, e^{-y}e^{-x})^{-1}, e^{-y}e^{-x}) f_2(e^x, e^{-y}e^{-x})}(x)\\ &= \mathrm{Ad}_{f_1(\alpha_{f_2}(e^x), \alpha_{f_2}(e^{-y}e^{-x}))}(\alpha_{f_2}(x))\\ &= \mathrm{Ad}_{\alpha_{f_2}(f_1(e^x, e^{-y}e^{-x}))}(x) = \theta(\alpha_{f_2} \circ \alpha_{f_1})(x),\end{aligned}$$

and

$$\begin{aligned}\theta(a_{g_1 * g_2})(x) &= \mathrm{Ad}_{(g_1 * g_2)(x, -x-y)}(x)\\ &= \mathrm{Ad}_{g_1(g_2(x, -x-y) x g_2(x, -x-y)^{-1}, -x-y) g_2(x, -x-y)}(x)\\ &= \mathrm{Ad}_{g_1(a_{g_2}(x), a_{g_2}(-x-y))}(a_{g_2}(x))\\ &= a_{g_2}(g_1(x, -x-y) x g_1(x, -x-y)^{-1}) = \theta(a_{g_2} \circ a_{g_1})(x).\end{aligned}$$

Remark **36.** — The Lie algebra morphism corresponding to $g \mapsto a_g^{-1}$ is the morphism $\nu : \mathfrak{grt}_1 \to \mathfrak{krv}$ from [AT2], given by $\psi(x, y) \mapsto (\psi(x, -x-y), \psi(y, -x-y))$.

6.4. *Torsor properties of the Duflo formal series (proof of Proposition 10).* — We have already proved that $\mathrm{M}_1(\mathbf{k}) \overset{\Phi \mapsto \mu_\Phi}{\to} \mathrm{SolKV}(\mathbf{k})$ and $\mathrm{SolKV}(\mathbf{k}) \overset{\mathrm{Duf}}{\to} u^2\mathbf{k}[[u]]$ are morphisms of torsors. On the other hand, it follows from [E] that $\mathrm{M}_1(\mathbf{k}) \overset{\Phi \mapsto \log \Gamma_\Phi}{\to} \{r \in u^2\mathbf{k}[[u]] | r_{ev}(u) = -\frac{u^2}{24} + \cdots\}$ is a morphism of torsors and from Proposition 7 that the diagram of Proposition 10 commutes. Proposition 10 follows.

For later use, let us make the group morphism $\mathrm{GT}_1(\mathbf{k}) \to u^3\mathbf{k}[[u^2]]$ underlying $\Phi \mapsto \log \Gamma_\Phi$ explicit.

Lemma **37.** — *For $f \in \mathrm{GT}_1(\mathbf{k})$, there is a unique $\Gamma_f \in \exp(u^3\mathbf{k}[[u^2]])$ such that*

$$[\log f(e^x, e^y)] = 1 - \frac{\Gamma_f(-\bar{x})\Gamma_f(-\bar{y})}{\Gamma_f(-\bar{x}-\bar{y})};$$

in the l.h.s., we use the isomorphism $\hat{\mathfrak{f}}_2'/\hat{\mathfrak{f}}_2'' \simeq \overline{xy}\mathbf{k}[[\bar{x}, \bar{y}]]$ given by (class of $(\mathrm{ad}\, x)^k(\mathrm{ad}\, y)^l([x, y])$) $\leftrightarrow$ $\bar{x}^{k+1}\bar{y}^{l+1}$. The map $\mathrm{GT}_1(\mathbf{k}) \to u^3\mathbf{k}[[u^2]]$, $f \mapsto \log \Gamma_f$ is a group morphism and $\Gamma_{f * \Phi} = \Gamma_f \Gamma_\Phi$ for any $f \in \mathrm{GT}_1(\mathbf{k})$, $\Phi \in \mathrm{M}_1(\mathbf{k})$.

Proof. — The map $\mathfrak{f}_2 \to \mathbf{k}[\bar{x}, \bar{y}]$, $\psi \mapsto (y\partial_y \psi(x,y))^{\mathrm{ab}}$ also induces an isomorphism $\hat{\mathfrak{f}}_2'/\hat{\mathfrak{f}}_2'' \simeq \overline{xy}\mathbf{k}[[\bar{x}, \bar{y}]]$, which takes the class $(\mathrm{ad}\, x)^k (\mathrm{ad}\, y)^l ([x, y])$ to $(-1)^{k+l+1}\bar{x}^{k+1}\bar{y}^{l+1}$. So for $\psi(x,y) \in \hat{\mathfrak{f}}_2'$, we have $(y\partial_y\psi(x,y))^{\mathrm{ab}}(\bar{x}, \bar{y}) = -[\psi](-\bar{x}, -\bar{y})$ (where $\psi \mapsto [\psi]$ is the map $\hat{\mathfrak{f}}_2' \to \hat{\mathfrak{f}}_2'/\hat{\mathfrak{f}}_2'' \simeq \overline{xy}\mathbf{k}[[\bar{x}, \bar{y}]]$).

So (4) may be rewritten

$$[\log \Phi](\bar{x}, \bar{y}) = 1 - \frac{\Gamma_\Phi(-\bar{x} - \bar{y})}{\Gamma_\Phi(-\bar{x})\Gamma_\Phi(-\bar{y})}.$$

If now $\psi, \alpha \in \hat{\mathfrak{f}}_2'$, we have $\psi(e^{-\alpha} x e^{\alpha}, y) \in \hat{\mathfrak{f}}_2'$ and $[\psi(e^{-\alpha} x e^{\alpha}, y)] = (1 - [\alpha(x,y)]) \times [\psi(x,y)]$. Indeed, when $\psi(x,y) = (\mathrm{ad}\, x)^k (\mathrm{ad}\, y)^l ([x,y])$, one checks that the part of $\psi(e^{-\alpha} x e^{\alpha}, y)$ containing α more than twice lies in $\hat{\mathfrak{f}}_2''$, and the part containing it once has the same class as $(\mathrm{ad}\, x)^k (\mathrm{ad}\, y)^l ([[-\alpha, x], y])$.

If now $f \in \mathrm{GT}_1(\mathbf{k})$, we have $(f * \Phi)(x,y) = \Phi(x,y) f(\Phi^{-1}(x,y) e^x \Phi(x,y), e^y)$, so

$$\begin{aligned}[\log(f * \Phi)(x,y)] &= [\log \Phi(x,y)] + [\log f(\Phi^{-1}(x,y) e^x \Phi(x,y), e^y)] \\ &= [\log \Phi(x,y)] + [\log f(e^x, e^y)] \\ &\quad - [\log \Phi(x,y)][\log f(e^x, e^y)],\end{aligned}$$

so

$$1 - [\log(f * \Phi)(x,y)] = (1 - [\log \Phi(x,y)])(1 - [\log f(e^x, e^y)]). \tag{23}$$

If we fix $\Phi_0 \in \mathrm{M}_1(\mathbf{k})$ and set $\Gamma_f(u) := \Gamma_{f*\Phi_0}(u)/\Gamma_{\Phi_0}(u)$, then we get

$$1 - [\log f(e^x, e^y)] = \frac{\Gamma_f(-\bar{x})\Gamma_f(-\bar{y})}{\Gamma_f(-\bar{x} - \bar{y})}$$

as wanted. Moreover, (23) implies that $\Gamma_{f*\Phi} = \Gamma_f \Gamma_\Phi$, which also implies that $f \mapsto \Gamma_f$ is a group morphism. □

7. Analytic aspects

In this section, the base field $\mathbf{k}$ is $\mathbf{R}$ or $\mathbf{C}$. The main result of this section is the proof of Theorem 5, which says that a solution of the original KV conjecture may be constructed using the Knizhnik–Zamolodchikov associator.

7.1. *Analytic germs.* — We set $\mathbf{R}_+\{\{x\}\} := \{f \in \mathbf{R}_+[[x]] | f$ has positive radius of convergence$\}$ and $\mathbf{R}_+\{\{x\}\}_0 := \{f \in \mathbf{R}_+\{\{x\}\} | f(0) = 0\}$. If $f, g \in \mathbf{R}_+[[r]]$, we write $f \preceq g$ iff $g - f \in \mathbf{R}_+[[r]]$. We define $f \preceq g$ similarly when $f, g \in \mathbf{R}_+[[r_1, \ldots, r_n]]$.

Let V, E be finite dimensional vector spaces and let $|\cdot|_\mathrm{V}, |\cdot|_\mathrm{E}$ be norms on V, E. The space of E-valued formal series on V is $\mathrm{E}[[\mathrm{V}]] = \{f = \sum_{n \geq 0} f_n, f_n \in \mathrm{S}^n(\mathrm{V}^*) \otimes \mathrm{E}\}$;

we define $\mathrm{E}[[\mathrm{V}]]_0 \subset \mathrm{E}[[\mathrm{V}]]$ by the condition $f_0 = 0$. For $f_n \in \mathrm{S}^n(\mathrm{V}^*) \otimes \mathrm{E}$, viewed as an homogeneous polynomial $\mathrm{V} \to \mathrm{E}$, we set $|f_n| := \sup_{v \neq 0}(|f_n(v)|_{\mathrm{E}}/|v|_{\mathrm{V}}^n)$. An analytic germ on V valued in E (at the neighborhood of 0) is a series $f \in \mathrm{E}[[\mathrm{V}]]$, such that $|f|(r) := \sum_{n \geq 0} |f_n| r^n \in \mathbf{R}_+\{\{r\}\}$. We denote by $\mathrm{E}\{\{\mathrm{V}\}\} \subset \mathrm{E}[[\mathrm{V}]]$ the subspace of analytic germs, and set $\mathrm{E}\{\{\mathrm{V}\}\}_0 := \mathrm{E}[[\mathrm{V}]]_0 \cap \mathrm{E}\{\{\mathrm{V}\}\}$.

