GEOMETRY OF KÄHLER METRICS AND FOLIATIONS BY HOLOMORPHIC DISCS

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Dedicated to Professor E. Calabi for his 80th birthday

CONTENTS

1. Introduction and main results

The purpose of this paper is to establish a completely new partial regularity theory on certain homogeneous complex Monge–Ampere (HCMA) equations. Our par-

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tial regularity theory will be obtained by studying foliations by holomorphic curves and their relations to homogeneous complex Monge–Ampere equations. As an application, we prove the uniqueness of extremal Kähler metrics in each Kähler class and prove a necessary condition for the existence of extremal Kähler metrics: Namely, the existence of constant scalar curvature (cscK) metric implies the existence of a uniform lower bound of the K energy functional which in turns implies the semi-K stability of the underlying polarization (in the algebraic case). Further applications will be discussed in our forthcoming papers.

1.1. *A brief tour of extremal Kähler metrics*

According to Calabi [4], a Kähler metric is called *extremal* if the complex gradient vector field of its scalar curvature is holomorphic. When this vector field vanishes, it is called constant scalar curvature Kähler (cscK) metric. It follows from the standard Hodge theory that any cscK metric must be Kähler–Einstein (KE) in canonical Kähler class.

In the 50's, E. Calabi proposed the problem of studying existence of Kähler–Einstein metrics on compact Kähler manifolds with definite first Chern class (We always use C_1 to denote the first Chern class in this paper.). In 1976, S. T. Yau solved this famous Calabi conjecture when $C_1 = 0$. Around the same time, both T. Aubin and S. T. Yau independently proved the existence of KE metric on compact Kähler manifolds with $C_1 < 0$. The remaining case is technically more involved. In [26], the second named author proved that, in any Fano Kähler surface with reductive automorphism group, there always exists a KE metric in the canonical Kähler class. For higher dimensional Kähler manifold, he proved in [27] that the existence of KE metrics in Fano manifold is equivalent to an analytic stability of the underlying Kähler manifold. It remains open how this analytic stability is related to certain algebraic stability from geometric invariant theory (cf. [27], [29], [12], [24], etc.). In [4], Calabi also asked if there always exists an extremal Kähler metric in any Kähler class. There has been extensive study on the issues related to the extremal metrics today. However, not much progress was made on the general existence of extremal metrics via direct PDE method. Very little is known even in Kähler surfaces. One possible reason is that the corresponding equation is highly nonlinear and it is of 6th order in full generalities.

On the other hand, there have been many partial results on the uniqueness of extremal metrics. Using maximum principle, E. Calabi observed in 50's that KE metric is unique when $C_1 \leq 0$. In [1], Bando and Mabuchi proved that KE metrics are unique modulo holomorphic automorphisms when $C_1 > 0$. In [30], X. H. Zhu and the second named author proved uniqueness of Kähler–Ricci solitons (KRS) on any Kähler manifolds with $C_1 > 0$. Following a suggestion of Donaldson, the first named author proved in [9] uniqueness for cscK metric in any Kähler class when $C_1 \leq 0$.

In [12], S. K. Donaldson proved that cscK metric is unique in any rational Kähler class on any projective manifold without non-trivial holomorphic vector fields[1].

In this paper, we prove

Theorem **1.1.1.** — *Let* $(M, [\omega])$ *be a compact Kähler manifold with a Kähler class* $[\omega] \in H^2(M, \mathbf{R}) \cap H^{1,1}(M, \mathbf{C})$. *Then there is at most one extremal Kähler metric with Kähler class* $[\omega]$ *modulo holomorphic transformations. Namely, if* ω_1 *and* ω_2 *are two extremal Kähler metrics with the same Kähler class, then there is a holomorphic transformation* σ *such that* $\sigma^* \omega_1 = \omega_2$.

In [20], T. Mabuchi introduced the K energy function: For any ϕ with $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$, set

$$\mathbf{E}_\omega(\phi) = -\int_0^1 \int_M \dot{\phi}(s(\omega_{\phi_t}) - \mu)\omega_{\phi_t}^n \wedge dt,$$

where ω_{ϕ_t} is any path in $[\omega]$ joining ω and ω_ϕ. Here we use $s(\omega_{\phi_t})$ to denote the scalar curvature and μ to denote its average. Then,

$$\mu = \frac{[C_1(M)] \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Theorem **1.1.2.** — *Let* $(M, [\omega])$ *be a compact Kähler manifold with a cscK metric. Then* $\mathbf{E}_\omega(\phi) \geq 0$ *for any* ϕ *with* $\omega_\phi > 0$.

Theorem 1.1.2 was proved first for KE metrics on Fano Kähler manifolds [1] (cf. [27]). It was first generalized by the first named author [9] to the case of cscK metric in any Kähler manifold with $C_1 \leq 0$[2]. This theorem can be also generalized to the case of extremal Kähler metrics if we modify the K energy function accordingly. It also answers partially a conjecture posed by the second named author earlier. Namely, $(M, [\omega])$ has a cscK metric in $[\omega]$ if and only if the K-energy is proper[3] in the space of Kähler metrics in $[\omega]$. Combining Theorem 1.1.2 with results in [29] and [24], we prove

Corollary **1.1.3.** — *Let* (M, L) *be a polarized algebraic manifold, that is,* M *is algebraic and* L *is a positive line bundle. If there is a cscK metric with Kähler class equal to* $c_1(L)$. *Then* (M, L) *is asymptotically K-semistable or CM-semistable in the sense of* [27] *(also see* [29]*)*[4].

[1] During preparation of this paper, we learned from T. Mabuchi that he is able to remove the assumption on non-existence of holomorphic vector fields in the special case of projective manifolds.

[2] After we finished the first draft of this paper, we learned that S. K. Donaldson proved this theorem in the case of projective manifolds without holomorphic vector fields [13]. His method is completely different from ours.

[3] A function is called proper if it dominates some suitable norm function on the Kähler potential.

[4] According to [24], the CM-stability (semistability) is equivalent to the K-stability (semistability).

1.2. *Space of Kähler metrics*

In next few subsections, we will explain main ideas to prove the first two theorems. First, let us briefly discuss a direct approach suggested by S. K. Donaldson. This method is used in [9] first by the first named author. It follows from Hodge theory that the space of Kähler metrics with Kähler class $[\omega]$ can be identified with the space of Kähler potentials

$$\mathcal{H}_\omega = \{\phi \mid \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0 \text{ on } \mathrm{M}\}/\sim.$$

Here $\phi_1 \sim \phi_2$ if and only if $\phi_1 = \phi_2 + c$ for some constant c. We will drop the subscript ω if no possible confusion may occur. A tangent vector in $\mathrm{T}_\phi\mathcal{H}_\omega$ is just a function ϕ_0 such that

$$\int_{\mathrm{M}} \phi_0 \omega_\phi^n = 0.$$

Its norm in the L^2-metric on $\mathcal{H}_\omega$ is given by (cf. [20])

$$\|\phi_0\|_\phi^2 = \int_{\mathrm{M}} {\phi_0}^2 \omega_\phi^n.$$

A straightforward computation shows that the geodesic equation of this L^2 metric is

$$\phi''(t) - \frac{1}{2}\langle d\phi', d\phi'\rangle_\phi = 0.$$

Here $\langle\cdot,\cdot\rangle_\phi$ denotes the natural inner product on $\mathrm{T}^*\mathrm{M}$ induced by the Kähler metric ω_ϕ, $\phi(t)$, $t \in [0,1]$ denotes a continuous path in $\mathcal{H}_\omega$; while ϕ', ϕ'' denote the partial derivatives of ϕ on variable t. Set $\phi(t,\theta,x) = \phi(t)(x)$ for any $t \in [0,1]$, any $\theta \in \mathrm{S}^1$ and any $x \in \mathrm{M}$. Then, the path $\{\phi(t)(t \in [0,1])\}$ represents a geodesic segment if and only if the function ϕ on $[0,1] \times \mathrm{S}^1 \times \mathrm{M}$ satisfies the HCMA equation

$$(\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\phi)^{n+1} = 0, \quad \text{on } \Sigma \times \mathrm{M}, \tag{1.1}$$

where $\Sigma = [0,1] \times \mathrm{S}^1$; $\pi_1 : \Sigma \times \mathrm{M} \mapsto \Sigma$ and $\pi_2 : \Sigma \times \mathrm{M} \mapsto \mathrm{M}$ are the natural projections.

In [11], Donaldson conjectured that the geodesic segment is always smooth between any two smooth Kähler potentials in $\mathcal{H}_\omega$. He also pointed out that the K-energy is convex along any smooth geodesic segment; moreover, both Theorem 1.1.1 and 1.1.2 would follow once his conjecture is established. However, this turns out to be a very difficult problem and it remains open until now. In fact, one can consider (1.1) over a general Riemann surface Σ with boundary condition $\phi = \phi_0$ along $\partial\Sigma$, where ϕ_0 is a smooth function on $\partial\Sigma \times \mathrm{M}$ such that $\phi_0(z,\cdot) \in \mathcal{H}_\omega$ for each $z \in \partial\Sigma$.[5]

[5] We often regard ϕ_0 as a smooth map from $\partial\Sigma$ into $\mathcal{H}_\omega$.

This solution has also natural geometric interpretation. Solution to (1.1) can be regarded as the infinite dimensional analogue of the WZW equation for maps from Σ into $\mathcal{H}_\omega$ (cf. [11]).[6] The following theorem was proved by the first named author in [9] which plays a fundamental role in this paper.

Theorem **1.2.1** [9]. — *For any smooth map* $\phi_0 : \partial\Sigma \to \mathcal{H}_\omega$*, there exists a unique* $C^{1,1}$ *solution* ϕ *of* (1.1) *such that* $\phi = \phi_0$ *on* $\partial\Sigma$ *and* $\phi(z,\cdot) \in \overline{\mathcal{H}_\omega}$ *for each* $z \in \Sigma$.[7]

The lack of sufficient regularity limits the geometric application of Theorem 1.2.1. Note that complex Monge–Ampere equations have been studied extensively by many famous authors (cf. [17], [18], [2], etc.). However, the regularity for homogeneous complex Monge–Ampere equations beyond $C^{1,1}$ has been missing in the vast literatures. Indeed, there are some setting that solutions to HCMA equation with smooth Dirichlet boundary data are only $C^{1,1}$. Here is a simple and well known example: Let Σ be the unit ball in $\mathbf{C}^2$ and define

$$u = \begin{cases} 0 & \text{if} \quad |z_1|^2, |z_2|^2 \le \frac{1}{2}; \\ \left(\frac{1}{2} - |z_1|^2\right)^2 & \text{if} \quad |z_1|^2 \ge \frac{1}{2}; \\ \left(\frac{1}{2} - |z_2|^2\right)^2 & \text{if} \quad |z_2|^2 \ge \frac{1}{2}; \end{cases}$$

then $(\partial\bar\partial u)^2 = 0$ on Σ and $u|_{\partial\Omega}$ is smooth, but u is only $C^{1,1}$. Note that the solution is unique in this case so that there is no hope to find solution with better regularity in general!

This example illustrates very powerfully that better regularity beyond $C^{1,1}$ is in general false. Nonetheless, we believe Donaldson's conjecture on smoothness of geodesic is likely to be correct because the rich geometry structure presented in this setting. One distinguished feature in our setting is that boundary map $\phi_0 : \partial\Sigma \to \mathcal{H}_\omega$ is strictly pluri-subharmonic along the boundary (modified by ω). This fact plays a crucial role in our approach.

1.3. *Partial regularity of HCMA equation*

The main task in this paper is to establish a partial regularity theory for HCMA equation (1.1) in the case that Σ is a unit disk in $\mathbf{C}$. To do this, we need to introduce notions of smoothness of different "degree." These different layers of smoothness are in one-one correspondence to various notions of smoothness of the moduli space of holomorphic discs which we will discuss in next subsection. In a way, we encourage the readers to read this two sections side by side. While there is extensive literatures

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[6] Original WZW equation is for maps from a Riemann surface into a Lie group.

[7] Here $\overline{\mathcal{H}_\omega}$ denotes the closure of $\mathcal{H}_\omega$ in any $C^{1,\alpha}(\Sigma \times M)$-topology ($\forall\, \alpha \in (0,1)$).

in the subject of HCMA equation, this seems to be the first partial regularity theory for HCMA equations.

Notations. — Suppose that ϕ is a $C^{1,1}$ solution of HCMA equation (1.1), we denote by $\mathscr{R}_\phi$ the set of all $(z, x) \in \Sigma \times M$ near which ϕ is smooth and $\omega_\phi|_{\{z\}\times M} > 0$. We call $\mathscr{R}_\phi$ the **regular part** of ϕ. It is open, but *a priori*, the regular part might be empty. In this regular part, we may introduce a distribution $\mathscr{D}_\phi \subset T(\Sigma \times M)$:

$$\mathscr{D}_\phi|_{(z,x)} = \{v \in T_z\Sigma \times T_xM \mid i_v(\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\phi) = 0\}, \quad (z, x) \in \mathscr{R}_\phi. \tag{1.2}$$

Here i_v denotes the interior product. Since the form is closed, $\mathscr{D}_\phi$ is integrable. Let $\mathscr{V}$ be any subset of $\Sigma \times M$. For our purpose in this paper, $\mathscr{V}$ usually denotes an open dense subset of $\Sigma \times M$. We say that $\mathscr{R}_\phi$ is **saturated** in $\mathscr{V}$ if every maximal integral sub-manifold of $\mathscr{D}_\phi$ in $\mathscr{R}_\phi \cap \mathscr{V}$ is a disk and relatively closed in $\mathscr{V}$. By nature of product manifold, we may write any vector in $\mathscr{D}_\phi$ as

$$\frac{\partial}{\partial z} + X \in \mathscr{D}_\phi|_{(z,x)}, \quad \text{where } X \in T_x^{1,0}M. \tag{1.3}$$

Definition **1.3.1.** — *A solution ϕ of* (1.1) *is called partially smooth if it satisfies the following three conditions*

1. *It has a uniform $C^{1,1}$-bounded on $\Sigma \times M$ and $\mathscr{R}_\phi$ is saturated in $\Sigma \times M$;*
2. *The regular part $\mathscr{R}_\phi \bigcap (\partial\Sigma \times M)$ is open and dense in $\partial\Sigma \times M$;*
3. *The varying volume form $\omega^n_{\phi(z,\cdot)}$ can be extended to $\Sigma^0 \times M$ as a continuous (n, n) form on $\Sigma^0 \times M$, where $\Sigma^0 = (\Sigma \backslash \partial\Sigma)$.*

Theorem **1.3.2.** — *Suppose that Σ is a unit disk. For every smooth map $\phi_0 : \partial\Sigma \to \mathscr{H}_\omega$, there exists a unique partially smooth solution to HCMA equation* (1.1).

Theorem 1.3.2 improves the regularity of the $C^{1,1}$ solution by the first named author in [9]: it must be smooth over some open subset of $\Sigma \times M$ which is also dense in $\partial\Sigma \times M$. We expect this density property holds in interior as well.

Set $\mathscr{S}_\phi = \Sigma \times M \backslash \mathscr{R}_\phi$.