If $f \in \mathrm{E}\{\{\mathrm{V}\}\}$ and $\alpha = \sum_{n \geq 0} \alpha_n r^n \in \mathbf{R}_+[[r]]_0$, we say that α is a dominating series for f if $|f_n| \leq \alpha_n$ for any n; we write this as $|f(v)|_{\mathrm{E}} \preceq \alpha(|v|_{\mathrm{V}})$.

If $\mathrm{V}_1, \ldots, \mathrm{V}_k$ are finite dimensional vector spaces with norms $|\cdot|_{\mathrm{V}_1}, \ldots, |\cdot|_{\mathrm{V}_k}$, then we equip $\mathrm{V}_1 \oplus \cdots \oplus \mathrm{V}_k$ with the norm $|(v_1, \ldots, v_k)| := \sup_k |v_i|_{\mathrm{V}_i}$. If f is an analytic germ on $\mathrm{V}_1 \oplus \cdots \oplus \mathrm{V}_k$ valued in E, we decompose $f = \sum_{\mathbf{n} \in \mathbf{N}^k} f_{\mathbf{n}}$, where $f_{\mathbf{n}} : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{E}$ is the $\mathbf{n}$-multihomogeneous component of f. We then set

$$|f_{\mathbf{n}}| := \sup_{(x_1,\ldots,x_k) \in \prod_i (\mathrm{V}_i - \{0\})} |f_{\mathbf{n}}(x_1, \ldots, x_k)|_{\mathrm{E}} / |x_1|_{\mathrm{V}_1}^{n_1} \ldots |x_k|_{\mathrm{V}_k}^{n_k}.$$

Then f is an analytic germ iff $|f|(r_1, \ldots, r_n) := \sum_{\mathbf{n}} |f_{\mathbf{n}}| r_1^{n_1} \cdots r_k^{n_k} \in \mathbf{R}_+[[r_1, \ldots, r_k]]$ converges in a polydisc. If $\alpha = \sum_{n_1,\ldots,n_k \geq 0} \alpha_{n_1,\ldots,n_k} r_1^{n_1} \cdots r_k^{n_k} \in \mathbf{R}_+[[r_1, \ldots, r_k]]$, we write $|f(v_1, \ldots, v_k)|_{\mathrm{E}} \preceq \alpha(|v_1|_{\mathrm{V}_1}, \ldots, |v_k|_{\mathrm{V}_k})$ if for each $\mathbf{n}$, $|f_{\mathbf{n}}(v_1, \ldots, v_k)|_{\mathrm{E}} \leq \alpha_{\mathbf{n}}(|v_1|_{\mathrm{V}_1}, \ldots, |v_k|_{\mathrm{V}_k})$.

Let now $\mathfrak{g}$ be a finite dimensional Lie algebra; let $|\cdot|$ be a norm on $\mathfrak{g}$; let $\mathrm{M} > 0$ be such that the identity $|[x, y]| \leq \mathrm{M}|x||y|$ holds.

The specialization to $\mathfrak{g}$ of the Campbell–Baker–Hausdorff series is a series $x * y = \mathrm{cbh}(x, y) \in \mathfrak{g}[[\mathfrak{g} \times \mathfrak{g}]]_0$.

Lemma **38.** — (1) *The CBH series is an analytic germ* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$; *we have* $|x * y| \preceq \frac{1}{\mathrm{M}} f(\mathrm{M}(|x| + |y|))$, *where* $f(u) = \int_0^u -\frac{\ln(2-e^v)}{v} dv$.
(2) $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto e^{\mathrm{ad}\, x}(y)$ *is an analytic germ, and* $|e^{\mathrm{ad}\, x}(y)| \preceq e^{\mathrm{M}|x|}|y|$.

Proof. — (1) is proved as in [Bk], not making use of the final estimate $\frac{1}{r+s} \leq 1$. (2) follows immediately from $|(\mathrm{ad}\, x)^n(y)| \leq \mathrm{M}^n |x|^n |y|$. □

7.2. $\mathrm{Taut}_n^{an}(\mathfrak{g})$ *and* $\mathfrak{tder}_n^{an}(\mathfrak{g})$. — We set $\mathrm{Taut}_n(\mathfrak{g}) := \{(a_1, \ldots, a_n) | a_i \in \mathfrak{g}[[\mathfrak{g}^n]]_0\}$ and define on this set a product by $(a_1, \ldots, a_n)(b_1, \ldots, b_n) := (c_1, \ldots, c_n)$, where

$$c_i(x_1, \ldots, x_n) := b_i(e^{\mathrm{ad}\, a_1(x_1,\ldots,x_n)}(x_1), \ldots, e^{\mathrm{ad}\, a_n(x_1,\ldots,x_n)}(x_n)) * a_i(x_1, \ldots, x_n).$$

This equips $\mathrm{Taut}_n(\mathfrak{g})$ with a group structure. We set $\mathrm{Taut}_n^{an}(\mathfrak{g}) := \{(a_1, \ldots, a_n) | a_i \in \mathfrak{g}\{\{\mathfrak{g}^n\}\}_0\}$.

Proposition **39.** — $\mathrm{Taut}_n^{an}(\mathfrak{g})$ *is a subgroup of* $\mathrm{Taut}_n(\mathfrak{g})$.

Proof. — Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ belong to $\mathrm{Taut}_n^{an}(\mathfrak{g})$. Let $\alpha(r), \beta(r) \in \mathbf{R}_+\{\{r\}\}_0$ be germs such that the identities $|a_i(x_1, \ldots, x_n)| \preceq \alpha(\sup_i |x_i|)$, $|b_i(x_1, \ldots, x_n)| \preceq \beta(\sup_i |x_i|)$ hold. Then

$$\begin{aligned}|c_i(x_1,\dots,x_n)| &\preceq f_{\mathrm{M}}(|a_i(x_1,\dots,x_n)|+|b_i(e^{\mathrm{ad}\,a_1}(x_1),\dots,e^{\mathrm{ad}\,a_n}(x_n))|)\\ &\preceq f_{\mathrm{M}}(\alpha(\sup_i|x_i|)+\beta(e^{\mathrm{M}\alpha(\sup_i|x_i|)}\sup_i|x_i|))=\gamma(\sup_i|x_i|),\end{aligned}$$

where $f_{\mathrm{M}}(u)=\frac{1}{\mathrm{M}}f(\mathrm{M}u)$ and $\gamma(r)=f_{\mathrm{M}}(\alpha(r)+e^{\mathrm{M}\alpha(r)}\beta(r))$ has nonzero radius of convergence. Here we use the compatibility of norms with composition: namely, if $f\in \mathrm{E}[[\mathrm{V}_1\times\cdots\times\mathrm{V}_n]]_0$ and $g_i\in\mathrm{V}_i[[\mathrm{W}]]_0$, with $|f(v_1,\dots,v_n)|\preceq\alpha(|v_1|,\dots,|v_n|)$ and $|g_i(w)|\preceq\beta_i(|w|)$, then $h:=f\in(g_1,\dots,g_n)\in\mathrm{E}[[\mathrm{W}]]_0$ and $|h(w)|\preceq\alpha\circ(\beta_1,\dots,\beta_n)(|w|)$. We also use the non-decreasing properties of elements of $\mathbf{R}_+[[r_1,\dots,r_n]]_0$ (i.e., if $\mathrm{F}\in\mathbf{R}_+[[u_1,\dots,u_k]]_0$ and $u_i,u_i'\in\mathbf{R}_+[[r_1,\dots,r_l]]_0$ with $u_i\preceq u_i'$, then $\mathrm{F}(u_1,\dots)\preceq\mathrm{F}(u_1',\dots)$. So $(a_1,\dots,a_n)(b_1,\dots,b_n)\in\mathrm{Taut}_n^{an}(\mathfrak{g})$.

If now $(a_1,\dots,a_n)\in\mathrm{Taut}_n^{an}(\mathfrak{g})$, then its inverse $(b_1,\dots,b_n)$ in $\mathrm{Taut}_n(\mathfrak{g})$ is uniquely determined by the identities

$$b_i(x_1,\dots,x_n)=-a_i(e^{\mathrm{ad}\,b_1(x_1,\dots,x_n)}(x_1),\dots,e^{\mathrm{ad}\,b_n(x_1,\dots,x_n)}(x_n)).$$

Let us show that each $b_i(x_1,\dots,x_n)$ is an analytic germ. For this, we define inductively the sequence $b^{(k)}=(b_1^{(k)},\dots,b_n^{(k)})$ by $b^{(0)}=(0,\dots,0)$, and

$$b_i^{(k+1)}(x_1,\dots,x_n)=-a_i(e^{\mathrm{ad}\,b_1^{(k)}(x_1,\dots,x_n)}(x_1),\dots,e^{\mathrm{ad}\,b_n^{(k)}(x_1,\dots,x_n)}(x_n)).$$

One checks that $b^{(k)}=b^{(k-1)}+\mathrm{O}(x^k)$, so the sequence $(b^{(k)})_{k\geq 0}$ converges in the formal series topology; the limit b is then the inverse of $a=(a_1,\dots,a_n)$.

Let us now set $\beta_k:=\sup_i|b_i^{(k)}|$ (if $u_i(r)=\sum_{k\geq 0}u_{i,k}r^k\in\mathbf{R}_+[[r]]$ is a finite family, we set $(\sup_i u_i)(r):=\sum_{k\geq 0}(\sup_i u_{i,k})r^k$). We then have

$$|b_i^{(k+1)}(x_1,\dots,x_n)|\preceq\alpha(\sup_i|e^{\mathrm{ad}\,b_i^{(k)}(x_1,\dots,x_n)}(x_i)|)\preceq\alpha(e^{\mathrm{M}\beta_k(\sup_i|x_i|)}\sup_i|x_i|),$$

so $\beta_{k+1}(r)\preceq\alpha(e^{\beta_k(r)}r)$.

We now define a sequence $(\gamma_k)_{k\geq 0}$ of elements of $\mathbf{R}_+[[r]]_0$ by $\gamma_0=0$,

$$\gamma_{k+1}(r)=\alpha(e^{\mathrm{M}\gamma_k(r)}r).$$

As the exponential function, mutiplication by r and α are non-decreasing, we have $\beta_k\preceq\gamma_k$. On the other hand, we have $\gamma_k(r)=\gamma_{k-1}(r)+\mathrm{O}(r^k)$, so the sequence $(\gamma_k)_k$ converges in $\mathbf{R}_+[[r]]_0$ (one also checks that this sequence is non-decreasing). Its limit γ then satisfies

$$\gamma(r)=\alpha(e^{\mathrm{M}\gamma(r)}r). \tag{24}$$

It is easy to show that (24) determines $\gamma(r)\in\mathbf{R}[[r]]_0$ uniquely. On the other hand, the function $(\gamma,r)\mapsto\gamma-\alpha(e^{\mathrm{M}\gamma}r)=:\mathrm{F}(\gamma,r)$ is analytic at the neighborhood of $(0,0)$, with differential at this point $\partial_\gamma\mathrm{F}(0,0)d\gamma+\partial_r\mathrm{F}(0,0)dr=d\gamma-\mathrm{M}\alpha'(0)dr$. We may then apply the implicit function theorem and use the fact that the $d\gamma$-component of $d\mathrm{F}(0,0)$

is nonzero to derive the existence of an analytic function $\gamma_{an}(r)$ satisfying (24). By the uniqueness of solutions of (24), we get that the expansion of γ_{an} is γ, so $\gamma \in \mathbf{R}_+\{\{r\}\}_0$.

Now $|b_i^{(k)}(x_1, \ldots, x_n)| \preceq \beta_k(\sup_i |x_i|) \preceq \gamma_k(\sup_i |x_i|) \preceq \gamma(\sup_i |x_i|)$, so by taking the limit $k \to \infty$, $|b_i(x_1, \ldots, x_k)| \preceq \gamma(\sup_i |x_i|)$, which implies that $b_i \in \mathfrak{g}\{\{\mathfrak{g}^n\}\}_0$, as wanted. $\square$

According to [AT2], we have a bijection

$$\kappa : \underline{\mathrm{Taut}}_n(\mathbf{k}) \to \underline{\mathfrak{tder}}_n^{\mathbf{k}}, \quad g \mapsto \ell - \theta(g)\ell\theta(g)^{-1},$$

where ℓ is the derivation given by $x_i \mapsto x_i$.

Set $\mathfrak{tder}_n(\mathfrak{g}) := \{(u_1, \ldots, u_n) | u_i(x_1, \ldots, x_n) \in \mathfrak{g}[[\mathfrak{g}^n]]_0\}$, and $\mathfrak{tder}_n^{an}(\mathfrak{g}) := \{(u_1, \ldots, u_n) | u_i \in \mathfrak{g}\{\{\mathfrak{g}^n\}\}_0\} \subset \mathfrak{tder}_n(\mathfrak{g})$. We have maps $\underline{\mathrm{Taut}}_n(\mathbf{k}) \to \mathrm{Taut}_n(\mathfrak{g})$, $\underline{\mathfrak{tder}}_n^{\mathbf{k}} \to \mathfrak{tder}_n(\mathfrak{g})$ induced by the specialization of formal series.

Lemma **40.** — (1) *There exists a map* $\kappa_{\mathfrak{g}} : \mathrm{Taut}_n(\mathfrak{g}) \to \mathfrak{tder}_n(\mathfrak{g})$, *such that the diagram*

$$\begin{array}{ccc} \underline{\mathrm{Taut}}_n(\mathbf{k}) & \xrightarrow{\kappa} & \underline{\mathfrak{tder}}_n^{\mathbf{k}} \\ \downarrow & & \downarrow \\ \mathrm{Taut}_n(\mathfrak{g}) & \xrightarrow{\kappa_{\mathfrak{g}}} & \mathfrak{tder}_n(\mathfrak{g}) \end{array}$$

commutes.

(2) *This map restricts to a map* $\kappa_{\mathfrak{g}}^{an} : \mathrm{Taut}_n^{an}(\mathfrak{g}) \to \mathfrak{tder}_n^{an}(\mathfrak{g})$.

Proof. — (1) If $a_i, b_i \in \hat{\mathfrak{f}}_n^{\mathbf{k}}$ are such that $g = (e^{b_1}, \ldots, e^{b_n})$, $g^{-1} = (e^{a_1}, \ldots, e^{a_n})$, then $\kappa(g) = u = (u_1, \ldots, u_n)$, with

$$u_i(x_1, \ldots, x_n) = \left(\frac{1 - e^{\mathrm{ad}\, a_i}}{\mathrm{ad}\, a_i}(\dot{a}_i)\right)(e^{\mathrm{ad}\, b_1(x_1,\ldots,x_n)}(x_1), \ldots, e^{\mathrm{ad}\, b_n(x_1,\ldots,x_n)}(x_n))$$

and $\dot{a}_i = \ell(a_i) = \frac{d}{dt}_{|t=1} a_i(tx_1, \ldots, tx_n)$. So we define $\kappa_{\mathfrak{g}}$ by the same formula, where $\dot{a}_i$ is now defined as $\frac{d}{dt}_{|t=1} a_i(tx_1, \ldots, tx_n)$ (or $\sum_{k\geq 0} k a_i^k$, where a_i^k is the degree n part of a_i).

(2) If the functions a_i, b_i are analytic germs, then so is $\dot{a}_i$ and therefore also each u_i. $\square$

Recall also from [AT2] that if $\mu \in \mathrm{Taut}_2(\mathbf{k})$, $\mu(x * y) = x + y$ and $\mathrm{J}(\mu) = \langle r(x) + r(y) - r(x+y)\rangle$ (i.e., $\mu \in \mathrm{SolKV}(\mathbf{k})$), then $u := -\kappa(\mu^{-1}) = (\mathrm{A}(x,y), \mathrm{B}(x,y))$ satisfies:

(KV1) $x + y - y * x = (1 - e^{-\mathrm{ad}\, x})(\mathrm{A}(x,y)) + (e^{\mathrm{ad} y} - 1)(\mathrm{B}(x,y))$,
(KV3) $j(u) = \langle \phi(x) + \phi(y) - \phi(x * y)\rangle$, where $\phi(t) = tr'(t)$.