Definition **1.3.3.** — *We say that a solution ϕ of HCMA equation* (1.1) *is almost smooth if*

1. *It is partially smooth.*
2. *The distribution $\mathscr{D}_\phi$ extends to a continuous distribution in an open dense and saturated set $\tilde{\mathscr{V}} \subset \Sigma \times M$, such that the complement $\tilde{\mathscr{S}_\phi}$ of $\tilde{\mathscr{V}}$ has Whitney extension property (WEP)*[8]. *The set $\tilde{\mathscr{S}_\phi}$ is referred as the* **singular part** *of ϕ.*
3. *The leaf vector field X is uniformly bounded in $\tilde{\mathscr{V}}$.*

[8] A closed subset $S \subset \Sigma \times M$ of measure 0 is *WEP* if for any continuous function in $\Sigma \times M$ which is $C^{1,1}$ on $\Sigma \times M \setminus S$ can be extended to a $C^{1,1}$ function on $\Sigma \times M$. Notice that any set of codimension 2 or higher is automatically has this property.

Note that $\mathscr{S} \setminus \tilde{\mathscr{S}}$ is in general not empty. The reason we don't want to refer $\mathscr{S}$ as singular part since the corresponding foliation (we will discuss in next subsection) might behave nicely in $\mathscr{S} \setminus \tilde{\mathscr{S}}$. A smooth solution is certainly an almost smooth solution of (1.1). For a sequence of almost smooth solutions whose boundary values converge in certain smooth topology, then it converges to a partially smooth solution in weak $C^{1,1}$-topology.

Theorem **1.3.4.** — *Suppose that* Σ *is a unit disk. For any* $C^{k,\alpha}$ *map* $\phi_0 : \partial\Sigma \to \mathscr{H}_\omega$ *(*$k \geq 2,\ 0 < \alpha < 1$*) and for any* $\epsilon > 0$, *there exists a* $\phi_\epsilon : \partial\Sigma \to \mathscr{H}_\omega$ *which admits an almost smooth solution to HCMA equation* (1.1) *with boundary value* ϕ_ϵ *such that*

$$\|\phi_0 - \phi_\epsilon\|_{C^{k,\alpha}(\partial\Sigma\times M)} < \epsilon.$$

This partial regularity result is sharp in light of the singular solution suggested in [12]. Using estimates developed in later sections, we can prove that for any sequence of almost smooth solutions whose boundary values converge in smooth $C^{k,\alpha}$ topology, then a subsequence will converge to a partially smooth solution in weak $C^{1,1}$-topology. Thus, Theorem 1.3.2 follows Theorem 1.3.4[9].

It is well known that the K energy function is convex along any smooth geodesic segment. In this paper, we generalize this to disc version geodesic solution: for any almost smooth solution to the disc version of geodesic equation, the K energy functional is sub-harmonic function when restricted to this disc family of Kähler metrics. More precisely, we have

Theorem **1.3.5.** — *Suppose that* ϕ *is a partially smooth solution to* (1.1)*. For every point* $z \in \Sigma$, *let* $\mathbf{E}_\omega(z)$ *be the K-energy (or modified K energy) evaluated at* $\phi(z,\cdot) \in \mathscr{H}_\omega$. *Then* $\mathbf{E}_\omega$ *is a bounded sub-harmonic function on* Σ *in the sense of distribution, moreover, we have the following*

$$\int_{\mathscr{R}_\phi} \left|\mathbf{D}\frac{\partial\phi}{\partial\bar{z}}\right|^2_{\omega_{\phi(z,\cdot)}} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \wedge \omega_{\phi(z,\cdot)}{}^n\, dz d\bar{z} \leq \int_{\partial\Sigma} \left.\frac{\partial\mathbf{E}_\omega}{\partial\mathbf{n}}\right|_{\partial\Sigma} ds,$$

where ds *is the length element of* $\partial\Sigma$ *and for any smooth function* Θ, $\mathbf{D}\Theta$ *denotes the* $(2,0)$*-part of* Θ*'s Hessian with respect to the metric* $\omega_{\phi(z,\cdot)}$. *The equality holds if* ϕ *is almost smooth.*

Theorem 1.1.2 follows from this theorem. The proof in smooth case is straightforward, but requires considerable care (comparing to proof of the convexity property of the K energy functional along a smooth geodesic segment). For readers' convenience, we include a proof of smooth case in Section 6.1.

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[9] It is believed that for any smooth boundary map $\phi_0 : \partial\Sigma \mapsto \mathscr{H}_\omega$, the corresponding $C^{1,1}$ solution is almost smooth. It is also interesting to estimate precise size of $\mathscr{S}_\phi$.

Proposition **1.3.6.** — *If there are two constant scalar curvature metrics (resp. two extremal Kähler metrics), then there exists a path in* $\overline{\mathcal{H}_\omega}$ *of* $C^{1,1}$*-functions* ϕ_t $(0 \le t \le 1)$ *which connects those two metrics, such that the K-energy (resp. modified K energy) achieves its minimum at every* ϕ_t *along the path.*

It was conjectured by the first named author that any $C^{1,1}$ minimizer of the K energy function must also be smooth (cf. [9], [8]). In this paper, we will confirm this conjecture in the case that the $C^{1,1}$ minimizers arises from Proposition 1.3.6. A key step is to prove a partial C^1-regularity for the varying volume form of any $C^{1,1}$ K-energy minimizer. Theorem 1.1.1 follows from this partial regularity result, Proposition 1.3.6 and Theorem 1.3.4.

The main part of this paper is devoted to proving technical results described here. We believe that our techniques developed in this paper can be applied to other more general setting when studying the regularity problem for HCMA equation.

1.4. *Ideas for proof of Theorem 1.3.4*

It has been known for a long time that solutions of the homogeneous complex Monge–Ampere equation are closely related to foliations by holomorphic curves (cf. [19], [25], [12]). In [25], S. Semmes formulated the Dirichlet problem for (1.1) in terms of a foliation by holomorphic curves with boundary in a totally real submanifold of the complex cotangent bundle of the underlying manifold.

Associated with each Kähler class $[\omega]$, S. Semmes [25] (cf. [12]) constructed a complex manifold $\mathcal{W}_{[\omega]}$ (locally it consists of pieces of T_*M) with a holomorphic (2,0)-form Θ. There is a natural projection $\pi : \mathcal{W}_{[\omega]} \mapsto M$ by simply forgetting the second component. He observed that for any $\phi \in \mathcal{H}_\omega$, we can associate a Lagrange sympletic submanifold Λ_ϕ in $\mathcal{W}_{[\omega]}$ such that

$$\Theta|_{\Lambda_\phi} = -\sqrt{-1}\omega_\phi, \tag{1.4}$$

that is, $\mathrm{Re}(\Theta)|_{\Lambda_\phi} = 0$ and $-\mathrm{Im}(\Theta)|_{\Lambda_\phi} = \omega_\phi > 0$. Locally, Λ_ϕ is simply the graph of $\partial(\rho + \phi)$ where $\omega = \sqrt{-1}\partial\bar\partial\rho$. This means that Λ_ϕ is an exact Lagrangian symplectic submanifold of $\mathcal{W}_{[\omega]}$ with respect to Θ. Conversely, given an exact Lagrangian symplectic submanifold Λ of $\mathcal{W}_{[\omega]}$, one can construct a smooth function ϕ such that $\Lambda = \Lambda_\phi$. Hence, Kähler metrics in the Kähler class $[\omega]$ are in one-to-one correspondence with exact Lagrangian symplectic submanifolds of $\mathcal{W}_{[\omega]}$.

Given $\phi_0 : \partial\Sigma \mapsto \mathcal{H}_\omega$, define

$$\bar\Lambda_{\phi_0} = \{(\tau, v) \in \partial\Sigma \times \mathcal{W}_{[\omega]} \mid v \in \Lambda_{\phi_0(\tau)}\}. \tag{1.5}$$

One can show that $\bar\Lambda_{\phi_0}$ is a totally real sub-manifold in $\Sigma \times \mathcal{W}_{[\omega]}$. So it makes sense to study the *moduli* space $\mathcal{M}_{\phi_0}$ of all holomorphic disks in $\Sigma \times \mathcal{W}_{[\omega]}$ with boundary in $\bar\Lambda_{\phi_0}$. Its significance is clear from the following result from [25] (also see [12]).

Proposition **1.4.1.** — *Assume that* Σ *is simply connected. For any boundary map* $\phi_0 : \partial\Sigma \to \mathcal{H}_\omega$*, there is a solution* $\phi \in C^\infty(\Sigma, \mathcal{H}_\omega)$ *of* (1.1) *with boundary value* ϕ_0 *if and only if there is a smooth family of holomorphic maps* $h_x : \Sigma \mapsto \mathcal{W}_{[\omega]}$ *parametrized by* $x \in M$ *satisfying: (1)* $\pi_2(h_x(z_0)) = x$*, where* z_0 *is a given point in* $\Sigma\backslash\partial\Sigma$*; (2)* $h_x(\tau) \in \Lambda_{\phi_0(\tau)}$ *for each* $\tau \in \partial\Sigma$ *and* $x \in M$*; (3) For each* $z \in \Sigma$*, the map* $\gamma_z(x) = \pi_2(h_x(z))$ *is a diffeomorphism of* M.

In [12], S. Donaldson used this fact to study deformations of smooth solutions for (1.1) as the boundary value varies. This inspired us to study foliations by holomorphic disks in order to have a partial regularity theory for (1.1). Theorem 1.3.4 will be proved by establishing existence of foliations by holomorphic disks with relatively mild singularities. More precisely, we will show that for a generic boundary value, there is an open set in the moduli space of holomorphic disks which generates a foliation on $\Sigma \times M\backslash S$ for a closed subset S of codimension at least one.

Now let us fix a generic boundary value ϕ_0 and study the corresponding *moduli* $\mathcal{M}_{\phi_0}$ of holomorphic disks. First it follows from the Index theorem that the expected dimension of this moduli is $2n$. Recall that a holomorphic disk u is regular if the linearized $\bar\partial$-operator $\bar\partial_u$ has vanishing cokernel. The moduli space is smooth near a regular holomorphic disk. Following [12], we call u super-regular if there is a basis $s_1, ..., s_{2n}$ of the kernel of $\bar\partial_u$ such that $d\pi(s_1)(x), ..., d\pi(s_{2n})(x)$ span $T_{u(x)}M$ for every $x \in \Sigma$, where $\pi : \Sigma \times \mathcal{W}_{[\omega]} \mapsto \Sigma \times M$ is the natural projection. We call u almost super-regular if $d\pi(s_1)(x), ..., d\pi(s_{2n})(x)$ span $T_{u(x)}M$ for every $x \in \Sigma\backslash\partial\Sigma$. Clearly, the set of super-regular disks is open.

Semmes and Donaldson consider only the case where the Moduli space is super regular in the sense of Donaldson and its relation to a smooth solution to HCMA equation (1.1). In order to establish a correspondence with the so called *almost smooth solution* to HCMA equation (1.1), we need introduce "nearly smooth foliation."

Definition **1.4.2.** — *A nearly smooth foliation* $\mathcal{F}_{\phi_0}$ *associated to a boundary value* ϕ_0 *is an open subset* $\mathcal{U}_{\phi_0}$ *of super-regular disks in* $\mathcal{M}_{\phi_0}$ *with properties described below. Let ev be the evaluation map from* $\mathcal{M}_{\phi_0}$ *to* $\Sigma \times \mathcal{W}_{[\omega]}$*. The collection of holomorphic discs*

$$\{\pi \circ ev(f) \mid f \in \mathcal{U}_{\phi_0}\}$$

foliated an open-dense set $\mathcal{V}_{\phi_0}$ *of* $\Sigma \times M$ *such that*

1. *This foliation can be extended to be a continuous foliation by holomorphic disks in an open set* $\tilde{\mathcal{V}}_{\phi_0} \subset \Sigma^0 \times M$ *such that it admits a continuous lifting in* $\Sigma \times \mathcal{W}_M$*;*
2. *The complement of* $\tilde{\mathcal{V}}_{\phi_0}$ *in* $\Sigma \times M$ *is WEP;*
3. *The leaf vector field (cf. definition below) induced by the foliation in* $\mathcal{V}_{\phi_0}$ *is uniformly bounded.*

We note that the name *nearly smooth foliation* is a bit misleading. It is not a foliation in the total space $\Sigma \times \mathcal{W}_{[\omega]}$, but a foliation by holomorphic discs for an open and dense subset in $\Sigma \times M$.

Definition **1.4.3.** — *For each $(z, x) \in \mathscr{V}_{\phi_0}$, the complex tangential direction of the image of holomorphic discs in $\mathscr{U}_{\phi_0}$ is called leaf vector field. It takes the form*

$$\frac{\partial}{\partial z} + \mathrm{X}, \quad \text{where } \mathrm{X} \in \mathrm{T}^{1,0}_{(z,x)}(\{z\} \times \mathrm{M})$$

Sometimes, we also call X *as the leaf vector field in $\mathscr{V}_{\phi_0}$.*

Proposition 1.4.1 has the following generalization.

Theorem **1.4.4.** — *Almost smooth solutions of* (1.1) *are in one-to-one correspondence with nearly smooth foliations. Moreover, if ϕ_0 is generic, the corresponding almost smooth solution ϕ has additional properties: ω_ϕ is a smooth* (1, 1) *form in $\Sigma \times \mathrm{M} \setminus \tilde{\mathscr{S}_\phi}$ and the singular set $\tilde{\mathscr{S}_\phi}$ has codimension at least 2 in each slice $\{z\} \times \mathrm{M}$, $\forall z \in \Sigma^0$.*[10]

Thus, in order to prove Theorem 1.3.4, we only need to show the following

Theorem **1.4.5.** — *For a generic boundary value ϕ_0, there is a nearly smooth foliation associated to ϕ_0 generated by an open set $\mathscr{U}_{\phi_0}$ of super regular discs in the moduli space $\mathscr{M}_{\phi_0}$. Moreover, the set of holomorphic disks which are neither super-regular nor almost super-regular has codimension at least two in the closure of $\mathscr{U}_{\phi_0}$ in $\mathscr{M}_{\phi_0}$.*

The idea for proving Theorem 1.4.5 is outlined as follows. Let ϕ_0 be a generic boundary value such that $\mathscr{M}_{\phi_0}$ is smooth. This follows from a result of Oh on transversality. By the same transversality argument, one can show that there is a generic path ϕ_t $(0 \leq t \leq 1)$ such that $\phi_1 = 0$ and the total moduli $\tilde{\mathscr{M}} = \bigcup_{t\in[0,1]} \mathscr{M}_{\phi_t}$ is smooth. Moreover, we may assume that $\mathscr{M}_{\phi_t}$ are smooth for all t except finitely many $t_1, \ldots, t_{\mathrm{N}}$ where the *moduli* space may have isolated singularities. It follows from Semmes and Donaldson's work – Proposition 1.4.1 that $\mathscr{M}_{\phi_1}$ has at least one connected component which gives a foliation for $\Sigma \times \mathrm{M}$. We want to show that this component will deform to a connected component of $\mathscr{M}_{\phi_0}$ which generates a nearly smooth foliation. Assume that ϕ is the unique $\mathrm{C}^{1,1}$-solution of (1.1) with boundary value ϕ_t for some $t \in [0, 1]$. Let f be any holomorphic disk in the connected component of $\mathscr{M}_{\phi_t}$ which generates the corresponding foliation.

Using the $\mathrm{C}^{1,1}$ bound on ϕ, one can have a uniform area[11] bound on holomorphic disks in $\mathscr{M}_{\phi_t}$. It follows from an extension of Gromov's compactness theorem that any sequence of such holomorphic disks has a subsequence which converges to a holomorphic disk together with possibly finitely many bubbles. These bubbles which

[10] The corresponding nearly smooth foliations have additional properties and will be called almost super-regular foliations (cf. Section 3.3).