Let Φ_{KZ} be the KZ associator, $\tilde{\Phi}_{\mathrm{KZ}}(a, b) := \Phi_{\mathrm{KZ}}(a/(2\pi \mathrm{i}), b/(2\pi \mathrm{i})) \in \mathrm{M}_1(\mathbf{C})$ and $\mu_{\mathrm{KZ}} := \mu_{\tilde{\Phi}_{\mathrm{KZ}}}$. Let $u_{\mathrm{KZ}} := \kappa(\mu_{\mathrm{KZ}}^{-1})$. Then $\mathrm{J}(\mu_{\mathrm{KZ}}) = \langle r_{\mathrm{KZ}}(x) + r_{\mathrm{KZ}}(y) - r_{\mathrm{KZ}}(x * y)\rangle$, where $r_{\mathrm{KZ}}(u) = -\sum_{n\geq 2}(2\pi \mathrm{i})^{-n}\zeta(n)u^n/n$, therefore

$$j(u_{\mathrm{KZ}}) = \langle \phi_{\mathrm{KZ}}(x) + \phi_{\mathrm{KZ}}(y) - \phi_{\mathrm{KZ}}(x * y)\rangle,$$

where $\phi_{\mathrm{KZ}}(u) = -\sum_{n\geq 2}(2\pi \mathrm{i})^{-n}\zeta(n)u^n$. Now the real part of this function (obtained by taking the real part of the coefficients of u^n) is

$$\phi^{\mathbf{R}}_{\mathrm{KZ}}(u) = \frac{1}{2}\left(\frac{u}{e^u - 1} - 1 + \frac{u}{2}\right).$$

Let us now set $u_{\mathbf{R}} := (\mathrm{A}_{\mathbf{R}}(x,y), \mathrm{B}_{\mathbf{R}}(x,y))$, where the real part is taken with respect to the natural real structure on $\mathfrak{f}_2^{\mathbf{C}}$. Then by the linearity of (KV1), (KV3), we have:

$$\text{(KV1)} \quad x + y - y * x = (1 - e^{-\operatorname{ad} x})(\mathrm{A}_{\mathbf{R}}(x,y)) + (e^{\operatorname{ad} y} - 1)(\mathrm{B}_{\mathbf{R}}(x,y))$$

$$\text{(KV3)} \quad j(u_{\mathbf{R}}) = \frac{1}{2}\left\langle \frac{x}{e^x - 1} + \frac{y}{e^y - 1} - \frac{x*y}{e^{x*y} - 1} - 1 \right\rangle.$$

7.3. *Analytic aspects to the KV conjecture (proof of Theorem 5).* — Recall that $\log \tilde{\Phi}_{\mathrm{KZ}} \in \hat{\mathfrak{f}}_2^{\mathbf{C}}$. We denote the specialization of this series to the Lie algebra $\mathfrak{g}$ as $(\log \tilde{\Phi}_{\mathrm{KZ}})^{\mathfrak{g}} \in \mathfrak{g}[[\mathfrak{g}^2]]_0$.

Proposition **41.** — $(\log \tilde{\Phi}_{\mathrm{KZ}})^{\mathfrak{g}}$ *is an analytic germ, i.e.,* $(\log \tilde{\Phi}_{\mathrm{KZ}})^{\mathfrak{g}} \in \mathfrak{g}\{\{\mathfrak{g}^2\}\}_0$.

Proof. — Recall that $\mathrm{A}_2 = \mathrm{U}(\mathfrak{f}_2^{\mathbf{C}})$ is the free associative algebra in a, b. For $x \in \mathrm{A}_2$, set

$$|x|_{\mathrm{A}_2} := \sup_{\mathrm{N}\geq 1} \sup_{m_1,m_2 \in \mathrm{M}_{\mathrm{N}}(\mathbf{C}) | \|m_1\|, \|m_2\| \leq 1} \|x(m_1, m_2)\|.$$

Here $\|\cdot\|$ is an algebra norm on $\mathrm{M}_{\mathrm{N}}(\mathbf{C})$. Then $|x|_{\mathrm{A}_2}$ is $\leq \sum_{\mathrm{I}\in \bigsqcup_{n\geq 0}\{0,1\}^n} |x_{\mathrm{I}}|$, where $x = \sum_{\mathrm{I}} x_{\mathrm{I}} e_{\mathrm{I}}$, and for $\mathrm{I} = (i_1, \ldots, i_n)$, $e_{\mathrm{I}} = e_{i_1} \cdots e_{i_n}$, $e_0 = a$, $e_1 = b$. It follows from the Amitsur–Levitsky theorem [AL] that $(|x|_{\mathrm{A}_2} = 0) \Rightarrow (x = 0)$; indeed, by this theorem, $x(m_1, m_2) = 0$ for $m_1, m_2 \in \mathrm{M}_{\mathrm{N}}(\mathbf{C})$ implies: (a) that x is in the 2-sided ideal generated by $ab - ba$ if $\mathrm{N} = 1$; (b) that $x = 0$ if $\mathrm{N} > 1$. It follows that $|.|_{\mathrm{A}_2}$ is an algebra norm[4] on A_2, in particular $|xy|_{\mathrm{A}_2} \leq |x|_{\mathrm{A}_2}|y|_{\mathrm{A}_2}$.

We then define a vector space norm $|.|_{\mathfrak{f}_2}$ on $\mathfrak{f}_2^{\mathbf{C}}$ by $|x|_{\mathfrak{f}_2} := |x|_{\mathrm{A}_2}$; we have $|[x,y]|_{\mathfrak{f}_2} \leq 2|x|_{\mathfrak{f}_2}|y|_{\mathfrak{f}_2}$.

For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbf{N}^d$, and f a formal series on $(\mathfrak{f}_2)^d$ (resp., $\mathbf{R}^d$), we denote by $f(\xi_1, \ldots, \xi_d)_{\mathbf{n}}$ (resp., $f(t_1, \ldots, t_d)_{\mathbf{n}}$) the $\mathbf{n}$-multihomogeneous part of f, which we view as a multihomogeneous polynomial on $(\mathfrak{f}_2^{\mathbf{C}})^d$ (resp., $\mathbf{R}^d$).

Lemma **42.** — *For any* $\mathbf{n}$, *we have the identity*

$$|\log(e^{\xi_1} \cdots e^{\xi_d})_{\mathbf{n}}|_{\mathfrak{f}_2} \leq ((\log(2 - e^{t_1 + \cdots + t_d})^{-1})_{\mathbf{n}})_{t_1 = |\xi_1|_{\mathfrak{f}_2}, \ldots, t_d = |\xi_d|_{\mathfrak{f}_2}}.$$

[4] We will not use $(|x|_{\mathrm{A}_2} = 0) \Rightarrow (x = 0)$, so our proof of Proposition 41 is independent of the Amitsur–Levitsky theorem.

Proof of lemma. — We have for any $\mathbf{n}$, $|\xi_1^{n_1}\cdots\xi_d^{n_d}|_{A_2} \le |\xi_1|_{\mathfrak{f}_2}^{n_1}\cdots|\xi_d|_{\mathfrak{f}_2}^{n_d}$ so

$$|(e^{\xi_1}\cdots e^{\xi_d}-1)_{\mathbf{n}}|_{A_2} \le ((e^{t_1+\cdots+t_d}-1)_{\mathbf{n}})_{t_1=|\xi_1|_{\mathfrak{f}_2},\dots,t_d=|\xi_d|_{\mathfrak{f}_2}}.$$

Then $\log(e^{\xi_1}\cdots e^{\xi_d})_{\mathbf{n}} = \sum_{k\ge 1}\frac{(-1)^{k+1}}{k}\sum_{(\mathbf{n}_1,\dots,\mathbf{n}_k)|\mathbf{n}_1+\cdots+\mathbf{n}_k=\mathbf{n}}(e^{\xi_1}\cdots e^{\xi_d}-1)_{\mathbf{n}_1}\cdots(e^{\xi_1}\cdots e^{\xi_d}-1)_{\mathbf{n}_k}$ so

$$\begin{aligned}|\log(e^{\xi_1}\cdots e^{\xi_d})_{\mathbf{n}}|_{A_2} &\le \Bigg(\sum_{k\ge 1}\frac{1}{k}\sum_{\mathbf{n}_1+\cdots+\mathbf{n}_k=\mathbf{n}}(e^{t_1+\cdots+t_d}-1)_{\mathbf{n}_1}\cdots \\ &\qquad \times(e^{t_1+\cdots+t_d}-1)_{\mathbf{n}_d}\Bigg)_{t_1=|\xi_1|_{\mathfrak{f}_2},\dots,t_d=|\xi_d|_{\mathfrak{f}_2}} \\ &= \Bigg(\sum_{k\ge 1}\frac{1}{k}((e^{t_1+\cdots+t_d}-1)^k)_{\mathbf{n}}\Bigg)_{t_1=|\xi_1|_{\mathfrak{f}_2},\dots,t_d=|\xi_d|_{\mathfrak{f}_2}} \\ &= ((\log(2-e^{t_1+\cdots+t_d})^{-1})_{\mathbf{n}})_{t_1=|\xi_1|_{\mathfrak{f}_2},\dots,t_d=|\xi_d|_{\mathfrak{f}_2}}. \qquad \square\end{aligned}$$

Let $a(t)$ be a function $[0,1]\to\hat{\mathfrak{f}}_2^{\mathbf{C}}$ of the form $a(t)=\sum_{k\ge 1}a_k(t)$, where $a_k(t)\in\mathfrak{f}_2^{\mathbf{C}}[k]$ (here k is the total degree in a, b) and $\int_0^1|a_k(t)|_{\mathfrak{f}_2}dt<\infty$. Let u_0, u_1 be solutions of $u'(t)=a(t)u(t)$ with $u_0(0)=u_1(1)=1$, and $\mathrm{U}:=u_1^{-1}u_0$.

Lemma **43.** — *For* $n\ge 1$, *let* $(\log\mathrm{U})_n$ *the degree* n (*in* a, b) *part of* $\log\mathrm{U}$. *Then*

$$\sum_{n\ge 1}|(\log\mathrm{U})_n|_{\mathfrak{f}_2}r^n \preceq \log(2-e^{\sum_{k\ge 1}r^k\int_0^1|a_k(t)|_{\mathfrak{f}_2}dt})^{-1}.$$

Proof of lemma. — Let $\mathrm{Lie}(n)$ be the multilinear part of $\mathfrak{f}_n^{\mathbf{C}}$ in the generators $x_1,\dots,x_n$. We denote by $w_n(x_1,\dots,x_n)\in\mathrm{Lie}(n)$ the multilinear part of $\log(e^{x_1}\cdots e^{x_n})$.