[11] We actually calculate area of the image of disks in $\Sigma \times \mathrm{M}$.

occur in the interior are holomorphic spheres, while bubbles in the boundary might be holomorphic spheres or disks. We will show that no bubbles can actually occur. According to E. Calabi and X. X. Chen [6], this infinite dimensional space $\mathscr{H}_\omega$ is non-positively curved in the sense of Alexanderov. Heuristically speaking, we can exploit this curvature condition to rule out the existence of interior bubbles. One can also rule out boundary bubbles by using the non-positivity and totally real property of the boundary condition. Since there are no bubbles, the Fredholm index of holomorphic disks is invariant under the limiting process. This is an important fact needed in our doing deformation theory.

In order to get a nearly smooth foliation, we need to prove that the *moduli* space has an open set of super-regular holomorphic disks for each t. First we observe that the set of super-regular disks is open. Moreover, using the transversality arguments, one can show that for a generic path ϕ_t, the closure of all super-regular disks in each $\mathscr{M}_{\phi_t}$ is either empty or forms an irreducible component. This implies the openness. It remains to prove that each *moduli* has at least one super-regular disk. It is done by using capacity estimates and curvature estimate along super-regular holomorphic disks (cf. Sections 4 and 5 for details).

1.5. *Organization*

In Section 2, we establish the correspondence between homogeneous complex Monge–Ampere equations and foliations by holomorphic curves. The goal is to prove Theorem 1.4.4. The proof is based on a local version of Semmes' construction. Semmes's construction is global in nature and was rediscovered in Donaldson's work [12]. In Section 3, we show necessary transversality results. In particular, we show that the set of boundary values such that corresponding *moduli* space $\mathscr{M}$ induces an almost super regular foliation is generically open. In Section 4, we study the deformation of holomorphic disks arising from a smooth solution to a homogenous complex Monge–Ampere equation. This is a local theory which is used in Sections 2, 3 and later sections as well. In Section 5, we prove the set of boundary values such that corresponding *moduli* space $\mathscr{M}$ induces an almost super regular foliation is closed. This will be done by proving a volume ratio estimate via a capacity argument. In Section 6, we will prove that the K energy function is sub-harmonic when restricted to a disk family of almost smooth solutions, which in turns implies that the K energy function is always bounded from below. For readers' convenience, we will first give a proof of that the K energy is sub-harmonic in the case of smooth solutions. In Section 7, we derive a partial C^1-regularity for the vertical volume form of any $C^{1,1}$ K-energy minimizer. We need to introduce a notion of weak Kähler–Ricci flow to derive this *a priori* estimate. In Section 8, we prove the uniqueness result for extremal Kähler metrics.

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2. Foliations and the homogenous complex Monge–Ampere equation

In this section, we discuss the correspondence between homogeneous complex Monge–Ampere equations and foliations by holomorphic disks. We will prove Theorem 1.4.4.

2.1. *Semmes' construction*

In [25], Semmes associated a complex manifold $\mathscr{W}_{[\omega]}$ to each Kähler class $[\omega]$: Let $\{U_i, i \in \mathscr{I}\}$ be a covering of M such that $\omega|_{U_i} = \sqrt{-1}\partial\bar{\partial}\rho_i$, where $\mathscr{I}$ is an index set. For any $x = y \in U_i \cap U_j (i, j \in \mathscr{I})$, we identify $(x, v_i) \in T^*U_i$ with $(y, v_j) \in T^*U_j$ if $v_i = v_j + \partial(\rho_i - \rho_j)$. Then $\mathscr{W}_{[\omega]}$ consists of all these equivalence classes of $[x, v_i]$. There is a natural map $\pi : \mathscr{W}_{[\omega]} \mapsto T^*M$, assigning $(x, v_i) \in T^*U_i$ to $(x, v_i - \partial\rho_i)$. Then the complex structure on T^*M pulls back to a complex structure on $\mathscr{W}_{[\omega]}$. Moreover, there is also a canonical holomorphic 2-form Θ on $\mathscr{W}_{[\omega]}$, in terms of canonical local coordinates z_α, ξ_α $(\alpha = 1, \ldots, n)$ of T^*U_i, such that

$$\Theta = dz_\alpha \wedge d\xi_\alpha.$$

Now for any $\phi \in \mathscr{H}_\omega$, we can define a submanifold Λ_ϕ in $\mathscr{W}_{[\omega]}$: For any coordinate chart U on which ω can be written as $\sqrt{-1}\partial\bar{\partial}\rho$, we define $\Lambda_\phi|_U \subset \mathscr{W}_{[\omega]}$ to be the graph of $\partial(\rho + \phi)$ in T^*U. Clearly, this Λ_ϕ is independent of the choice of coordinate chart U. A straightforward computation shows

$$\Theta|_{\Lambda_\phi} = -\sqrt{-1}\omega_\phi, \tag{2.1}$$

that is, $\mathrm{Re}(\Theta)|_{\Lambda_\phi} = 0$ and $-\mathrm{Im}(\Theta)|_{\Lambda_\phi} = \omega_\phi > 0$. This means that Λ_ϕ is an exact Lagrangian symplectic submanifold of $\mathscr{W}_{[\omega]}$ with respect to Θ. Conversely, given an exact Lagrangian symplectic submanifold Λ of $\mathscr{W}_{[\omega]}$, we have a smooth function $\phi \in \mathscr{H}_{[\omega]}$

such that $\Lambda = \Lambda_\phi$. Hence, Kähler metrics with Kähler class $[\omega]$ are in one-to-one correspondence with exact Lagrangian symplectic submanifolds in $\mathscr{W}_{[\omega]}$.

This is discussed briefly in our introduction. We refer readers to both [25] and [12] for more details. For the readers' convenience, let us briefly explain the proof of Proposition 1.4.1. Let ϕ be a solution of (1.1) on $\Sigma \times M$ such that $\phi(z,\cdot) \in \mathscr{H}_\omega$ for any $z \in \Sigma$. Recall that there is an induced distribution $\mathscr{D}_\phi \subset T(\Sigma \times M)$ by

$$(\mathbf{2.2}) \qquad \mathscr{D}_\phi|_p = \left\{ v \in T_p(\Sigma \times M) \;\middle|\; i_v\left(\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\phi\right) = 0 \right\}, \quad p \in \Sigma \times M.$$

It is a holomorphic integrable distribution. If Σ is simply-connected and $\phi(z,\cdot) \in \mathscr{H}_\omega$ for each $z \in \Sigma$, then the leaf of $\mathscr{D}_\phi$ containing (z_0, x) is the graph of a holomorphic map $f_x : \Sigma \mapsto M$ with $f_x(z_0) = x$. If we write $f_x(z) = \sigma_z(x)$, we obtain a family of diffeomorphisms σ_z of M with $\sigma_{z_0} = \mathrm{Id}_M$. Now for any fixed z we have a Kähler form $\omega + \sqrt{-1}\partial\bar{\partial}\phi(z,\cdot)$ on M and hence a section $s_z : M \mapsto \mathscr{W}_{[\omega]}$ whose image is the exact Lagrangian symplectic graph $\Lambda_{\phi(z,\cdot)}$. Then $h_x(z) = s_z(f_x(z))$ and $\gamma_z(x) = f_x(z)$ as required. This process can be reversed. Since we have to carry out this reversed process in the proof of Theorem 1.4.4, we omit details here and refer the readers either to [25], [12] or to the next subsection if they are interested in the proof of the converse part of Proposition 1.4.1.

2.2. *Local uniqueness for HCMA equation* (1.1)

One of our crucial new development is that Semmes' arguments can be made local along super-regular holomorphic disks. In this subsection, we will first introduce the notion of compatible solutions to HCMA equation. We then prove the uniqueness of compatible solutions for (1.1) near any super-regular disk.

Given a boundary value ϕ_0 on $\partial\Sigma \times M$. Suppose that $\mathscr{F}_{\phi_0}$ is a nearly smooth foliation (cf. Definition 1.4.2). An open subset $\mathscr{O} \subset \Sigma \times M$ is called **saturated** with respect to $\mathscr{F}_{\phi_0}$ if the image of any super disc in $\mathscr{U}_{\phi_0} \subset \mathscr{F}_{\phi_0}$ intersects with $\mathscr{O}$, then it lies entirely in $\mathscr{O}$. A solution ϕ of (1.1) with boundary value ϕ_0 in an open subset $\mathscr{O} \subset \Sigma \times M$ is called **compatible** with this foliation $\mathscr{F}_{\phi_0}$ if

1. $\mathscr{O}$ is saturated with respect to $\mathscr{F}_{\phi_0}$;
2. $\omega + \sqrt{-1}\partial\bar{\partial}\phi(z,\cdot)$ is a family of Kähler metrics on $\mathscr{O} \cap (\{z\} \times M)$;
3. The kernel of $\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\phi$ lies in the distribution induced by $\mathscr{F}_{\phi_0}$.

Indeed, ϕ solves (1.1) with partial boundary value problem in $\mathscr{O}$. Sometimes, we refer this as **germ** of HCMA equation (1.1) associated to $\mathscr{F}_{\phi_0}$.

Theorem **2.2.1.** — *Two compatible solutions of* (1.1) *with respect to the nearly smooth foliation* $\mathscr{F}_{\phi_0}$ *coincides along the intersection of their domains.*

The novelty of this theorem is that two compatible solutions only agree partially along the boundary.

We will adopt the notations from previous sections. First we recall the integrable distribution

$$\mathscr{D}_\phi|_{(z,x)} = \left\{ v \in T_z\Sigma \times T_x M \;\middle|\; i_v\left(\pi_2^*\omega + \sqrt{-1}\partial\bar\partial\phi\right) = 0 \right\}, \quad (z,x) \in \mathscr{R}_\phi. \tag{2.3}$$

Here i_v denotes the interior product. Since $\mathscr{R}_\phi$ is saturated, every maximal integral submanifold of $\mathscr{D}_\phi$ in $\mathscr{R}_\phi$ is a disk and closed in $\Sigma \times M$.

Lemma **2.2.2.** — *For any $f \in \mathscr{U}_{\phi_0}$, suppose that $\mathscr{O}_f$ is a saturated open neighborhood of the image of the map $\pi \circ ev$ of f in $\Sigma \times M$. Suppose that ϕ_f is a solution of* (1.1) *on $\mathscr{O}_f$ compatible with $\mathscr{F}_{\phi_0}$. Then, for any $\tilde f \in \mathscr{U}_{\phi_0}$ near f such that the image of $\pi \circ ev(\tilde f)$ lies completely in $\mathscr{O}_f$, we have*

$$\frac{\partial^2}{\partial z \partial \bar z}(\phi_f(\pi \circ \tilde f(z))) = -|\partial(\pi \circ ev(\tilde f))|^2_\omega(\pi \circ ev(\tilde f(z))), \quad \forall z \in \Sigma, \tag{2.4}$$

and

$$\phi_f(\pi \circ ev(\tilde f(z))) = \phi_0(\pi \circ ev(\tilde f(z))), \quad \forall z \in \partial\Sigma. \tag{2.5}$$

This lemma implies that ϕ_f is uniquely determined by only the geometric conditions along the image of each leaf. Theorem 2.2.1 follows from this lemma.

Theorem 2.2.1 allows us to construct solution locally around each super regular discs first. Then patch them together to obtain a global solution of HCMA equation (1.1) in $\mathscr{V}_{\phi_0}$. We then have to argue that this is indeed a solution of (1.1) in $\Sigma \times M$. Even in the case of Donaldson and Semmes, this approach might also be interesting to go thorough.

2.3. *Almost smooth solutions $\Leftrightarrow$ nearly smooth foliations*

In this subsection, we establish the equivalence between almost smooth solutions of (1.1) and nearly smooth foliations. This generalizes Semmes' construction. We will adopt notations from previous subsections.

Proposition **2.3.1.** — *An almost smooth solution to* (1.1) *with boundary value ϕ_0 induces a nearly smooth foliation associated to ϕ_0.*

Proof. — Let ϕ be an almost smooth solution with boundary value ϕ_0 : $\partial\Sigma \mapsto \mathscr{H}_\omega$. For every point $(z,x) \in \mathscr{R}_\phi$, there is a unique holomorphic map $f \in \mathscr{M}_{\phi_0}$ whose corresponding map $\pi \circ f : \Sigma \mapsto \Sigma \times M$ passes through (z,x). The property that

$\mathscr{R}_\phi$ is saturated implies that $\pi \circ f$ is a holomorphic disk and extends to the boundary of $\Sigma \times M$. According to Donaldson [12], such a holomorphic disk is super-regular. All these super-regular disks from $\mathscr{R}_{\phi_0}$ give rise to this open set $\mathscr{U}_{\phi_0} \subset \mathscr{M}_{\phi_0}$. Here $\mathscr{R}_{\phi_0}$ corresponds to $\mathscr{V}_{\phi_0}$ in the definition of nearly smooth foliations; $\tilde{\mathscr{V}}$ in the definition of almost smooth solution is direct correspondence to $\tilde{\mathscr{V}}_{\phi_0}$ in the definition of nearly smooth conditions. The other two conditions of a nearly smooth foliation can be verified in a straightforward fashion as well. In other words, an almost smooth solution indeed induces a nearly smooth foliation $\mathscr{F}_{\phi_0}$. □

Theorem 1.4.4 follows from the above proposition and the following.

Theorem **2.3.2.** — *If $\mathscr{F}_{\phi_0}$ is a nearly smooth foliation (cf. Definition 1.4.2) associated to a boundary value $\phi_0 : \partial\Sigma \to \mathscr{H}_\omega$, then there is an almost smooth solution ϕ to* (1.1) *with boundary value ϕ_0.*

The rest of this subsection is devoted to prove this theorem. Let $\mathscr{U}_{\phi_0}$ be the open subsets of $\mathscr{M}_{\phi_0}$ and $\mathscr{V}_{\phi_0}$ be an open dense subset of $\Sigma \times M$ foliated by image of the map $\pi \circ ev$ of all holomorphic discs in $\mathscr{U}_{\phi_0}$. By definition, the induced foliation in $\mathscr{V}_{\phi_0}$ can be extended to be a continuous foliation by holomorphic disks in an open and dense subset $\tilde{\mathscr{V}}_{\phi_0}$ such that it admits a continuous lifting to $\Sigma \times \mathscr{W}_M$. Moreover, $\Sigma \times M \setminus \tilde{\mathscr{V}}_{\phi_0}$ has WEP.

Proposition **2.3.3.** — *There is a smooth family of non-degenerated, closed* $(1,1)$ *forms* $\tilde{\omega}(z,\cdot)$ *defined on $\mathscr{V}_{\phi_0} \cap (\{z\} \times M)$ and a closed* $(1,1)$ *form Ω in $\mathscr{V}_{\phi_0}$ such that*

1. $\tilde{\omega} = \omega_{\phi_0}$ in $\partial\Sigma \times M$, wherever $\tilde{\omega}$ is defined;
2. The restriction of $\tilde{\omega}$ to each leaf is a constant form;
3. Ω is defined by the following conditions:

$$\Omega|_{\{z\}\times M} = \tilde{\omega}, \quad \text{and} \quad i_{\frac{\partial}{\partial z}+X}\Omega = 0,$$

where X is the leaf vector field in $\mathscr{V}_{\phi_0}$ induced by the nearly smooth foliation (cf. Definition 1.4.3).