Let now α_n be the coefficient of $t_1\cdots t_n$ in the expansion of $\log(2-e^{t_1+\cdots+t_n})^{-1}$ (this is also the nth derivative at $t=0$ of $\log(2-e^t)^{-1}$). Specializing Lemma 42 for $\mathbf{n}=(1,\dots,1)$, we get the identity

$$|w_n(\xi_1,\dots,\xi_n)|_{\mathfrak{f}_2}\le\alpha_n|\xi_1|_{\mathfrak{f}_2}\cdots|\xi_n|_{\mathfrak{f}_2}$$

for $\xi_1,\dots,\xi_n\in\mathfrak{f}_2^{\mathbf{C}}$.

Now $\log\mathrm{U}$ expands as

$$\log\mathrm{U}=\sum_{n\ge 0}\int_{0<t_1<\dots<t_n<1}w_n(a(t_1),\dots,a(t_n))dt_1\cdots dt_n$$

(see e.g. [EG]). It follows that

$$(\log\mathrm{U})_k=\sum_{n\ge 0}\sum_{k_1,\dots,k_n|\sum_i k_i=k}\int_{0<t_1<\dots<t_n<1}w_n(a_{k_1}(t_1),\dots,a_{k_n}(t_n))dt_1\cdots dt_n$$

and therefore

$$|(\log \mathrm{U})_k|_{\mathfrak{f}_2} \leq \sum_{n\geq 0} \alpha_n \sum_{k_1,\ldots,k_n | \sum_i k_i = k} \int_{0<t_1<\ldots<t_n<1} |a_{k_1}(t_1)|_{\mathfrak{f}_2} \cdots \times |a_{k_n}(t_n)|_{\mathfrak{f}_2} dt_1 \cdots dt_n.$$

Now the generating series for the r.h.s. is $\log(2 - e^{\sum_{k\geq 1} r^k \int_0^1 |a_k(t)|_{\mathfrak{f}_2} dt})^{-1}$, proving the result. $\square$

According to [Dr], Section 2, if we set

$$a(t) := \sum_{k\geq 0, l\geq 1} \frac{1}{k!l!(2\pi \mathrm{i})^{k+l+1}} \frac{(-\log(1-t))^k (-\log t)^l}{t-1} (\mathrm{ad}\, b)^k (\mathrm{ad}\, a)^l (b),$$

then $\tilde{\Phi}_{\mathrm{KZ}} = \mathrm{U}$. We have $|(\mathrm{ad}\, b)^k (\mathrm{ad}\, a)^l (b)|_{\mathfrak{f}_2} \leq k + l + 2 \leq 2^{k+l+1}$, so

$$|a_n(t)| \leq \sum_{k\geq 0, l\geq 1, k+l+1=n} \frac{1}{\pi^{k+l+1} k!l!} \frac{(-\log(1-t))^k (-\log t)^l}{1-t}.$$

Then we have the inequality of formal series in r

$$\sum_{n\geq 1} r^n \int_0^1 |a_n(t)|_{\mathfrak{f}_2} dt \preceq \int_0^1 \sum_{k\geq 0, l\geq 1} \frac{r^{k+l+1}}{\pi^{k+l+1} k!l!} \frac{(-\log(1-t))^k (-\log t)^l}{1-t} dt$$
$$= \frac{r}{\pi} \int_0^1 (1-t)^{-1-\frac{r}{\pi}} (t^{-\frac{r}{\pi}} - 1) dt.$$

Now the identity $\int_0^1 t^a (1-t)^b dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$, valid for $\Re(a), \Re(b) > -1$, implies that if $\Re(r) < 0$, then

$$\frac{r}{\pi} \int_0^1 (1-t)^{-1-\frac{r}{\pi}} (t^{-\frac{r}{\pi}} - 1) dt = \frac{1}{2} \left(1 - \frac{\Gamma(1-2r)^2}{\Gamma(1-4r)} \right).$$

This implies that the radius of convergence of $\frac{r}{\pi} \int_0^1 (1-t)^{-1-\frac{r}{\pi}} (t^{-\frac{r}{\pi}} - 1) dt$ is $1/4$, so this series belongs to $\mathbf{R}_+\{\{r\}\}_0$. Plugging this in Lemma 43, we get

$$\sum_{n\geq 0} |(\log \tilde{\Phi}_{\mathrm{KZ}})_n|_{\mathfrak{f}_2} r^n \preceq \log(2 - e^{\frac{1}{2}(1 - \frac{\Gamma(1-2r)^2}{\Gamma(1-4r)})})^{-1},$$

where the series in the r.h.s. lies in $\mathbf{R}_+\{\{r\}\}_0$ (being a composition of two series in $\mathbf{R}_+\{\{r\}\}_0$).

Let us now prove that $(\log \tilde{\Phi}_{\mathrm{KZ}})^{\mathfrak{g}} \in \mathfrak{g}\{\{\mathfrak{g}^2\}\}_0$ is an analytic germ. By Ado's theorem, there exists a injective morphism $\rho : \mathfrak{g} \to \mathrm{M_N}(\mathbf{k})$, where $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, hence an injective morphism $\tilde{\rho} : \mathfrak{g} \to \mathrm{M_N}(\mathbf{C})$. Equip $\mathfrak{g}$ with the norm $|x|_{\mathfrak{g}} := \|\tilde{\rho}(x)\|$. We recall that all the norms on $\mathfrak{g}$ are equivalent, so it will suffice to prove analyticity w.r.t. $|\cdot|_{\mathfrak{g}}$.

The degree n part of the series $(\log\tilde\Phi_{\mathrm{KZ}})^{\mathfrak g}$ is the specialization to $\mathfrak g$ of $(\log\tilde\Phi_{\mathrm{KZ}})_n$. Now if $\psi\in\mathfrak f_2[n]$ and $\psi^{\mathfrak g}:\mathfrak g\times\mathfrak g\to\mathfrak g$ is its specialization to $\mathfrak g$, we have $|\psi^{\mathfrak g}(x,y)|_{\mathfrak g}=\|\psi(\tilde\rho(x),\tilde\rho(y))\|\le|\psi|_{\mathfrak f_2}\sup(\|\tilde\rho(x)\|,\|\tilde\rho(y)\|)^n=|\psi|_{\mathfrak f_2}\sup(|x|_{\mathfrak g},|y|_{\mathfrak g})^n$, therefore $|\psi^{\mathfrak g}|\le|\psi|_{\mathfrak f_2}$. We then have

$$\sum_{n\ge0}|(\log\tilde\Phi_{\mathrm{KZ}})_n^{\mathfrak g}|r^n\preceq\sum_{n\ge0}|(\log\tilde\Phi_{\mathrm{KZ}})_n|_{\mathfrak f_2}r^n\preceq\log(2-e^{\frac12(1-\frac{\Gamma(1-2r)^2}{\Gamma(1-4r)})})^{-1};$$

together with the fact that the series in the right has positive radius of convergence, this implies the analyticity of the series $(\log\tilde\Phi_{\mathrm{KZ}})^{\mathfrak g}$. □

Proposition 41, together with the local analyticity of the CBH series, implies that the specialization of $\mu_{\tilde\Phi_{\mathrm{KZ}}}$ belongs to $\mathrm{Taut}_2^{an}(\mathfrak g)$. It follows that $\mathrm A(x,y)$, $\mathrm B(x,y)$ are analytic germs, and so

(KV2) $(\mathrm A^{\mathbf R},\mathrm B^{\mathbf R})$ is an analytic germ $\mathfrak g^2\to\mathfrak g^2$.

All this implies that $(\mathrm A^{\mathbf R},\mathrm B^{\mathbf R})$ is a solution of the 'original' KV conjecture (as formulated in [KV]) and proves 1) in Theorem 5.

Let us now prove Theorem 5, 2). One checks easily that if $(\mathrm A,\mathrm B)$ is a solution of the 'original' KV conjecture, then $(\mathrm A_s,\mathrm B_s):=(\mathrm A+s(\log(e^xe^y)-x),\mathrm B+s(\log(e^xe^y)-y))$ is a family of solutions. In fact, if $\mu\in\mathrm{SolKV}(\mathbf k)$ and $(\mathrm A,\mathrm B)=-\kappa(\mu^{-1})$, then $(\mathrm A_s,\mathrm B_s)=-\kappa(\mu_{-s}^{-1})$, where $\mu_s:=\mathrm{Inn}(e^{s(x+y)})\circ\mu$; this corresponds to the action of 'trivial', degree 1 element of $\mathfrak{krv}$ on $\mathrm{SolKV}(\mathbf k)$ (see [AT2]).