Proof. — This is a local theorem. The proof can be found in [25], [12]. □

Next we want to show that $\Omega = \pi_2^*\omega_0 + i\bar{\partial}\partial\phi$ for a *compatible* solution ϕ of HCMA equation (1.1) in $\Sigma \times M$. We want to construct this potential function for any small open and saturated neighborhood around the image of any super regular disc in $\mathscr{U}_{\phi_0}$. Theorem 2.2.1 implies that two different locally defined *compatible* solutions of (1.1) must agree with each other on the overlap of their domains of definition. Since $\mathscr{V}_{\phi_0}$ is dense in $\Sigma \times M$, this defines ϕ in $\Sigma \times M$ by taking limit. The final step is to show that ϕ is uniformly $C^{1,1}$ in $\mathscr{V}_{\phi_0}$ – therefore it solve HCMA equation (1.1) globally with appropriate boundary condition. This step is not needed if we start from an super regular moduli space.

Proposition **2.3.4.** — *For any super regular leaf f, there exists a smooth function ϕ defined in a small tubular neighborhood $\mathscr{O}_f$ (which is saturated with respect to $\mathscr{F}_{\phi_0}$) of $\pi \circ ev(f) \subset \Sigma \times M$ such that*

$$\Omega = \pi_2^* \omega_0 + i\bar{\partial}\partial\phi, \quad \text{on } \mathscr{O}_f \subset \Sigma \times M \tag{2.6}$$

$$\phi = \phi_0, \quad \text{on } \mathscr{O}_f \cap (\partial\Sigma \times M). \tag{2.7}$$

Remark **2.3.5.** — The potential function ϕ can be also defined in a tubular neighborhood $\mathscr{O}_f$ of an almost super regular leaf f. The function is smooth in $\mathscr{O}_f \cap (\Sigma^0 \times M)$.

Proof. — Recall that $M = \bigcup_{i \in \mathscr{I}} U_i$ and $\rho_i (i \in \mathscr{I})$ is the local defining potential function for ω_0. For any point $(z, x) \in \mathscr{V}_{\phi_0}$, suppose that $x \in U_i$ for some $i \in \mathscr{I}$. In local coordinates, write

$$\omega_0 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \rho_i}{\partial w_\alpha \partial w_{\bar{\beta}}} dw^\alpha \wedge dw^{\bar{\beta}}, \quad \tilde{\omega} = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 (\rho_i + \phi)}{\partial w_\alpha \partial \bar{w}_\beta} dw^\alpha \wedge dw^{\bar{\beta}}.$$

We can express the image of Σ as[12]

$$\{(z, x = f(z), \xi(x)), \quad \forall z \in \Sigma\},$$

where

$$\xi(x) = \partial(\phi + \rho_i).$$

Since $\xi(x)$ is uniquely determined by image of an open set of super regular disks in $\Sigma \times \mathscr{W}_M$, then ϕ is uniquely determined by ξ, or by the structure of $\mathscr{W}_M$, up to a constant in $U_i \subset M$. In particular, in $(\partial\Sigma \times M) \cap \mathscr{V}_{\phi_0}$, we have $\phi = \phi_0$ modular some function in z locally.

By definition of the closed $(1, 1)$ form Ω, we may write

$$\Omega = \omega_0 + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \phi}{\partial w_\alpha \partial \bar{w}_\beta} dw_\alpha d\bar{w}_\beta + \sum_{\alpha=1}^{n} \zeta^\alpha dw_\alpha d\bar{z} + \sum_{\beta=1}^{n} \zeta^{\bar{\beta}} dz dw_{\bar{\beta}} + h_f dz d\bar{z}.$$

[12] Here $x = \pi_2 \circ \pi \circ f(z)$ in the formula. However, for notation simplicity, we simplify it as $x = f(z)$. This convention will be used later. This should cause no confusion.

The goal is to show first that

$$\zeta^\alpha = \frac{\partial^2\phi}{\partial\bar{z}\partial w_\alpha}, \quad \forall\alpha\in[1,n].$$

Since

$$i_{\frac{\partial}{\partial z}+\mathrm{X}}\Omega = 0 \quad \text{and} \quad \mathrm{X} = \sum_1^n \eta^\alpha \frac{\partial}{\partial w^\alpha},$$

we have

$$\zeta^\alpha + \left(g_{0,\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial w^\alpha \partial w^{\bar{\beta}}}\right)\eta^{\bar{\beta}} = 0, \quad \forall\alpha = 1,2,\dots,n.$$

Since $\xi(f(z))$ is a holomorphic function of z, we have

$$\begin{aligned} 0 &= \frac{\partial\xi^\alpha}{\partial\bar{z}} \\ &= \frac{\partial^2\phi}{\partial w^\alpha\partial\bar{z}} + \frac{\partial^2(\phi+\rho_i)}{\partial w^\alpha\partial w^{\bar{\beta}}}\frac{\partial f^{\bar{\beta}}}{\partial\bar{z}} \\ &= \frac{\partial^2\phi}{\partial w^\alpha\partial\bar{z}} + \left(g_{0,\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial w^\alpha\partial w^{\bar{\beta}}}\right)\eta^{\bar{\beta}}. \end{aligned}$$

Then

$$\zeta^\alpha = \frac{\partial^2(\phi+\rho_i)}{\partial w^\alpha\partial\bar{z}}, \quad \forall\alpha = 1,2,\dots,n.$$

Consequently, Ω takes the form:

$$\begin{aligned} \omega_0 &+ \sum_{\alpha,\beta=1}^n \frac{\partial^2\phi}{\partial w_\alpha\partial\bar{w}_\beta}dw_\alpha d\bar{w}_\beta + \sum_{\alpha=1}^n \frac{\partial^2\phi}{\partial\bar{z}\partial w_\alpha}dw_\alpha d\bar{z} \\ &+ \sum_{\beta=1}^n \frac{\partial^2\phi}{\partial z\partial w_{\bar{\beta}}}dz dw_{\bar{\beta}} + h_f dz d\bar{z}. \end{aligned}$$

By choosing an appropriate gauge in z direction, we claim that

$$h_f = \frac{\partial^2\phi}{\partial z\partial\bar{z}}.$$

Note that we can not change the value of h_f as we will see soon below. However, we can modify ϕ by some functions of z to make the above equation holds: first locally along a super regular leaf; then, globally along a super regular leaf.

Recalled that

$$\frac{\partial^2 \phi}{\partial w^\alpha \partial \bar{z}} = -\left(g_{0,\alpha\bar{\beta}} + \frac{\partial^2 \phi}{\partial w^\alpha \partial w^{\bar{\beta}}} \right) \eta^{\bar{\beta}}, \quad \forall \alpha = 1, 2, ..., n.$$

Using this equation and the fact that $\Omega^{n+1} = 0$, we have

$$\begin{aligned} 0 &= \left(h_f - g_\varphi^{\alpha\bar{\beta}} \cdot \frac{\partial^2 \phi}{\partial w^\alpha \partial \bar{z}} \cdot \frac{\partial^2 \phi}{\partial w^{\bar{\beta}} \partial z} \right) \cdot \tilde{\omega}^n \\ &= \left(h_f - \left(g_{0,\alpha\bar{\beta}} + \frac{\partial^2 \phi}{\partial w^\alpha \partial w^{\bar{\beta}}} \right) \eta^{\bar{\beta}} \eta^\alpha \right) \cdot \tilde{\omega}^n. \end{aligned}$$

Since $\tilde{\omega}$ is non-degenerate in $\mathcal{O}_f$, then

$$h_f = \left(g_{0,\alpha\bar{\beta}} + \frac{\partial^2 \phi}{\partial w^\alpha \partial w^{\bar{\beta}}} \right) \eta^{\bar{\beta}} \eta^\alpha$$

is uniquely determined as well.

However (note $z = w_{n+1}$),

$$\Omega - \sum_{\alpha,\beta=1}^{n+1} \sqrt{-1} \frac{\partial^2 (\rho_i + \phi)}{\partial w_\alpha \partial w_{\bar{\beta}}} dw_\alpha dw_{\bar{\beta}}$$

is a closed form. Consequently,

$$l_f dz d\bar{z} = \left(h_f - \frac{\partial^2 \phi}{\partial z \partial \bar{z}} \right) dz d\bar{z}$$

is a closed form on $\mathcal{O}_f$. Therefore, l_f is a function of z only. Locally, we can replace ϕ by $\phi + \mathrm{K}_f(z)$ where

$$\frac{\partial^2 \mathrm{K}_f}{\partial z \partial \bar{z}} = -l_f.$$

After such a replacement, in each local coordinate chart in $\mathcal{O}_f$, we can choose the potential function ϕ uniquely, up to a harmonic function on z only. This follows from the fact that

$$\frac{\partial^2 \phi}{\partial z \partial \bar{z}} = h_f$$

where h_f is uniquely determined by geometric data of super regular disks in $\mathcal{O}_f$.

Because of the unique extension property of harmonic functions, ϕ is uniquely determined in $\mathcal{O}_f$ by a global harmonic function in the z direction. Choose such a potential function in $\mathcal{O}_f$ now. Notice that in $(\partial\Sigma \times \mathrm{M}) \cap \mathcal{O}_f$, we have $\omega_z|_{\partial\Sigma} = \omega_{\phi_0}$. Then, we can set

$$\phi(z, \cdot) = \phi_0(z, \cdot) + m_f(z), \quad \forall z \in \partial\Sigma$$

where $m_f(z)$ is a function of z in Σ. Note that m_f may not be harmonic function.

Choose a function K_{ff} as a function of z only such that

$$\frac{\partial^2 K_{ff}}{\partial z \partial \bar{z}} = 0$$

and

$$K_{ff}|_{\partial\Sigma} = m_f.$$

This Dirichlet problem has a unique solution. Now replace ϕ by $\phi - K_f$. Then Ω can be re-written as $\pi_2^*\omega_0 + \sqrt{-1}\bar{\partial}\partial\phi$ in a tubular neighborhood of $\pi \circ f(\Sigma)$ in $\Sigma \times M$ such that $\phi = \phi_0$ in $(\partial\Sigma \times M) \cap \mathcal{O}_f$. □

Now ϕ satisfies (1.1) on $\mathcal{V}_{\phi_0}$. Now we wish to extend it to solve the same equation in $\Sigma \times M$. The key step is to prove a global $C^{1,1}$ bound for ϕ in $\mathcal{V}_{\phi_0}$. The first step is to prove the positivity of $\tilde{\omega}$. By the proceeding proposition, we can really denote

$$\tilde{\omega} = \omega_\phi.$$

Proposition **2.3.6.** — *As a closed* $(1, 1)$ *form in* M *direction, we have* $\omega_\phi > 0$ *when restricted to* M *in* $\mathcal{V}_{\phi_0}$.

Proof. — For any point $(z, x') \in \mathcal{V}_{\phi_0}$, there exists a holomorphic leaf $f \in \mathcal{U}_{\phi_0}$ such that $\pi \circ ev(z, f) = (z, x')$. For any $z \in \partial\Sigma$,

$$\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_0(z, \cdot) > 0.$$

In other words, $\omega_\phi|_{\pi\circ ev(\partial\Sigma, f)} > 0$. However, $\omega_\phi|_{\pi\circ ev(\Sigma, f)}$ is a constant form. Thus, ω_ϕ is strictly positive for any $(z, x') \in \mathcal{V}_{\phi_0}$. □

It follows from this proposition that ω_ϕ defines a smooth Kähler metric along M direction in $\mathcal{V}_{\phi_0}$. Next we want to show that this metric has a uniform L^∞ bound in $\mathcal{V}_{\phi_0}$. In a local coordinate chart, write

$$\begin{cases} \omega_0 = \sum_{\alpha,\beta=1}^{n} g_{0,\alpha\bar{\beta}} dw^\alpha dw^{\bar{\beta}}, & \omega_\phi = \sum_{\alpha,\beta=1}^{n} g_{\phi,\alpha\bar{\beta}} dw^\alpha dw^{\bar{\beta}}, \\ g_{\phi,\alpha\bar{\beta}} = g_{0,\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial w^\alpha \partial w^{\bar{\beta}}}, & \forall \alpha, \beta = 1, 2, \ldots, n. \end{cases}$$

For any super regular leaf f and for any $z \in \Sigma$, the restricted bundle $T^{1,0}_{f(z)}M$ at $f(z)$ is a trivial holomorphic bundle over Σ with complex rank n. Restriction of g_ϕ to this $T^{1,0}M$ bundle induces a Hermitian metric on this bundle. Denote by $F^\alpha_\beta (1 \leq \alpha, \beta \leq n)$ the curvature of this Hermitian metric. We have the following formula (cf. Section 4),

$$F^r_\alpha = -\partial_{\bar{z}}\left(g_\phi^{r\bar{\delta}}\partial_z g_{\phi,\alpha\bar{\delta}}\right) \tag{2.8}$$

$$= -\frac{\partial\eta^r}{\partial w_{\bar{i}}} \cdot \frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} \leq 0. \tag{2.9}$$

In particular, the curvature is always semi-negative. Now we are ready to state the maximum principle.

Proposition **2.3.7** *(Maximum principle along leaves). — The Kähler metric g_ϕ is uniformly bounded from above in each leaf in $\mathcal{U}_{\phi_0}$.*

Proof. — For any $f \in \mathcal{U}_{\phi_0}$, the restricted $T^{1,0}M$ bundle on $\pi \circ f(\Sigma)$ is a trivial holomorphic vector bundle over Σ. For any (z, p) in this leaf, pick any n-frame $s_1, s_2, \ldots, s_n \in T_p^{1,0}M$ and extend these vectors over the disk as a frame of n holomorphic sections in this $T^{1,0}M$ bundle. We still denote as $\{s_1, s_2, \ldots, s_n\}$. For any $c = (c_1, c_2, \ldots, c_n) \in \mathbf{C}^n$, define a section in $T^{1,0}M$ bundle as

$$s(c) = \sum_{i=1}^{n} c_i s_i.$$

It is easy to see that

$$\inf_{c \in \mathbf{C}^n} \inf_{z \in \Sigma} g_0(s(c), s(c)) > c_0(f) \cdot \|c\|^2, \quad \text{where } \|c\|^2 = \sum_{i=1}^{n} |c_i|^2$$

and $c_0(f)$ depends on g_0 and the embedding of f only. To prove the maximum principle for metric g_ϕ along the leaf, we just need to show that $g_\phi(s(c), s(c))(1 \leq \alpha \leq n)$ has a uniform upper bound for any $c \in \mathbf{C}^n$. If the upper bound is achieved on the boundary, then the claim is proved since $g_\phi = g_{\phi_0}$ in $\partial\Sigma \times M$. If the maximum is attained at some interior point $(z, p) \in \Sigma^0 \times M$, choose an appropriate coordinate in T_pM. We may assume

$$g_{\phi,\alpha\bar\beta}(z, p) = \delta_{\alpha\beta}(\forall \alpha, \beta = 1, 2, \ldots, n), \quad \partial_z g_\phi|_{(z,p)} = \bar\partial_z g_\phi|_{(z,p)} = 0.$$

At this point (z, p), we have

$$\begin{aligned}\bar\partial_z \partial_z g_\phi(s(c), s(c)) &= g_\phi(\partial_z s(c), \partial_z s(c)) + (\bar\partial_z \partial_z g_\phi)(s(c), s(c)) \\ &= g_\phi(\partial_z s(c), \partial_z s(c)) - F(s(c), s(c)) \geq 0.\end{aligned}$$

This shows that the maximum must achieve in the boundary. □

Using this proposition, we can prove

Theorem **2.3.8.** — *There exists a uniform upper bound for the Kähler metric g_ϕ in $\mathcal{V}_{\phi_0}$.*

Proof. — To prove our theorem, we need to show that

$$\sup_{c \in \mathbf{C}^n} \sup_{\bar{\mathcal{U}}_{\phi_0}} \frac{g_\phi(s(c), s(c))}{g_0(s(c), s(c))} \leq C$$

for some uniform constant C. However,

$$\inf_{c\in\mathbf{C}^n}\inf_{\bar{\mathscr{U}}_{\phi_0}} g_0(s(c), s(0)) > c_0(g_0, \bar{\mathscr{U}}_{\phi_0})\cdot\|c\|^2$$

where c_0 is a constant depending only on the *moduli* space and g_0. Thus, it is sufficient to prove the following

$$\sup_{c\in\mathbf{C}^n}\sup_{\bar{\mathscr{U}}_{\phi_0}} g_\phi(s(c), s(c)) \le \mathrm{C}\cdot\|c\|^2.$$

Now, $g_\phi = g_{\phi_0}$ in the boundary and the maximum principle along the leave implies the existence of the upper bound C. Our theorem is then proved. □

Finally, we have

Theorem **2.3.9.** — *ϕ is uniformly* $\mathrm{C}^{1,1}$ *in* $\Sigma\times\mathrm{M}$*, smooth in $\mathscr{V}_{\phi_0}$ such that it solves*

$$\left(\pi_2^*\omega_0+\sqrt{-1}\partial\bar\partial\phi\right)^{n+1}=0,\quad\text{in}\quad\Sigma\times\mathrm{M}. \tag{2.10}$$

Moreover, this solution coincides with the solution established in [9].

Proof. — We already know that ϕ is smooth in $\mathscr{V}_{\phi_0}$ and has uniformly $\mathrm{C}^{1,1}$ upper bound. Note that all holomorphic discs which lie in $\mathscr{V}_{\phi_0}$ can be continuously lifted to $\Sigma\times\mathscr{W}_\mathrm{M}$ over $\tilde{\mathscr{V}}_{\phi_0}$. In view of the definition of $\mathscr{W}_\mathrm{M}$, this amounts to saying that $\partial\phi|_\mathrm{M}$ is continuous in $\tilde{\mathscr{V}}_{\phi_0}$. More importantly, (2.4) and (2.5) make perfect sense in $\tilde{\mathscr{V}}_{\phi_0}$ which force the Kähler potential ϕ to be a continuous function in $\tilde{\mathscr{V}}_{\phi_0}$. In particular, this also implies that $\partial_z\phi$ is a continuous function in $\tilde{\mathscr{V}}_{\phi_0}$. Note that

$$\partial_z\phi=\frac{\partial\phi}{\partial z}+\mathrm{X}^\alpha\frac{\partial\phi}{\partial w^\alpha},$$

where X is the leaf vector field. Thus, $\frac{\partial\phi}{\partial z}$ is continuous in $\tilde{\mathscr{V}}_{\phi_0}$. since $\partial\phi|_\mathrm{M}$ and X are continuous in $\tilde{\mathscr{V}}_{\phi_0}$. Consequently, ϕ is a C^1 continuous function in $\tilde{\mathscr{V}}_{\phi_0}$. Since $\tilde{\mathscr{S}}_\phi = \Sigma\times\backslash\tilde{\mathscr{V}}_{\phi_0}$ has WEP and ϕ is uniformly $\mathrm{C}^{1,1}$ bounded in $\mathscr{V}_{\phi_0}$, then ϕ can be extended as a global $\mathrm{C}^{1,1}$ function in $\Sigma\times\mathrm{M}$. It follows that there is a sequence of Kähler potential $\{\phi_m\in\mathscr{H}_\omega, m\in\mathbf{N}\}$ such that $\phi_m\to\phi$ in weak $\mathrm{C}^{1,1}(\Sigma\times\mathrm{M})$ topology. Moreover, the convergence is smooth in any compact subset of $\mathscr{V}_{\phi_0}$. Consequently, for any test function ψ, we have

$$\lim_{m\to\infty}\int_{\Sigma\times\mathrm{M}}\psi\omega_{\phi_m}^{n+1}=0.$$

Suppose φ is the $C^{1,1}$ solution given by the first author in [9], we have

$$\begin{aligned}
0 &= \lim_{m\to\infty}\int_{\Sigma\times M}(\varphi-\phi)\big(\omega_{\varphi}^{n+1}-\omega_{\phi_m}^{n+1}\big)\\
&= \lim_{m\to\infty}\int_{\Sigma\times M}(\varphi-\phi)(\omega_{\varphi}-\omega_{\phi_m})\Big(\sum_{i=0}^{n}\omega_{\varphi}^{i}\wedge\omega_{\phi_m}^{n-i}\Big)\\
&= -\lim_{m\to\infty}\int_{\Sigma\times M}\sqrt{-1}\partial(\varphi-\phi)\wedge\bar\partial(\varphi-\phi_m)\Big(\sum_{i=0}^{n}\omega_{\varphi}^{i}\wedge\omega_{\phi_m}^{n-i}\Big)\\
&= -\int_{\mathscr{V}_{\phi_0}}\sqrt{-1}\partial(\varphi-\phi)\wedge\bar\partial(\varphi-\phi)\Big(\sum_{i=0}^{n}\omega_{\varphi}^{i}\wedge\omega_{\phi}^{n-i}\Big).
\end{aligned}$$

It follows that $\phi=\varphi$ in $\mathscr{V}_{\phi_0}$ since ω_ϕ is smooth in $\mathscr{V}_{\phi_0}$. Since $\mathscr{V}_{\phi_0}$ is dense in $\Sigma\times M$, it follows that the solution we constructed coincide with solution established in the first author's paper [9]. □

3. Deformation of holomorphic disks with totally real boundary

3.1. *Local analysis of holomorphic disks*

For any boundary map $\phi_0:\partial\Sigma\to\mathscr{H}$, there is a $2n+1$-dimensional totally real submanifold $\bar\Lambda_{\phi_0}=\bigcup_{z\in\partial\Sigma}(\{z\}\times\Lambda_{\phi_0(z,\cdot)})$ in $\Sigma\times\mathscr{W}_M$. Consider the *moduli* space $\mathscr{M}_{\phi_0}$ of all of holomorphic disks

$$\rho:(\Sigma,\partial\Sigma)\to(\Sigma\times\mathscr{W}_M,\bar\Lambda_{\phi_0})$$

with vanishing normal *Maslov* index. In this section, we will use term "holomorphic map" or "(holomorphic) disc" interchangeably when no confusion arisen.

It is well known that the normal bundle over $\rho(\Sigma)$ in $\Sigma\times\mathscr{W}_M$ is always trivial and we will denote it by

$$\begin{pmatrix}\mathbf{C}^{2n}\\ \big\downarrow\pi\\ \rho(D)\end{pmatrix}.$$

For any $z=e^{i\theta}(0\le\theta\le 2\pi)$, let $\mathbf{R}^{2n}(e^{i\theta})$ be the totally real subspace $\rho^*T_{\rho(e^{i\theta})}(\bar\Lambda_{\phi_0})$ of $T_{\rho(e^{i\theta})}\mathbf{C}^{2n}=\mathbf{C}^{2n}$. Consider all $H^{1,2}$-sections $s:\Sigma\to\mathbf{C}^{2n}$ such that $s(e^{i\theta})\in\mathbf{R}^{2n}(e^{i\theta})$. The linearized operator of ρ is given by

$$\bar\partial_z : H^{1,2}(\Sigma,\mathbf{C}^{2n})\to L^2(\Sigma,\mathbf{C}^{2n}).$$

This is a Fredholm operator, so we can compute its index

$$\mathrm{index}(\bar{\partial}_z) = \dim \mathrm{Ker}(\bar{\partial}_z) - \dim \mathrm{Coker}(\bar{\partial}_z).$$

This indice is invariant under deformation of holomorphic disks. Denote the normal *Maslov* indice of ρ as $\mu(\bar{\partial}_z)$. Then, we have the following (cf. [31]):

$$\mathrm{indice}(\bar{\partial}_z) = \mu(\bar{\partial}_z) + 2n = 2n.$$

Thus, the kernel of $\bar{\partial}_z$ is of dimension at least $2n$. Recalled that a holomorphic disk ρ is regular in the sense of the Fredholm theory if the cokernel of $\bar{\partial}_z$ vanishes. In other words, it is regular if the kernel of $\bar{\partial}_z$ has real dimension $2n$.

For every disk $\rho : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}_{\mathrm{M}}, \bar{\Lambda}_{\phi_0})$, we have a loop of $2n$-dimensional real subspaces $\{\mathbf{R}^{2n}(e^{i\theta}) \mid 0 \leq \theta \leq 2\pi\}$ in $\mathbf{C}^{2n}$. By fixing a real $\mathbf{R}^{2n}$ subspace in $\mathrm{T}_{\rho(1)}\mathscr{W}_{\mathrm{M}}$, this induces a map from $\partial\Sigma$ to $\mathrm{GL}(2n, \mathbf{C})/\mathrm{GL}(2n, \mathbf{R})$. In general, this map may not be lifted to a map from $\partial\Sigma$ to $\mathrm{Gl}(2n, \mathbf{C})$. However, this property of being able to be lifted to $\mathrm{C}^\infty(\partial\Sigma, \mathrm{GL}(2n, \mathbf{C}))$ is invariant under continuous deformation of the disk (including varying boundary conditions). A disk is called *trivial* if all real subspaces $\mathbf{R}^{2n}(e^{i\theta})$ are equal to a constant $2n$-dimensional real subspace (independent of $e^{i\theta}$). For a trivial disk, its induced loop always admits a lifting to $\mathrm{GL}(2n, \mathbf{C})$. Therefore, if a disk is path-connected to a trivial disk, then its induced loop must admit a lifting to an map $\mathrm{C}^\infty(\partial\Sigma, \mathrm{GL}(2n, \mathbf{C}))$. We call this an *associated loop* of the disk ρ. It is clear that *associated loop* is defined up to multiplication by $\mathscr{L}^+\mathrm{GL}(2n, \mathbf{C})$ on the left. Here $\mathscr{L}\mathrm{GL}(2n, \mathbf{C}) = \mathrm{C}^\infty(\partial\Sigma, \mathrm{GL}(2n, \mathbf{C}))$, while $\mathscr{L}^+\mathrm{GL}(2n, \mathbf{C}) \subset \mathrm{C}^\infty(\partial\Sigma, \mathrm{GL}(2n, \mathbf{C}))$ is the set of loops which can be extended to a holomorphic map $\mathrm{C}^\infty(\Sigma, \mathrm{GL}(2n, \mathbf{C}))$. In this paper, we only consider discs which are path connected to a trivial disc. For these holomorphic discs, it is natural to consider the partial indices which are independent of the lifting. According to [31], [16] and [21], using a special form of Birkhoff factorization, we have

Theorem **A**[13]**.** — *Let* $\tilde{\rho} : \mathrm{S}^1 = \partial\Sigma \to \mathbf{R}^{2n}(\theta)$ *be a loop of totally real* $2n$ *dimensional sub-spaces in* $\mathbf{C}^{2n}$. *Suppose that* $\tilde{\rho}$ *is induced by some holomorphic disk* $\rho : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}_{\mathrm{M}}, \bar{\Lambda}_{\phi_0})$. *Then, this loop map may be represented as*

$$\tilde{\rho}(z) = \Theta(z)\Lambda(z)^{\frac{1}{2}} \cdot \mathbf{R}^{2n}, \quad z \in \partial\Sigma = \mathrm{S}^1,$$

where $\Theta(z) \in \mathscr{L}\mathrm{GL}(2n, \mathbf{C})$ *and* $\Lambda(z)$ *is a diagonal matrix:*

$$\Lambda(z) = [z^{k_1}, z^{k_2}, \ldots, z^{k_{2n}}], \quad \forall z \in \partial\Sigma.$$

Here $(k_1, k_2, \ldots, k_{2n})$ *is called* **partial indices** *of the loop* ρ. *Moreover, these partial indices have the following properties:*

---

[13] For the generic *Maslov* indice, this theorem was first obtained by [14] for complex surface. It was generalized to all dimensions in [16] with the assumption that all of the partial indices are non-negative. This last restriction was removed in [21]. The present statement follows the format in [21].

1. *Each individual partial indice is not invariant under continuous deformation. However, the total sum of all partial indices is precisely the normal Maslov indice. Then, the total sum is invariant under any continuous deformation. Thus,*

$$\sum_{i=1}^{2n} k_i = \mu = 0.$$

2. *A disk is Fredholm regular if and only if all of its partial indices* ≥ -1.

Using this theorem, Oh was able to reduce the equation for kernel vectors to a scalar equation:

$$u = \begin{cases} \frac{\partial \xi}{\partial \bar{z}} = 0, & \forall\, z \in \Sigma, \\ \xi(z) = z^{\frac{k_i}{2}} \cdot \mathbf{R}, & \forall\, z \in \mathrm{S}^1. \end{cases}$$

This equation has no solutions when $k_i \leq -1$. For $k_i \geq 0$, this equation has exactly $k_i + 1$ linearly independent solution while each solution is a polynomial in z with degree k_i.

Theorem **B.** — *Suppose* f *is a regular disk whose partial indice decomposition* $(k_1, k_2, \ldots, k_{2n})$ *contains exactly* $l \leq n$ *number of partial indices which equals* -1. *Then the kernel matrix of this disk has co-rank at least* l *everywhere in the interior of this disk.*

This can be easily derived from [16] and [21].

3.2. *The universal moduli space is regular*

Define

$$\mathscr{G} = \bigcup_{\phi_0 \in \mathrm{C}^\infty(\partial\Sigma, \mathscr{H})} \bar{\Lambda}_{\phi_0},$$

and

$$\Upsilon = \bigcup_{\phi_0 \in \mathrm{C}^\infty(\partial\Sigma, \mathscr{H})} \mathscr{M}_{\phi_0},$$

where $\mathscr{M}_{\phi_0}$ is the moduli space of all holomorphic disks with vanishing normal *Maslov* indice:

$$\rho : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}_{\mathrm{M}}, \bar{\Lambda}_{\phi_0}).$$

Clearly, $\mathscr{G}$ is an infinite dimensional manifold. There is a natural projection $p : \Upsilon \to \mathscr{G}$ such that for any $\phi_0 \in \mathrm{C}^\infty(\partial\Sigma, \mathscr{H})$, the moduli space $\mathscr{M}_{\phi_0}$ is mapped to $\bar{\Lambda}_{\phi_0}$.

Recall that the *Moduli* space is **smooth** if every holomorphic disk in this *Moduli* space is regular. It follows from the following lemma and the Sard–Smale transversality theorem that $\mathscr{M}_{\phi_0}$ is smooth for a generic boundary value ϕ_0.

Lemma **3.2.1**[14]**.** — *The universal moduli space* $p : \Upsilon \to \mathcal{G}$ *is smooth.*

Proof. — The tangent space of $\mathcal{G}$ at ϕ_0 can be considered as

$$T_{\phi_0}\mathcal{G} = C^\infty(\partial\Sigma \times M), \quad \forall \phi_0 \in C^\infty(\partial\Sigma, \mathcal{H}).$$

Let $\epsilon_k \to 0$ be a sequence of positive numbers which converges to zero. Denote $\bar{\epsilon} = (\epsilon_1, \epsilon_2, \ldots)$. Set

$$\|f\|_{\bar{\epsilon}} = \sum_{k=0}^{\infty} \epsilon_k \max_{x \in \partial\Sigma \times M} |D^k f(x)|.$$

This defines an $\bar{\epsilon}$-norm on

$$C^{\bar{\epsilon}}(\Lambda_{\phi_0}) = \{f \in C^\infty(\partial\Sigma \times M) \mid \|f\|_{\bar{\epsilon}} < \infty\}.$$

This norm has been introduced by Floer in a different context. Under this norm, $C^{\bar{\epsilon}}(\Lambda_{\phi_0})$ is a Banach space. We can choose $\bar{\epsilon}$ so that $C^{\bar{\epsilon}}(\Lambda_{\phi_0})$ is dense in $C^\infty(\partial\Sigma \times M)$ with respect to the L^2 norm.