Finally, let us prove Theorem 5, 3). Let σ be the antilinear automorphism of $\hat{\mathfrak f}_2^{\mathbf C}$ such that $\sigma(x)=-y$, $\sigma(y)=-x$. The series $\Phi_{\mathrm{KZ}}(x,y)$ is real, therefore $\overline{\tilde\Phi_{\mathrm{KZ}}(x,y)}=\tilde\Phi_{\mathrm{KZ}}(-x,-y)$ (the bar denotes the complex conjugation). This implies that $\theta(\mu_{\mathrm{KZ}})\sigma=\mathrm{Inn}(e^{-(x+y)/2})\sigma\theta(\mu_{\mathrm{KZ}})$. Using $\sigma\ell\sigma^{-1}=\ell$ and $\ell(x+y)=x+y$, we get

$$(\theta(\mu_{\mathrm{KZ}})\sigma\theta(\mu_{\mathrm{KZ}})^{-1})\ell(\theta(\mu_{\mathrm{KZ}})\sigma\theta(\mu_{\mathrm{KZ}})^{-1})^{-1}=\ell+\mathrm{inn}\left(\frac12(x+y)\right),$$

where $\mathrm{inn}(x+y)$ is the inner derivation $z\mapsto[x+y,z]$ of $\hat{\mathfrak f}_2^{\mathbf C}$. Using now $\theta(\mu_{\mathrm{KZ}})^{-1}(x+y)=\log(e^xe^y)$, we get

$$(\sigma\theta(\mu_{\mathrm{KZ}}^{-1}))\ell(\sigma\theta(\mu_{\mathrm{KZ}})^{-1})^{-1}=\theta(\mu_{\mathrm{KZ}})^{-1}\ell\theta(\mu_{\mathrm{KZ}})+\mathrm{inn}\left(\frac12\log(e^xe^y)\right).$$

Since $\sigma\ell\sigma^{-1}=\ell$, $\theta(\mu_{\mathrm{KZ}})^{-1}\ell\theta(\mu_{\mathrm{KZ}})-\ell=-(\mathrm A_{\mathrm{KZ}},\mathrm B_{\mathrm{KZ}})$ and $\mathrm{inn}(\frac12\log(e^xe^y))=(\frac12(\log(e^xe^y)-x),\frac12(\log(e^xe^y)-y))$, this implies

$$\sigma(\mathrm A_{\mathrm{KZ}},\mathrm B_{\mathrm{KZ}})\sigma^{-1}=(\mathrm A_{\mathrm{KZ}},\mathrm B_{\mathrm{KZ}})-\left(\frac12(\log(e^xe^y)-x),\frac12(\log(e^xe^y)-y)\right).$$

This implies

$$
\begin{aligned}
&(\mathrm{B}_{\mathrm{KZ}}(-y,-x),\mathrm{A}_{\mathrm{KZ}}(-y,-x))\\
&\quad=(\mathrm{A}_{\mathrm{KZ}}(x,y),\mathrm{B}_{\mathrm{KZ}}(x,y))-\left(\frac{1}{2}(\log(e^xe^y)-x),\frac{1}{2}(\log(e^xe^y)-y)\right).
\end{aligned}
$$

If now $(\mathrm{A}',\mathrm{B}') := (\mathrm{A}_{\mathrm{KZ}},\mathrm{B}_{\mathrm{KZ}}) - \frac{1}{4}(\log(e^xe^y)-x,\log(e^xe^y)-y)$, this implies

$$
(\mathrm{B}'(-y,-x),\mathrm{A}'(-y,-x))=(\mathrm{A}'(x,y),\mathrm{B}'(x,y)),
$$

which by taking real parts implies $(\mathrm{B}_{-1/4}(-y,-x),\mathrm{A}_{-1/4}(-y,-x)) = (\mathrm{A}_{-1/4}(x,y),\mathrm{B}_{-1/4}(x,y))$, proving Theorem 5, 3).

Acknowledgements

We are grateful to V. G. Drinfeld and to D. Bar-Natan who posed the question of how to construct explicit solutions of the KV problem in terms of associators. We thank D. Calaque and G. Massuyeau for discussions. We thank referees for comments which helped improve the exposition. The research of A. A. was supported in part by the grants 200020-120042 and 200020-121675 of the Swiss National Science Foundation, and the research of C.T. was supported by CNRS.

Appendix A: The morphism $\mathrm{GT}_1(\mathbf{k}) \to \mathrm{KV}(\mathbf{k})$, cocycle identities and profinite versions

We will show:

Proposition **44.** — *For $f \in \mathrm{GT}_1(\mathbf{k})$, α_f defined in Theorem 9 satisfies the cocycle identity*

$$
f(\log x_{12},\log x_{23})\circ\alpha_f^{\widetilde{12},3}\circ\alpha_f^{1,2}=\alpha_f^{1,\widetilde{23}}\circ\alpha_f^{2,3} \tag{25}
$$

in $\mathrm{Taut}_3(\mathbf{k})$ (*see Section* 3.4 *and the end of the Introduction*).

The group $\mathrm{GT}_1(\mathbf{k})$ admits profinite and pro-l versions. We show that:

Proposition **45.** — *The morphism $f \mapsto \alpha_f^{-1}$ admits variants in these setups, which fit in a commutative diagram*

$$
\begin{array}{ccccc}
\widehat{\mathrm{GT}}_1 & \to & \mathrm{GT}_1^l & \to & \mathrm{GT}_1(\mathbf{Q}_l)\\
\downarrow & & \downarrow & & \downarrow\\
\widehat{\mathrm{Taut}}_2 & \to & \mathrm{Taut}_{2,l} & \to & \mathrm{Taut}_2(\mathbf{Q}_l)
\end{array}
$$

and satisfy analogues of (25).

A.1 *Proof of Proposition 44.* — The action of $f \in \mathrm{GT}_1(\mathbf{k})$ on $\Phi \in \mathrm{M}_1(\mathbf{k})$ has been defined in the Introduction. Then $\mu_{f*\Phi} = \mu_\Phi \alpha_f$ and $\mu_\Phi(e^x e^y) = e^{x+y}$, hence $\mu_{f*\Phi}^{12,3}\mu_{f*\Phi}^{1,2} = \mu_\Phi^{12,3}\mu_\Phi^{1,2}\alpha_f^{\widetilde{12,3}}\alpha_f^{1,2}$ and $\mu_{f*\Phi}^{1,23}\mu_{f*\Phi}^{2,3} = \mu_\Phi^{1,23}\mu_\Phi^{2,3}\alpha_f^{\widetilde{1,23}}\alpha_f^{2,3}$.

Now

$$(f * \Phi)(t_{12}, t_{23})\mu^\Phi_{\bullet((\bullet\bullet)\bullet)} = \Phi(t_{12}, t_{23})f(t_{12}, \Phi^{-1}(t_{12}, t_{23})t_{23}\Phi(t_{12}, t_{23}))\mu^\Phi_{\bullet((\bullet\bullet)\bullet)}$$
$$= \Phi(t_{12}, t_{23})\mu^\Phi_{\bullet((\bullet\bullet)\bullet)}f(\log x_{12}, \log x_{23})$$

as $t_{12}\mu^\Phi_{\bullet((\bullet\bullet)\bullet)} = \mu^\Phi_{\bullet((\bullet\bullet)\bullet)}\log x_{12}$, $t_{23}\mu^\Phi_{\bullet(\bullet(\bullet\bullet))} = \mu^\Phi_{\bullet(\bullet(\bullet\bullet))}\log x_{23}$, and $\Phi(t_{12}, t_{23})\mu^\Phi_{\bullet((\bullet\bullet)\bullet)} = \mu^\Phi_{\bullet(\bullet(\bullet\bullet))}$.

Therefore

$$(f * \Phi)(t_{12}, t_{23})\mu_{f*\Phi}^{12,3}\mu_{f*\Phi}^{1,2} = (f * \Phi)(t_{12}, t_{23})\mu^\Phi_{\bullet((\bullet\bullet)\bullet)}\alpha_f^{\widetilde{12,3}}\alpha_f^{1,2}$$
$$= \Phi(t_{12}, t_{23})\mu^\Phi_{\bullet((\bullet\bullet)\bullet)}f(\log x_{12}, \log x_{23})\alpha_f^{\widetilde{12,3}}\alpha_f^{1,2},$$

while $\mu_{f*\Phi}^{1,23}\mu_{f*\Phi}^{2,3} = \mu_\Phi^{1,23}\mu_\Phi^{2,3}\alpha_f^{\widetilde{1,23}}\alpha_f^{2,3} = \mu^\Phi_{\bullet(\bullet(\bullet\bullet))}\alpha_f^{\widetilde{1,23}}\alpha_f^{2,3}$.

Proposition 44 then follows from $(f * \Phi)(t_{12}, t_{23})\mu_{f*\Phi}^{12,3}\mu_{f*\Phi}^{1,2} = \mu_{f*\Phi}^{1,23}\mu_{f*\Phi}^{2,3}$ and $\Phi(t_{12}, t_{23})\mu^\Phi_{\bullet((\bullet\bullet)\bullet)} = \mu^\Phi_{\bullet(\bullet(\bullet\bullet))}$.