Now fix $s > 1$ and define

$$\mathcal{F} = \mathcal{F}^s = H^{s+1}(\Sigma, \Sigma \times \mathcal{W}_M)$$

which is a Sobolev space of all maps $\omega : \Sigma \to \Sigma \times \mathcal{W}_M$ whose $(s+1)^{th}$ derivatives are in L^2. For any boundary map $\phi_0 : \partial\Sigma \to \mathcal{H}$, the totally real submanifold Λ_{ϕ_0} of $\Sigma \times \mathcal{W}_M$ is a point in $\mathcal{G}$. For any small r positive, we define a r-neighborhood of this point in $\mathcal{G}$ as:

$$\mathcal{N}(\Lambda_{\phi_0}) = \{\Lambda_{\phi_0+f} \mid \|f\|_{\bar{\epsilon}} < r \quad \text{and} \quad f \in C^\infty(\partial\Sigma \times M)\}.$$

The corresponding neighborhood of holomorphic disks is

$$\begin{aligned}\bar{\mathcal{M}} &= \mathcal{M}(\mathcal{N}(\Lambda_{\phi_0})) \\ &= \{(\rho, \Lambda_{\phi_0+f}) \mid \bar{\partial}\rho = 0,\ \rho|_{\partial\Sigma} \subset \Lambda_{\phi_0+f},\ \|f\|_{\bar{\epsilon}} < r\}.\end{aligned}$$

For each $\rho \in \mathcal{F}$, define the pulled back bundle as

$$\mathcal{B}_\rho = H^s(\rho^* T\mathcal{W}_M)$$

consisting of all H^s-sections of $\rho^* T\mathcal{W}_M$ on Σ. Set

$$\mathcal{B} = \bigcup_{\rho \in \mathcal{F}} \mathcal{B}_\rho = \bigcup_{\rho \in \mathcal{F}} H^s(\rho^* T\mathcal{W}_M).$$

[14] This was first carried out in [22] in the context of Lagrange/totally real submanifold. For convenience of readers, we include a proof of this transversality below.

This is a smooth bundle over $\mathscr{F}$. We further set

$$\Omega(\Lambda_{\phi_0}) := H^{s+\frac{1}{2}}(\partial\Sigma, \Sigma \times \mathscr{W}_M) \cap C^0(\partial\Sigma, \Lambda_{\phi_0}).$$

This is simply the space of $H^{s+\frac{1}{2}}$ maps from $\partial\Sigma$ to Λ_{ϕ_0}. By the trace theorem, for each map $\rho \in H^{s+1}(\Sigma, \Sigma \times \mathscr{W}_M)$, its boundary map $\rho|_{\partial\Sigma}$ lies in $\Omega(\Sigma \times \mathscr{W}_M) = H^{s+\frac{1}{2}}(\partial\Sigma, \Sigma \times \mathscr{W}_M)$. Now we define a map

$$\Delta : \mathscr{F} \times \mathscr{N}(\Lambda_{\phi_0}) \to \mathscr{B} \times \Omega(\Lambda_{\phi_0})$$

by

$$\Delta(\rho, \Lambda_{\phi_0+f}) = \left(\bar{\partial}\rho, \phi_{\phi_0+f}^{-1}(\rho|_{\partial\Sigma})\right)$$

where $\phi_{\phi_0+f} : \Lambda_{\phi_0} \to \Lambda_{\phi_0+f}$ identifies the small perturbation Λ_{ϕ_0+f} with Λ_{ϕ_0}. Denote by

$$X_f = \frac{d}{dt}\left(\phi_{\phi_0+tf}^{-1}\right)\Big|_{t=0} \in T_{\Lambda_{\phi_0}}\mathscr{N}(\Lambda_{\phi_0}).$$

Consequently, $T_{\Lambda_{\phi_0}}\mathscr{N}(\Lambda_{\phi_0})$ consists of all such fields X_f for $f \in C^\infty(\partial\Sigma \times M)$. Note that

$$\bar{\mathscr{M}} = \Delta^{-1}(\{0\} \times \Omega(\Lambda_{\phi_0})).$$

The goal here is to show that the map Δ is transverse to the submanifold at $\{0\} \times \Omega(\Lambda_{\phi_0}) \subset \mathscr{B} \times \Omega(\Sigma \times \mathscr{W}_M)$. Then it follows that $\bar{\mathscr{M}}$ is a smooth Banach submanifold of $\mathscr{F}^s \times \mathscr{N}(\Lambda_{\phi_0})$. Moreover, by the elliptic regularity theory, $\bar{\mathscr{M}}$ is a smooth Banach submanifold of $\mathscr{F}^s \times \mathscr{N}(\Lambda_{\phi_0})$ for all $s > 1$.

For any small $f \in C^\infty(\partial\Sigma \times M)$, we set $\phi = \phi_{\phi_0+f}$ for simplicity. To show the transversality, we need to show

(3.1)
$$\begin{aligned}\mathrm{Im}_\Delta\, T_{\rho,\Lambda_{\phi_0+f}}(\mathscr{F} \times \mathscr{N}(\Lambda_{\phi_0})) + \{0\} \oplus T_{\phi^{-1}(\rho|_{\partial\Sigma})}\Omega(\Lambda_{\phi_0}) \\ = T_{0,\phi^{-1}(\rho|_{\partial\Sigma})}(\mathscr{B} \times \Omega(\partial\Sigma \times \mathscr{W}_M)),\end{aligned}$$

where $(\rho, \Lambda_{\phi_0+f}) \in \mathscr{F} \times \mathscr{N}(\Lambda_{\phi_0})$. If $(\xi, X_f) \in T_{\rho,\Lambda_{\phi_0+f}}(\mathscr{F} \times \mathscr{N}(\Lambda_{\phi_0}))$, then a straightforward calculation shows that

$$\mathrm{Im}_\Delta(\xi, X_f) = (\bar{\partial}\xi, X_f - \xi|_{\partial\Sigma}).$$

Clearly, the LHS (left hand side) of (3.1) is a subspace of the RHS (right hand side). We need to show that the normal space to LHS in the RHS of (3.1) is null. Suppose that (r, α) is in such an normal space, that is, $(r, \alpha) \perp \mathrm{Im}_\Delta(\xi, X_f)$ and $(r, \alpha) \perp (\{0\} \oplus T_{\phi^{-1}(\rho|_{\partial\Sigma})}\Omega(\Lambda_{\phi_0}))$. The second condition implies

$$\alpha \in (T_{\phi^{-1}(\rho|_{\partial\Sigma})}\Omega(\Lambda_{\phi_0}))^\perp.$$

In other words, α represents some variation normal to $T_{\phi_{\phi_0}^{-1}(\rho|_{\partial\Sigma})}\Omega(\Lambda_{\phi_0})$. The first condition implies that

$$\int_\Sigma (\bar\partial\xi, r) + \int_{\partial\Sigma} (X_f - \xi|_{\partial\Sigma}, \alpha) = 0.$$

Integrating by parts, we have

$$\int_\Sigma (\xi, \nabla_J r) + \int_{\partial\Sigma} (\xi|_{\partial\Sigma}, e^{-i\theta} r - \alpha) d\theta + \int_{\partial\Sigma} (X_f - \xi|_{\partial\Sigma}, \alpha) d\theta = 0.$$

Thus

$$\nabla_J r = 0, \tag{3.2}$$

$$-\alpha|_{\partial\Sigma} + e^{-i\theta} r|_{\partial\Sigma} = 0, \tag{3.3}$$

$$\alpha^\perp = 0. \tag{3.4}$$

Equation (3.4) shows that α must be tangent to $T_{\phi^{-1}(\rho|_{\partial\Sigma})}\Omega(\Lambda_{\phi_0})$. On the other hand, α must be also normal to this space. Then, $\alpha = 0$. Consequently, $r|_{\partial\Sigma} = 0$ by (3.3). This, together with (3.2), implies that $r = 0$ in Σ. This completes the proof of transversality. □

The same arguments also show that for a generic path $\psi : [0, 1] \times \partial\Sigma \mapsto \mathscr{H}_\omega$, the total moduli $\bigcup_{t\in[0,1]} \mathscr{M}_|\psi(t, \cdot)$ is smooth.

3.3. *Selection of a generic path*

Next we turn our attention to variations of an arbitrary disk f in the universal *moduli* space of holomorphic disks. As before, for every disk, it induces a map from $\partial\Sigma$ to the space of totally real $2n$ plane in $\mathbf{C}^{2n}$. Since all disks concerned are path connected to a trivial disk, this induced map can be lifted to a map from the universal *moduli* space of holomorphic disks to the loop space $\mathscr{L}GL(2n, \mathbf{C})$. It is well defined up to some normalization of the induced normal bundle of $\rho^* T_w \mathscr{W}_M$ over Σ. In other words, it is a map from a holomorphic disk to $\mathscr{L}GL(2n, \mathbf{C})/\mathscr{L}^+GL(2n, \mathbf{C})$. Define a fiber bundle $\mathscr{C}$ over $\mathscr{F}$ such that each fibre is isomorphic to

$$\mathscr{L}GL(2n, \mathbf{C})/\mathscr{L}^+GL(2n, \mathbf{C}).$$

This defines a natural map from the universal moduli space $\bar{\mathscr{M}}$ to this fibre bundle

$$\star : \mathscr{G} \to \mathscr{C}$$

which simply maps each holomorphic disk to its associated loop in $\mathscr{L}\mathrm{Gl}(2n, \mathbf{C})/\mathscr{L}^+\mathrm{Gl}(2n, \mathbf{C})$.

It is well known that $\mathscr{L}\mathrm{GL}(2n, \mathbf{C})/\mathscr{L}^+\mathrm{GL}(2n, \mathbf{C})$ admits a smooth stratification of loops by its partial indice $k = (k_1, k_2, ..., k_{2n})$. A somewhat lengthy calculation[15] shows

Lemma **3.3.1.** — *For the smooth stratification of $\mathscr{L}\mathrm{GL}(2n, \mathbf{C})/\mathscr{L}^+\mathrm{GL}(2n, \mathbf{C})$ by its partial indices $k = (k_1, k_2, ..., k_{2n})$, the real codimension of each component indexed by k is*

$$d = \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (k_i - k_j - \lambda_{ij})$$

where $k_1 \geq k_2 \geq \cdots \geq k_{2n}$. Moreover,

$$\lambda_{ij} = \begin{cases} 1 & \text{if } k_i > k_j \text{ and } i < j, \\ 0 & \text{otherwise.} \end{cases}$$

Let S_0, S_1, S_2, S be the set of loops whose partial indices satisfy:

1. All partial indices in S_0 are equal to 0;
2. All partial indices are of the form $(1, 0, ..., 0, -1)$ in S_1;
3. At least two of the partial indices equal to -1 in S_2, but no partial indice is ≤ -2;
4. At least one partial indice in S is less or equal to -2.

According to Lemma 3.3.1, S_0 is in generic position, while the real codimension for S_1 is 1. For $S \subset \mathscr{L}\mathrm{GL}(2n, \mathbf{C})/\mathscr{L}^+\mathrm{GL}(2n, \mathbf{C})$, suppose that $k_j \geq k_{2n}$ and $k_{2n} \leq -2$, $\forall\, i \in [1, 2n]$. Then the codimension is:

$$\begin{aligned} d &= \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (k_i - k_j - \lambda_{ij}) \\ &\geq \sum_{i=1}^{2n} (k_i - k_{2n} - \lambda_{i(2n)}) \\ &= \sum_{i=1}^{2n} k_i + \sum_{i=1}^{2n} (-k_{2n}) - \sum_{i=1}^{2n} \lambda_{i(2n)} \\ &\geq 0 + \sum_{i=1}^{2n} 2 - \sum_{i=1}^{2n-1} 1 = 2n + 1. \end{aligned}$$

[15] A proof can be founded in Section 9.

For S_2, we can assume $k_{2n-1} = k_{2n} = -1$. Thus,

$$\begin{aligned} d &= \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} (k_i - k_j - \lambda_{ij}) \\ &\geq \sum_{i=1}^{2n-1} (k_i - k_{2n-1} - \lambda_{i(2n-1)}) + \sum_{i=1}^{2n-2} (k_i - k_{2n} - \lambda_{i(2n)}) \\ &= 2\sum_{i=1}^{2n-2} k_i + \sum_{i=1}^{2n-2} (-k_{2n} - k_{2n-1}) - \sum_{i=1}^{2n-2} (\lambda_{i(2n)} + \lambda_{i(2n-1)}) \\ &= 2(k_{2n-1} + k_{2n}) + \sum_{i=1}^{2n-2} (-k_{2n} - k_{2n-1}) - \sum_{i=1}^{2n-2} (\lambda_{i(2n)} + \lambda_{i(2n-1)}) \\ &\geq 4 + 2(2n-2) - 2(2n-2) = 4. \end{aligned}$$

According to Lemma 3.3.1, we have

$$\begin{aligned} \mathscr{L}\mathrm{GL}(2n, \mathbf{C})/\mathscr{L}^+\mathrm{GL}(2n, \mathbf{C}) &= S_0 \cup S_1 \cup S_2 \cup S \\ &= S_0 \cup (S_1^{a.s.} \cup S_1^{n.a.s.}) \cup S_2 \cup S. \end{aligned}$$

Here $S_1^{a.s.}$ denotes all of the holomorphic disks in S_1 which are super regular at $z = 0$ and

$$S_1 = S_1^{a.s.} \cap S_1^{n.a.s.}.$$

It is straightforward to check that $S_1^{n.a.s.}$ has at least codimension 1 in S_1.

Proposition **3.3.2.** — *This map* $\star$ *is a submersion at any embedded disk of* $\bar{\mathscr{M}}$.

Proof. — We need to show that $\star$ is a submersion at a regular disk or at a non-regular but embedded disk. The first assertion follows from the fact that any regular holomorphic disk f with boundary in a totally real submanifold is stable under a small deformation of the boundary map. To be more explicit, let $f : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}_M, \bar{\Lambda}_{\phi_0})$ be a holomorphic disk with vanishing normal *Maslov* indice. If f is regular in the sense of Fredholm theory, then there is a holomorphic disk $f + \delta f$ such that its boundary lies in some $\mathbf{R}^{2n}(e^{i\theta}) + \delta P(e^{i\theta})$. For this family of holomorphic disks, the associated loop is exactly $\ell + \delta\ell$. Thus, "$\star$" is an submersion at the image (under $\star$) of every regular holomorphic disk.

Now suppose that $\star(f) = \ell \in S \subset \mathscr{C}$. Since $\mathscr{C}$ is a smooth infinite dimensional manifold which admits a smooth stratification by partial indices. More specifically, the space of loop matrices in $\mathrm{Gl}(2n, \mathbf{R})$, may be decomposed to a union of $S_0 \cup S_1 \cup S_2 \cup S$. Here we are only considering S which lie in a connected component of $S_1 \cup S_1 \cup S_2$.