A.2 *Proof of Proposition 45.* — Let us denote by $\widehat{\mathrm{G}}$ and G_l the profinite and pro-l completions of a group G. The set of equations defining the group $\mathrm{GT}_1(\mathbf{k})$ may be viewed as a map $\mathrm{F}_2(\mathbf{k}) \to \mathrm{F}_2(\mathbf{k})^2 \times \mathrm{PB}_4(\mathbf{k})$. Replacing it by maps $\widehat{\mathrm{F}}_2 \to \widehat{\mathrm{F}}_2^2 \times \widehat{\mathrm{PB}}_4$ and $\mathrm{F}_{2,l} \to (\mathrm{F}_{2,l})^2 \times \mathrm{PB}_{4,l}$, we define semigroups $\underline{\widehat{\mathrm{GT}}}_1$ and $\underline{\mathrm{GT}}_{1,l}$. We define $\widehat{\mathrm{GT}}_1$ and $\mathrm{GT}_{1,l}$ as the corresponding groups. We have natural maps $\widehat{\mathrm{GT}}_1 \to \mathrm{GT}_{1,l} \hookrightarrow \mathrm{GT}_1(\mathbf{Q}_l)$ (see [Dr]).

The definitions of the semigroup $\mathrm{Tauts}_{\mathrm{S}}$ and of the semigroup morphism $\theta : \mathrm{Tauts}_{\mathrm{S}} \to \mathrm{End}(\mathrm{F}_{\mathrm{S}})$ from Section 3.3 extend to the profinite and pro-l case. We denote by $\widehat{\mathrm{Tauts}}_{\mathrm{S}}$ and $\mathrm{Tauts}_{\mathrm{S},l}$ the corresponding semigroups. The contravariant functor structure of $\mathrm{S} \mapsto \widehat{\mathrm{Tauts}}_{\mathrm{S}}, \mathrm{Tauts}_{\mathrm{S},l}$ is defined as in Section 3.4.

Identity (25) can be proved directly, checking the identity on each of the generators of $\mathrm{F}_3(\mathbf{k})$ and using only the duality, hexagon and pentagon relations. Extending this proof to the profinite and pro-l cases, one shows that if $\tilde\alpha_f = (f(\mathrm{X}_1, \mathrm{X}_1^{-1}\mathrm{X}_1^{-1}), f(\mathrm{X}_2, \mathrm{X}_1^{-1}\mathrm{X}_1^{-1}))$, then $\tilde\alpha_f^{2,3}\tilde\alpha_f^{\widetilde{1,23}}f(x_{12}, x_{23}) = \tilde\alpha_f^{1,2}\tilde\alpha_f^{\widetilde{12,3}}$ and $\tilde\alpha_{ff'} = \tilde\alpha_{f'}\tilde\alpha_f$.

Appendix B: $\mu_{\Phi,\mathrm{O}}$ and its Jacobian

B.1 *Telescopic formulas.* — If $\mathrm{O} \in \mathrm{Ob}(\mathbf{PaB})$ has the form $\mathrm{O} = \bullet \otimes \mathrm{O}'$, with $|\mathrm{O}'| = n$, then one proves by using (17) that μ_{O} expresses directly in terms of μ_Φ, for example

$$\mu_{\bullet((((\bullet\bullet)(\bullet\bullet))(\bullet(\bullet\bullet)))(\bullet\bullet))} = \mu_\Phi^{1234567,89}\mu_\Phi^{1234,567}\mu_\Phi^{8,9}\mu_\Phi^{12,34}\mu_\Phi^{5,67}\mu_\Phi^{1,2}\mu_\Phi^{3,4}\mu_\Phi^{6,7}.$$

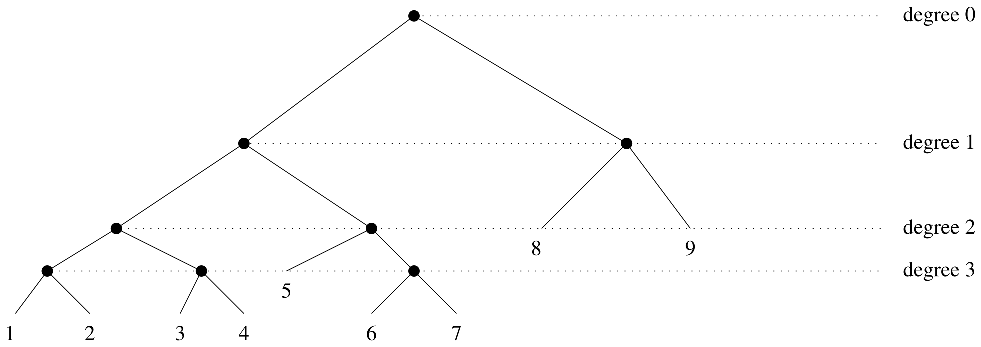

FIG. 4. — There are 8 nodes

The general formula is

$$\mu_{\bullet\otimes O'} = \prod_{n\geq 0} \prod_{\nu\in N(T'), d(\nu)=n} \mu_\Phi^{L(\nu),R(\nu)};$$

here T' is the binary planar rooted tree underlying O'; $N(T')$ is the set of its nodes; $d(\nu)$ is the degree of ν (distance to the root of the tree); $L(\nu)$, $R(\nu)$ is the set of left and right leaves of ν (these are disjoints subsets of $\{1, \dots, n\}$). The first product is taken according to increasing values of n (the order in the second product does not matter as the arguments of this product commute with each other). Here is the tree corresponding to the above example (Figure 4):

B.2 *Computation of Jacobians.* — Let $\mu_n := \mu_{\bullet(\bullet\dots(\bullet\bullet))}$. Then:

Proposition **46.** —$J(\mu_n) = \langle \sum_{i=1}^n \log\Gamma_\Phi(x_i) - \log\Gamma_\Phi(\sum_{i=1}^n x_i)\rangle$.

(We identified μ_n with its composition with $e^{x_i} \mapsto X_i$, which belongs to TAut_n.)

Proof. — We have $\mu_n = \mu_\Phi^{1,2\dots n} \circ \mu_\Phi^{2,3\dots n} \circ \dots \circ \mu_\Phi^{n-1,n}$. One then proves by descending induction on k that $J(\mu_\Phi^{k,k+1\dots n} \circ \dots \circ \mu_\Phi^{n-1,n}) = \langle \sum_{i=k}^n \log\Gamma_\Phi(x_i) - \log\Gamma_\Phi(\sum_{i=k}^n x_i)\rangle$, using the fact that the action of $\mu_\Phi^{k,k+1\dots n}$ on the various $\langle \log\Gamma_\Phi(x_i)\rangle$ as well as on $\langle \log\Gamma_\Phi(\sum_{i=k}^n x_i)\rangle$ is trivial. $\square$

If now $O \in \mathrm{Ob}(\mathbf{PaB})$ is arbitrary with $|O| = n+1$, then:

Proposition **47.** —$J(\mu_{\Phi,O}) = J(\mu_n) = \langle \sum_{i=1}^n \log\Gamma_\Phi(x_i) - \log\Gamma_\Phi(\sum_{i=1}^n x_i)\rangle$.

Proof. — We have $\mu_O = \mathrm{Ad}\,\Phi_{O_n,O} \circ \mu_n$, where $O_n = \bullet(\dots(\bullet\bullet))$. We then use the cocycle property of J, the above formula for $J(\mu_n)$, the fact that $J(\mathrm{Ad}\,g) = 0$ for $g \in \exp(\hat{\mathfrak{t}}_{n+1})$, and the following lemma:

Lemma **48.** — *If* $g \in \exp(\hat{\mathfrak{t}}_{n+1})$, *then* $(\mathrm{Ad}\,g)(x_1 + \dots + x_n)$ *is conjugate to* $x_1 + \dots + x_n$.

Proof of lemma. — Decompose $a \in \mathfrak{t}_{n+1}$ as $a_0 + a_1^{1,2,\ldots,n}$, with $a_0 \in \mathfrak{f}_n$ and $a_1 \in \mathfrak{t}_n$ (the map $a_1 \mapsto a_1^{1,2,\ldots,n}$ is the injection $\mathfrak{t}_n \to \mathfrak{t}_{n+1}$, $t_{ij} \mapsto t_{ij}$). Then $[t_{ij}, x_1 + \cdots + x_n] = 0$ for $i,j \in \{1, \ldots, n\}$, so $[a_1^{1,2,\ldots,n}, x_1 + \cdots + x_n] = 0$, so $[a, x_1 + \cdots + x_n] = [a_0, x_1 + \cdots + x_n]$. It follows that if $g \in \exp(\hat{\mathfrak{t}}_{n+1})$, there exists $x_g \in \exp(\hat{\mathfrak{f}}_n)$ such that $(\mathrm{Ad}\, g)(x_1 + \cdots + x_n) = x_g(x_1 + \cdots + x_n)x_g^{-1}$. □