Therefore, there always exists a path $\delta\ell(t)$ such that $\delta\ell(0) = 0$; and $\ell + \delta\ell(t) \in \mathscr{C} \setminus \mathrm{S}$ when $t \neq 0$. Consider

$$f : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}, \bigcup_{\theta \in \mathrm{S}^1} (\theta, \mathrm{L}(\theta))).$$

Here $\mathrm{L}(\theta)$ is a totally real sub-manifold in $(\theta, \mathscr{W})$ for any $\theta \in \mathrm{S}^1$. At the tangential level, $\mathrm{T}_{f(\theta)}\mathscr{W}_\mathrm{M}$ is a trivial $\mathbf{C}^{2n}$ bundle over Σ. Using this trivialization, we may assume

$$\mathrm{T}_{f(\theta)}\mathrm{L}(\theta) = \mathbf{R}^{2n}(\theta) = \mathrm{A}(\theta) \cdot \mathbf{R}^{2n}$$

for some $\mathbf{R}^{2n}$ fixed in $\mathrm{T}_{f(1)}\mathscr{W}_\mathrm{M}$. Here $\mathrm{A}(\theta) \in \mathrm{Gl}(2n, \mathbf{C})$. Clearly, ℓ can be lifted up to be a loop in $\mathrm{Gl}(2n, \mathbf{C})$ and

$$\ell(\theta) = \mathrm{A}(\theta), \quad \forall \theta \in \mathrm{S}^1.$$

The tangent space of $\mathscr{L}\mathrm{GL}(2n, \mathbf{C})$ at ℓ can be represented by a smooth 1-parameter family of loops of matrices $(-\epsilon \leq t \leq \epsilon)$:

$$\ell(t, \theta) = \mathrm{A}(\theta)(\mathrm{I} + t\mathrm{B}(\theta)), \quad \forall\ \theta \in \mathrm{S}^1.$$

The surjectivity at f is equivalent to the existence of a pre-image for this path $\ell(t, \theta)$ with respect to an arbitrary loop matrix B. Near a small tubular neighborhood of $\mathrm{L}(\theta) \in \mathscr{W}_\theta$, we define a product metric (so $\mathrm{L}(\theta)$ becomes totally geodesic in $\mathscr{W}_\mathrm{M}$). Call this metric g_θ. Define

$$\mathrm{L}(t, \theta) = exp_{f(\theta), g_\theta}((\mathrm{A}(\theta)(\mathrm{I} + t\mathrm{B}(\theta))) \cdot \mathbf{R}^{2n}),$$

where $(\mathrm{A}(\theta)(\mathrm{I}+t\mathrm{B}(\theta)))\cdot\mathbf{R}^n$ represents the n-dimensional plane spanned by it in $\mathrm{T}_{f(\theta)}\mathscr{W}_\mathrm{M}$. Clearly, $\mathrm{L}(0, \theta) = \mathrm{L}(\theta)$. Define $f(t)$ to be a family of disks in the total *moduli* space

$$f(t) : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}, \bigcup_{\theta \in \mathrm{S}^1} (\theta, \mathrm{L}(t, \theta))),$$

such that the image of each $f(t)$ is identified with f, but they represent a 1-parameter path of holomorphic disks in the total *moduli* space[16]. Clearly,

$$\star(f(t)) = \ell(t).$$

In other words, the map $\star$ is transversal to $\mathrm{S} \subset \mathscr{L}\mathrm{GL}(2n, \mathbf{C})/\mathscr{L}^+\mathrm{GL}(2n, \mathbf{C})$. ◻

Next we want to use this submersion map $\star$ to calculate the codimension of various components of the universal *moduli* space.

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[16] Consider all holomorphic discs in a complex manifold whose boundary lies in some totally real submanifold. It is possible to fix a holomorphic disc while deforming the totally really submanifold such that the boundary of this disc still lies in these "new boundary totally real submanifolds." One one hand, there is only one holomorphic disc since we never change it. On the other hand, in this total moduli space of holomorphic discs, it clearly represents a one parameter family of discs in a one parameter family of moduli space.

Note that $\star^{-1}\mathrm{S}_0, \star^{-1}\mathrm{S}_1, \star^{-1}\mathrm{S}_2$ and $\star^{-1}\mathrm{S}$ are smooth manifold or submanifold in $\bar{\mathscr{M}}$, where $\star^{-1}\mathrm{S}_0$ are the set of all super regular disks which is generic in $\bar{\mathscr{M}}$, $\star^{-1}\mathrm{S}_1$, $\star^{-1}\mathrm{S}_2$ are submanifolds of regular holomorphic disks in $\bar{\mathscr{M}}$ with real codimension at least 1 and 4. Finally, $\star^{-1}\mathrm{S}$ is the smooth submanifold of all irregular disks in $\bar{\mathscr{M}}$ with real codimension at least $2n+1$. This, together with the remark at the end of last section, implies

Theorem **3.3.3.** — *For any path $\psi : [0,1] \to \mathrm{C}^\infty(\partial\Sigma, \mathscr{H}_\omega)$ such that $\mathscr{M}_{\psi(0,\cdot)}$ contains a super regular disk with vanishing normal Maslov invariant, there exists a generic path (we still denote it by ψ), which is arbitrarily close to the original path, such that the total moduli $\bigcup_{0\le s\le 1}\{s\}\times\mathscr{M}_{\psi(s,\cdot)}$ is a smooth $2n+1$-dimensional manifold. Moreover, there is a connected component $\mathscr{M}^0_\psi$ of this total moduli such that the followings hold:*

1. *It contains super-regular disk(s) in the initial moduli $\mathscr{M}_{\psi(0,\cdot)}$;*
2. *The set of disks with partial indices $(0,0,...,0)$ in $\mathscr{M}^0_\psi$ is open and dense in this connected component;*
3. *The set of disks with partial indices $(1,0,0,...,0,-1)$ has codimension at least 1 in $\mathscr{M}^0_\psi$. The set of disks with partial indice $(1,0,...,0,-1)$ but not super-regular at $z=0$ has codimension 2 and higher;*
4. *The set of all other holomorphic disks has codimension 2 and higher;*
5. *There exist at most finitely many non-regular disks in the total moduli.*

3.4. *Almost super regular foliations*

In this subsection, we first introduce a new notion **almost super regular foliation**. An almost super regular foliation is certainly a nearly smooth foliation. Like a nearly smooth foliation, an almost super regular foliation is not a foliation in its target manifold! Rather, the image of its evaluation map foliated an open dense subset in $\Sigma\times\mathrm{M}$. Recalled the natural projection $\pi : \mathscr{W}_{[\omega]} \to \mathrm{M}$ by forgetting its second component. A regular disk f in $\mathscr{M}_{\phi_0}$ is called **super regular** at $z\in\Sigma$ if the Jacobi map of $\pi\circ ev$ is non-singular at $z\in\Sigma$. It is called **super regular** if it is super regular at every point of Σ. It is called **almost super regular** if it is super regular in Σ^0. Obviously, a super regular disc is necessary an almost super regular disc and an almost super regular disc is necessary a regular disc.

Definition **3.4.1.** — *For any boundary map $\phi_0 \in \mathrm{C}^\infty(\partial\Sigma, \mathscr{H}_\omega)$, an open and connected $2n$ dimensional subset $\mathscr{U}_{\phi_0}\subset\mathscr{M}_{\phi_0}$ is called an* **almost super regular foliation** *if*

1. *It is a nearly smooth foliation (cf. Definition 1.4.2);*
2. *Every disk in $\bar{\mathscr{U}}_{\phi_0}$ is regular except perhaps at most a set of finitely many disks. Furthermore, the set of almost super regular disks has co-dimension at least 1, while the set of all disks of other types has at least co-dimension 2 or higher.*

A moduli space $\mathcal{M}_{\phi_0}$ is called **super regular** if all its discs in one connected component consists only super regular discs. Clearly, for any boundary map $\phi_0 \in C^\infty(\partial\Sigma, \mathcal{H}_\omega)$, an almost super regular foliation $\mathcal{F}_{\phi_0}$ is super regular if $\mathcal{U}_{\phi_0} = \bar{\mathcal{U}}_{\phi_0} = \mathcal{M}_{\phi_0}$.

Proposition **3.4.2.** — *If $\mathcal{F}_{\phi_0}$ is an almost super regular foliation, then $\bar{\mathcal{U}}_{\phi_0}$ induces a foliation in $\Sigma^0 \times M$ via $\pi \circ ev$, except at most a set of codimension 2.*

Proof. — It is clear. □

Corollary **3.4.3.** — *For an almost super regular foliation, two disks intersect at most at subset of of $\Sigma \times M$ with codimension 2 or higher. In particular, no two super regular discs intersect in the interior of $\Sigma \times M$.*

This in turns implies

Corollary **3.4.4.** — *For any almost smooth solution ϕ of* (1.1) *which corresponds to an almost super regular foliation, the leaf vector field X which annihilate the Levi form $\pi_2^*\omega_0 + \sqrt{-1}\partial\bar\partial\phi$ is smooth in $\tilde{\mathcal{V}}_{\phi_0}$ and is uniformly bounded in $\Sigma^0 \times M$.*

Proposition **3.4.5.** — *For a generic boundary map $\phi_0 : \partial\Sigma \to \mathcal{H}_\omega$ such that every embedded disk in $\mathcal{M}_{\phi_0}$ is regular, then its connected component $\bar{\mathcal{U}}_{\phi_0} \subset \mathcal{M}_{\phi_0}$ is a smooth manifold without boundary.*

Proof. — For any sequence of holomorphic disks $f_k \in \mathcal{U}_{\phi_0}$, the leaf vector field X_k has uniform upper bound. It follows that there is a subsequence (which we still denoted as $\{f_k, k \in \mathcal{I}\}$), such that converges to a limiting embedded disk $f_\infty \in \mathcal{M}_{\phi_0}$. By our assumption, this limiting disk must be regular in the sense of Fredholm theory. In particular, f_∞ is an interior point of $\mathcal{M}_{\phi_0}$. Consequently, $\bar{\mathcal{U}}_{\phi_0}$ is compact without boundary. □

Proposition **3.4.6.** — *For a boundary map $\phi_0 : \partial\Sigma \to \mathcal{H}_\omega$ such that all but possibly finitely many disks in $\mathcal{M}_{\phi_0}$ are regular. Define $\tilde{\mathcal{U}}_{\phi_0}$ to be the set of all super regular and all almost super regular disks. Suppose*

1. *$\bar{\mathcal{U}}_{\phi_0} \setminus \tilde{\mathcal{U}}_{\phi_0}$ has codimension 2 or higher;*
2. *The evaluation map is continuous on $\bar{\mathcal{U}}_{\phi_0}$.*
3. *The covering index of $\pi \circ ev$ from $\mathcal{U}_{\phi_0}$ to its image is 1.*

Then $\mathcal{U}_{\phi_0}$ defines an almost super regular foliation. In particular, the covering indices for evaluation map is 1.

Proof. — Let N be the set of all regular discs in $\bar{\mathscr{U}}_{\phi_0}$. Consider the evaluation of N in the central fibre $\{0\} \times \mathrm{M}$. The evaluation map is locally covering map from generic points in N. Since $\partial \mathrm{N}$ is a set of isolated singular disks and the evaluation map is continuous at this set, then the image of N must be $\{0\} \times \mathrm{M}$ entirely. Since M is connected, the covering indices must be some positive constant $k \geq 1$ for generic point. By our third assumption, $k = 1$. Thus, there is only one connected component which defines an almost super regular foliation. □

Now, we introduce the notion of *partially smooth foliation*, which arises from limits of almost super regular foliations under convergence of boundary maps in suitable norms.

Definition **3.4.7.** — *For any boundary map $\phi_0 \in \mathrm{C}^\infty(\partial\Sigma, \mathscr{H}_\omega)$, an open $2n$ dimensional subset $\mathscr{U}_{\phi_0} \subset \mathscr{M}_{\phi_0}$ and closed subset $\tilde{\mathscr{U}}_{\phi_0} \subset \mathscr{M}_{\phi_0}$ is called a* **partially smooth foliation** *if the following conditions are met:*

1. $\mathscr{U}_{\phi_0} \subset \bar{\mathscr{U}}_{\phi_0} \subset \tilde{\mathscr{U}}_{\phi_0}$.
2. *Every disk in $\mathscr{U}_{\phi_0}$ is super regular.*
3. *The evaluation map $\pi \circ ev : \Sigma \times \bar{\mathscr{U}}_{\phi_0} \to \Sigma \times \mathrm{M}$ is a continuous onto map into its image where the image is dense in $\partial\Sigma \times \mathrm{M}$ Moreover, the image of $\tilde{\mathscr{U}}_{\phi_0}$ is $\Sigma \times \mathrm{M}$.*
4. *Any disk in $\mathscr{U}_{\phi_0}$ doesn't intersect with any other disk in $\tilde{\mathscr{U}}_{\phi_0}$ in $\Sigma^0 \times \mathrm{M}$.*

Recall that an almost smooth solution of the HCMA equation (1.1) corresponds to a nearly smooth foliation. One can view a *partially smooth foliation* as a sequential limit of *nearly smooth foliations*, while a partially smooth solution can be viewed as a sequential limit of almost smooth solutions. In this regard, a partially smooth solution corresponds conceptually to a partially smooth foliation.

3.4.1. *Open-denseness of almost super regular disks.* — In this subsection, we reformulate Theorem 3.3.3 in terms of a.s.r. or s.r. holomorphic disks. Let us first describe some properties of holomorphic disks with either partial indices $(0, 0, ..., 0)$ or $(1, 0, ..., 0, -1)$.

Theorem **3.4.8.** — *Given a connected component of $\mathscr{M}_{\phi_0}$ which consists of holomorphic disks with partial indices* $(0, 0, ..., 0)$. *If there exists at least one super regular disk in this connected component, then all disks in this component are super regular.*

Proof. — The set of super regular disks is open in the *moduli* space. Therefore, we just need to show that it is closed among disks with partial indices $(0, 0, ..., 0)$. Let $\{f_i : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}_{\mathrm{M}}, \bar{\Lambda}_{\phi_0})\}$ be a sequence of super regular disks such that $f_i \to f$

smoothly (cf. in $C^{2,\alpha}(\Sigma, \Sigma \times \mathscr{W}_M)$- norm[17]) in $\mathscr{M}_{\phi_0}$. We want to prove that if f has partial indices $(0, 0, ..., 0)$, then f is also super regular.