Remark **49.** — In [AT2], the Lie subalgebra $\mathfrak{sder}_n \subset \mathfrak{tder}_n$ of special derivations (normalized special in the terms of Ihara) was introduced: $\mathfrak{sder}_n = \{u \in \mathfrak{tder}_n | u(x_1 + \cdots + x_n) = 0\}$. Let $\widetilde{\mathfrak{sder}}_n$ be the intermediate Lie algebra $\widetilde{\mathfrak{sder}}_n = \{u \in \mathfrak{tder}_n | \exists u_0 \in \mathfrak{f}_{n-1} | u(x_1 + \cdots + x_n) = [u_0, x_1 + \cdots + x_n]\}$ (special derivations in Ihara's terms). So $\mathfrak{sder}_n \subset \widetilde{\mathfrak{sder}}_n \subset \mathfrak{tder}_n$. Then Lemma 48 says that we have a diagram

$$\begin{array}{ccccc} \mathfrak{t}_n & \to & \mathfrak{sder}_n & & \\ \downarrow & & \downarrow & & \\ \mathfrak{t}_{n+1} & \to & \widetilde{\mathfrak{sder}}_n & \hookrightarrow & \mathfrak{tder}_n \end{array}$$

Remark **50.** — Set $\mathrm{SolKV}_n(\mathbf{k}) := \{\mu_n \in \mathrm{TAut}_n | \mu_n(e^{x_1} \cdots e^{x_n}) = e^{x_1 + \cdots + x_n}$ and $\exists r \in u^2\mathbf{k}[[u]] | \mathrm{J}(\mu_n) = \langle r(\sum_i x_i) - \sum_i r(x_i) \rangle\}$. This is a torsor under the action of the groups

$$\mathrm{KV}_n(\mathbf{k}) := \Big\{ \alpha_n \in \mathrm{TAut}_n | \alpha_n(e^{x_1} \cdots e^{x_n}) = e^{x_1} \cdots e^{x_n}$$
$$\text{and } \exists \sigma \in u^2\mathbf{k}[[u]] | \mathrm{J}(\alpha) = \Big\langle \sigma(\log e^{x_1} \cdots e^{x_n}) - \sum_i \sigma(x_i) \Big\rangle \Big\}$$

and $\mathrm{KRV}_n(\mathbf{k})$, which is similarly defined (replacing $e^{x_1} \cdots e^{x_n}$ by $e^{x_1 + \cdots + x_n}$). These are prounipotent groups; the Lie algebra of $\mathrm{KRV}_n(\mathbf{k})$ is $\mathfrak{krv}_n := \{u \in \mathfrak{tder}_n | a(\sum_i x_i) = 0$ and $\exists s \in u^2\mathbf{k}[[u]] | j(a) = \langle s(\sum_i x_i) - \sum_i s(x_i) \rangle\}$. It contains as a Lie subalgebra $\mathfrak{krv}_n^0 := \{a \in \mathfrak{krv}_n | s = 0\}$, which is denoted $\mathfrak{kv}_n$ in [AT2]. One can prove that if $|\mathrm{O}'| = n$ and $\mathrm{O} = \bullet \otimes \mathrm{O}'$, the map $\mathrm{M}_1(\mathbf{k}) \to \mathrm{SolKV}_n(\mathbf{k})$, $\Phi \mapsto \mu_{\Phi,\mathrm{O}}$ is a morphism of torsors.

Appendix C: Computation of a centralizer

In this section, we compute the centralizer of t_{ij} in $\mathfrak{t}_n$. This result is used in the proof of Theorem 30.

Proposition **51.** — *Let $i < j \in [n]$. If $x \in \mathfrak{t}_n$ is such that $[x, t_{ij}] = 0$, then there exists $\lambda \in \mathbf{k}$ and $y \in \mathfrak{t}_{n-1}$ such that $x = \lambda t_{ij} + y^{ij,1,2,\ldots,\check{i},\ldots,\check{j},\ldots,n}$.*

Proof. — We may and will assume that $i = 1, j = 2$. We then prove the result by induction on n. It is obvious when $n = 2$. Assume that it has been proved at step $n - 1$ and let us prove it at step n. We have $\mathfrak{t}_n = \mathfrak{t}_{n-1} \oplus \mathfrak{f}_{n-1}$, where $\mathfrak{t}_{n-1}$ is the Lie subalgebra generated by the t_{ij}, $i \neq j \in \{1, \ldots, n-1\}$ and $\mathfrak{f}_{n-1}$ is freely generated by the $t_{1n}, \ldots, t_{n-1,n}$. Both $\mathfrak{t}_{n-1}$ and $\mathfrak{f}_{n-1}$ are Lie subalgebras of $\mathfrak{t}_n$, stable under the inner derivation $[t_{12}, -]$. Then if $x \in \mathfrak{t}_n$ is such that $[t_{12}, x] = 0$, we decompose $x = x' + f$, with $x' \in \mathfrak{t}_{n-1}, f \in \mathfrak{f}_{n-1}$, $[t_{12}, x'] = [t_{12}, f] = 0$. By the induction hypothesis, we have $x' = \lambda t_{12} + (y')^{12,3,\ldots,n-1}$, where $y' \in \mathfrak{t}_{n-2}$ and $\lambda \in \mathbf{k}$.

Let us set $x_i = t_{in}$ for $i = 1, \ldots, n-1$. The derivation $[t_{12}, -]$ of $\mathfrak{f}_{n-1}$ is given by $x_1 \mapsto [x_1, x_2]$, $x_2 \mapsto [x_2, x_1]$, $x_i \mapsto 0$ for $i > 2$. In terms of generators $y_1 = x_1, y_2 = x_1 + x_2$, $y_3 = x_3, \ldots, y_{n-1} = x_{n-1}$, it is given by $y_1 \mapsto [y_1, y_2], y_i \mapsto 0$ for $i > 1$.

Lemma **52.** — *The kernel of the derivation $y_1 \mapsto [y_1, y_2], y_i \mapsto 0$ for $i > 1$ of $\mathfrak{f}_{n-1}$ coincides with the Lie subalgebra $\mathfrak{f}_{n-2} \subset \mathfrak{f}_{n-1}$ generated by $y_2, \ldots, y_{n-1}$.*

Proof of Lemma. — Let us prove that the kernel of the induced derivation of $\mathrm{U}(\mathfrak{f}_{n-1})$ is $\mathrm{U}(\mathfrak{f}_{n-2})$. We have a linear isomorphism $\mathrm{U}(\mathfrak{f}_{n-1}) \simeq \bigoplus_{k\geq 1} \mathrm{U}(\mathfrak{f}_{n-2})^{\otimes k}$, whose inverse takes $u_1 \otimes \cdots \otimes u_k$ to $u_1 y_1 u_2 y_1 \cdots y_1 u_k$. The derivation $[t_{12}, -]$ of $\mathrm{U}(\mathfrak{f}_{n-1})$ is then transported to the direct sum of the endomorphisms of $\mathrm{U}(\mathfrak{f}_{n-2})^{\otimes k}$

$$u \mapsto (y_2^{(2)} + \cdots + y_2^{(k)})u - u(y_2^{(1)} + \cdots + y_2^{(k-1)}) \tag{26}$$

(this is 0 of $k = 1$; $y_2^{(i)} = 1^{\otimes i-1} \otimes y_2 \otimes 1^{\otimes k-i}$; we make use of the algebra structure of $\mathrm{U}(\mathfrak{f}_{n-2})^{\otimes k}$). Each of these endomorphisms has degree 1 for the filtration of $\mathrm{U}(\mathfrak{f}_{n-2})^{\otimes k}$ induced by the PBW filtration of $\mathrm{U}(\mathfrak{f}_{n-2})$ (the part of degree $\leq d$ of $\mathrm{U}(\mathfrak{f}_{n-2})$ for this filtration consists of combinations of products of $\leq d$ elements of $\mathfrak{f}_{n-2}$) and the associated graded endomorphism of $\mathrm{S}(\mathfrak{f}_{n-2})^{\otimes k}$ is the multiplication by $y_2^{(k)} - y_2^{(1)}$, which is injective if $k \geq 1$, so (26) is injective for $k \geq 1$; the kernel of the direct sum of maps (26) therefore coincides with the degree 1 part $\mathrm{U}(\mathfrak{f}_{n-2})$, which transports to $\mathrm{U}(\mathfrak{f}_{n-2}) \subset \mathrm{U}(\mathfrak{f}_{n-1})$. So the kernel of the derivation $[t_{12}, -]$ of $\mathrm{U}(\mathfrak{f}_{n-1})$ is $\mathrm{U}(\mathfrak{f}_{n-2})$. The kernel of the derivation $[t_{12}, -]$ of $\mathfrak{f}_{n-1}$ is then $\mathfrak{f}_{n-1} \cap \mathrm{U}(\mathfrak{f}_{n-2}) = \mathfrak{f}_{n-2}$. $\square$

End of proof of Proposition 51. — It follows that f expresses as $\mathrm{P}(t_{1n} + t_{2n}, t_{3n}, \ldots, t_{n-1,n})$. Then if we set $f' := \mathrm{P}(t_{1,n-1}, \ldots, t_{n-2,n-1})$, we get $f = (f')^{12,3,\ldots,n}$ so $x = x' + f = \lambda t_{12} + ((y')^{1,2,\ldots,n-1} + f')^{12,3,\ldots,n}$, as wanted. $\square$

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Manuscrit reçu le 31 mars 2009
Version révisée le 5 août 2010
Manuscrit accepté le 10 octobre 2010
publié en ligne le 30 octobre 2010.