Since f is regular, there is a small neighborhood $\mathscr{O}_f \subset \mathscr{M}_{\phi_0}$ of f such that $ev : \Sigma \times \mathscr{O}_f \to \Sigma \times \mathscr{W}_M$ is smooth. Put $F = \pi \circ ev$. Let $t_1, t_2, ..., t_{2n}$ be local coordinates of $\mathscr{O}_f$, write

$$s_k^{(i)}(z) = \frac{\partial ev}{\partial t_k}\bigg|_{ev(z,f)} \in T^{1,0}_{ev(z,f_i)}\mathscr{W}_M, \quad 1 \leq k \leq 2n.$$

Then, $\{s_k\}_{k=1}^{2n}$ is a basis of the Kernel space of the $\bar\partial$ operator. Moreover, at each image point $ev(z,f)$, the set of $2n$ vertical vectors $\{s_k\}_{k=1}^{2n}$ is also a basis of the "vertical" tangential subspace $T_{f(z)}\mathscr{W}_M$. Since $\mathscr{W}_M$ is locally the same as T^*M, its tangent space naturally splits into a TM part and the tangent space of the fibre direction. Denote by $\binom{u}{v}$ the corresponding two components of any kernel vector, where $u, v \in C^n$. Set the k-th kernel vector as

$$s_k^{(i)} = \begin{pmatrix} u_k^{(i)} \\ v_k^{(i)} \end{pmatrix}, \quad 1 \leq k \leq 2n.$$

According to Proposition 2.3.4, there exists a solution $\phi^{(i)}$ of (1.1) in $\pi \circ ev(\Sigma \times \mathscr{O}_f)$ with $\phi^{(i)}|_{\partial\Sigma\times M} = \phi_0|_{\partial\Sigma\times M}$. By Proposition 2.3.8, there is a uniform C such that

$$|\partial\bar\partial\phi^i| \leq C.$$

For any point (z, x) in the image of $\pi \circ ev(z, f)$, let U be a small open set of x in M. Then

$$G^{(i)} = \left(g_{0,\alpha\bar\beta} + \frac{\partial^2\phi^{(i)}}{\partial w_\alpha \partial\bar w_\beta}\right)_{n\times n} > 0, \quad S^{(i)} = \left(\frac{\partial^2(\rho + \phi^{(i)})}{\partial w_\alpha \partial w_\beta}\right),$$

where $\omega_0 = g_{0,\alpha\bar\beta}dw^\alpha d\bar w^\beta = \partial\bar\partial\rho$ in U. By a straightforward calculation, we have that for any $(z, w) \in f_i(\Sigma)$,

$$(\mathbf{3.5}) \qquad \left(v_k^{(i)}\right)_{n\times 1} = G^{(i)}_{n\times n} \cdot \left(\bar u_k^{(i)}\right)_{n\times 1} + S^{(i)}_{n\times n} \cdot \left(u_k^{(i)}\right)_{n\times 1}.$$

It is clear from here that $\{v_k^{(i)}\}_{n\times 1}$ is not tensorial in the usual sense since $S^{(i)}_{n\times n}$ is not. However,

$$(\mathbf{3.6}) \qquad \det\begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ v_1^{(i)} & v_2^{(i)} & \cdots & v_{2n}^{(i)} \end{pmatrix} = \det\begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ \bar u_1^{(i)} & \bar u_2^{(i)} & \cdots & \bar u_{2n}^{(i)} \end{pmatrix} \cdot \det G^{(i)}$$

[17] This regularity assumption is not optimal.

is both real and holomorphic. Note that the right hand side is a function independent of the choice of local coordinate in M. Thus, the left side is well defined in Σ and it must be a positive constant along the disc. Suppose this constant is c_i[18] in each holomorphic disk. Then,

$$\det\begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ \bar{u}_1^{(i)} & \bar{u}_2^{(i)} & \cdots & \bar{u}_{2n}^{(i)} \end{pmatrix} \cdot \det \mathrm{G}^{(i)} = c_i.$$

A more global view of (3.6) is

$$\frac{(\pi \circ ev)^* \omega^n_{\phi^{(i)}}}{dt_1 \wedge dt_2 \wedge \cdots \wedge dt_{2n}} = \frac{\omega_0^n}{dt_1 \wedge dt_2 \wedge \cdots \wedge dt_{2n}}\Big|_{z=z_0\in\partial\Sigma}.$$

Since f_i is a super regular disk,

$$\det\begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ \bar{u}_1^{(i)} & \bar{u}_2^{(i)} & \cdots & \bar{u}_{2n}^{(i)} \end{pmatrix} \neq 0 \tag{3.7}$$

holds everywhere in $f_i(\Sigma)$. Note that the left side of (3.7) is exactly the Jacobian of $\pi \circ ev$. By our assumptions, f is a disk with partial indices $(0, 0, ..., 0)$, that is, the kernel matrix is nowhere singular:

$$\det\begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ v_1^{(i)} & v_2^{(i)} & \cdots & v_{2n}^{(i)} \end{pmatrix} \neq 0$$

everywhere in $f(\Sigma)$. What we need to prove is that the inequality (3.7) hold everywhere in $f(\Sigma)$. For $ev(f_i, \Sigma)$, it is easy to see that this sequence of constants $\{c_i, i \in \mathcal{N}\}$ has both uniform upper and lower bound, provided that the limiting disk f has partial indices $(0, 0, ..., 0)$. Since $\det \mathrm{G}^{(i)} \leq \mathrm{C}$, we deduce

$$\det\begin{pmatrix} u_1 & u_2 & \cdots & u_{2n} \\ \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_{2n} \end{pmatrix} > 0$$

along the limiting disk f. This completes the proof of this theorem. □

However, we can squeeze a little more out from the arguments above. Let f be a holomorphic disk with partial indice $(1, 0, ..., 0, -1)$ which is super regular at $z = 0$. We claim that f is almost super regular. In fact, the condition implies that

$$\det\begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ \bar{u}_1^{(i)} & \bar{u}_2^{(i)} & \cdots & \bar{u}_{2n}^{(i)} \end{pmatrix}\Big|_{z=0} \neq 0.$$

[18] When restricted to each $\{z_0\} \times \mathrm{M}$, our normalization forces the first term on the right hand side to be positive.

On the other hand, according to (3.6), for each f_i, we have

$$c_i = \det \begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ v_1^{(i)} & v_2^{(i)} & \cdots & v_{2n}^{(i)} \end{pmatrix} = \det \begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ \bar{u}_1^{(i)} & \bar{u}_2^{(i)} & \cdots & \bar{u}_{2n}^{(i)} \end{pmatrix} \cdot \det G^{(i)}.$$

Using local deformation theory in next subsection (Corollary 4.2.11), $\log \det G^{(i)}$ is a subharmonic function in Σ. Moreover, it is uniformly bounded from above. Set

$$h_i = \log \det \begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ \bar{u}_1^{(i)} & \bar{u}_2^{(i)} & \cdots & \bar{u}_{2n}^{(i)} \end{pmatrix}$$

along $f_i(\Sigma)$. Then h_i $(i = 1, 2, \ldots)$ is a uniformly bounded subharmonic function on $f_i(\Sigma)$. Moreover, we have (cf. Proposition 4.2.4)

$$|\Delta_z h_i| = \left| -\Delta_z \log \det G^{(i)} \right| \leq C, \quad \forall z \in \Sigma^0.$$

Then Harnack inequality for harmonic function implies that either h_i approaches to $-\infty$ everywhere in any compact subset of Σ^0 or stays uniformly bounded in any compact subset of Σ^0. Since $h_i(0)$ is uniformly bounded, we have

$$\begin{aligned} \lim_{i\to\infty} h_i(z) &= h(z) \\ &= \log \det \begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_{2n}^{(i)} \\ \bar{u}_1^{(i)} & \bar{u}_2^{(i)} & \cdots & \bar{u}_{2n}^{(i)} \end{pmatrix} \Bigg|_{ev(f,\Sigma)} > -\infty, \quad \forall z \in \Sigma^0. \end{aligned}$$

Consequently, f is almost super regular. Thus, we have proved

Theorem **3.4.9.** — *If a holomorphic disk with partial indices* $(1, 0, \ldots, 0, -1)$ *is super regular at one interior point and it can be connected to disks of partial indices* $(0, 0, \ldots, 0)$*, then it is almost super regular.*

In view of these two theorems, we can reinterpret Theorem 3.3.3 as

Theorem **3.4.10.** — *Given any path* $\psi : [0, 1] \to C^\infty(\partial\Sigma, \mathcal{H}_\omega)$ *such that* $\mathcal{M}_{\psi(0,\cdot)}$ *contains a super regular disk with vanishing Maslov disk, there exists a generic path (still denoted by* ψ *for simplicity), which is arbitrarily close to the given path, such that for this new path, a connected component* $\mathcal{M}_\psi^0$ *of the total moduli space* $\bigcup_{0\leq s\leq 1}\{s\} \times \mathcal{M}_{\psi(s,\cdot)}$*, which contains the initial super regular disk, is a smooth* $2n+1$*-dimension manifold. Moreover, the followings hold*

1. *The set of super regular disks is open and dense in this connected component;*
2. *The set of almost super regular disks has codimension at least 1 in this component;*
3. *The set of disks, which are neither super regular nor almost super regular, has codimension at least 2;*
4. *There exist at most finitely many irregular disks in the total moduli space.*

Moreover, there exist at most finitely many points $0 < \bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_l < 1$ *such that for any* $t \neq \bar{t}_i$ $(1 \leq i \leq l)$*, all disks in* $\mathcal{M}_{\psi(t,\cdot)}$ *are regular and its subset of disks in* $\mathcal{M}^0_\psi$*, which are neither super regular nor almost super regular, has codimension at least 2. When* $t = \bar{t}_i$ *for some* i*,* $\mathcal{M}_{\psi(t,\cdot)} \cap \mathcal{M}^0_\psi$ *may either contain some isolated irregular disks or a subset of disks which are neither super regular nor almost super regular which has exactly codimension* 1.

3.4.2. *Almost super regular foliations along a generic path.* — In this subsection, we prove openness and closeness of almost super regular foliations along a generic path, which is assured by Theorem 3.4.10.

Theorem **3.4.11.** — *Let* $\psi : [0,1] \mapsto C^\infty(\Sigma, \mathcal{H}_\omega)$ *be a generic path with properties specified in Theorem 3.4.10. Suppose that* $\mathcal{M}_{\psi(0)} \cap \mathcal{M}^0_\psi$ *is connected and it defines an* **almost super regular foliation** *with boundary manifold* $\bar{\Lambda}_{\psi(0)}$*. Here* $\mathcal{M}^0_\psi$ *is the connected component defined in Theorem 3.4.10. Then for each* t*,* $\mathcal{M}_{\psi(t)} \cap \mathcal{M}^0_\psi$ *is connected and induces a foliation in an open dense subset of* $\Sigma \times M$*. Moreover, this component gives rise to an almost super regular foliation except at most a finite number of times.*

We first prove

Lemma **3.4.12.** — *For a sequence of* $\tau_i, i \in \mathbf{N}(\lim_{i\to\infty} \tau_i = \bar{t} \in (0,1])$ *such that* $\mathcal{F}_{\psi(\tau_i,\cdot)}$ *is a sequence of almost super regular foliations. Suppose that* ϕ_i *is the corresponding sequence of almost smooth solutions and* $\lim_{i\to\infty} \phi_i = \phi_\infty$*. Then,* ϕ_∞ *is a partially smooth solution of* (1.1) *and* $\mathcal{R}_{\phi_\infty}$ *(regular part of* ω_{ϕ_∞}*) is an open dense subset of* $\Sigma \times M$*. Moreover, there is a unique connected component of* $\mathcal{M}_{\psi(\bar{t},\cdot)}$ *which is the limit of* $\mathcal{F}_{\psi(\tau_i,\cdot)}$*. Either this component is regular in which case there is an almost super regular foliation* $\mathcal{F}_{\psi(t,\cdot)}$ *for* $t > \bar{t}$*; or this component is an almost super regular foliation itself.*

Proof. — We will assume some results in Section 5, such as, Theorem 5.0.17, Theorem 5.0.14, Formula 5.2 etc. Their proof has nothing to do with discussions in this section and will be given later.

Following Theorem 5.0.17, after passing to a subsequence, $\mathcal{F}_{\psi(\tau_i,0)}$ converges to a partially smooth foliation $\mathcal{F}_{\psi(\tau_\infty,\cdot)}$, where $\mathcal{U}_{\psi(\tau_\infty,\cdot)}$ denotes the set of all its super regular disks. Theorem 5.0.15 implies that there is at least one super regular disk which is the limit of a sequence of super regular disks in $\mathcal{F}_{\phi_0(\tau_i)}$ with uniformly bounded capacity (cf. Formula 5.2). Therefore, $\mathcal{U}_{\psi(\tau_\infty,\cdot)}$ is non-empty.

For convenience, let $\mathcal{B}$ be the union of all disks in $\mathcal{M}_{\psi(\tau_\infty,\cdot)}$ which are sequential limits of disks in $\mathcal{U}_{\phi_0(\tau_i)}$. By definition, any leaf in $\mathcal{B}$, is the limit of some sequence of disks in $\mathcal{M}_{\psi_{\tau_i}}$. Following Theorem 5.0.14, for any such sequence of disks, the corresponding sequence of leaf vector field in TM has a uniform upper bound on length. In particular, all leaves in $\mathcal{B}$ have a uniform upper bound on the length of their leaf vector fields. Consequently, any sequence in $\mathcal{B}$ must have a convergent subsequence where the limit is an embedded disk in $\mathscr{M}_{\tau_\infty}$. It follows that $\mathscr{B}$ is a closed, bounded set in the moduli space. On the other hand, by the choice of our generic path in Theorem 3.4.10, the moduli space at $t = \tau_\infty$ admits at most finitely many non-regular, embedded disks. Therefore, all disks in $\mathscr{B}$, except at most a finite number of disks, are regular. Consequently, the evaluation map is continuous everywhere in $\mathscr{B}$ and differentiable except at most a finite number of points (leaves).

Moreover, $\mathscr{U}_{\psi(\tau_\infty,\cdot)}$ is an open dense, and irreducible subset of $\mathscr{B}$. If $\tau_\infty \neq \bar{t}_k$, $(1 \leq k \leq l)$, then all disks in $\mathscr{B}$ are regular and the set of disks which are neither super regular nor almost super regular has codimension at least 2. In this case, $\mathscr{B} = \mathscr{F}_{\psi(\tau_\infty,\cdot)}$ is an almost super regular foliation. On the other hand, if $\tau_\infty = \bar{t}_k$ for some k, then either $\mathscr{B}$ contains a finite number of singular disks, or all disks in $\mathscr{B}$ are regular where the codimension for non-almost super regular or non-super regular disks may be 1. In the case that $\mathscr{B}$ contains a finite number of isolated disks, the codimension of non super regular disks or non almost super regular disks must have codimension 2 or higher. In this case, $\mathscr{B}$ defines an almost super regular foliation. The last remaining case is that $\mathscr{B}$ is regular but the set of non-super regular or non-almost super regular disks may have codimension 1. In this case, we can perturb this component $\mathscr{B}$ for $t - \bar{t} > 0$ small. Because covering index for evulation map is constant through continuous defomation, the covering index for super regular discs in $\mathscr{B}$ at time t must be 1. Following Proposition 3.4.6, the connected component after perturbation defines an almost super regular foliation for $t > \bar{t}$.

In all cases, it is easy to see that ϕ_∞ is smooth in an open dense subset $\mathscr{R}_{\phi_\infty}$ of $\Sigma \times$ M. Moreover, we can show that $\mathscr{B}$ is unique since the corresponding partially smooth solution in the limit is unique. This in particular implies that the limit $\mathscr{F}_{\psi(\tau_\infty,\cdot)}$ is independent of the time sequence $\tau_i \to \infty$. □

Now we return to prove our main theorem.

Proof. — To prove openness, we assume $\mathscr{F}_{\psi(\bar{t},\cdot)}$ is an almost super regular foliation. Here we follow the notations in Theorem 3.4.10. Without loss of generality, we may assume $\bar{t} \leq \bar{t}_1$. If $\bar{t} < \bar{t}_1$, then $\mathscr{M}_{\psi(\bar{t},\cdot)}$ is smooth. In particular, the connected component $\bar{\mathscr{U}}_{\phi(\bar{t},\cdot)}$ is smooth without boundary. Following from the standard deformation theory, this component will deform to a smooth component $\bar{\mathscr{U}}_{\psi(t,\cdot)}$ of $\mathscr{M}_{\psi(t,\cdot)}$ for $t - \bar{t} > 0$ small enough. By Theorems 3.4.8 and 3.4.10, $\bar{\mathscr{U}}_{\psi(t,\cdot)}$ induces an almost super regular foliation, so the openness follows in this case.

Now assume $\bar{t} = \bar{t}_1$. We want to show that for $\bar{t}_1 = \bar{t} < t < \bar{t}_2$, there is an almost super regular foliation $\mathscr{F}_{\psi(t,\cdot)}$.

By our choice of the generic path ψ, we may assume that there are a finite number of embedded, non-regular disks in $\bar{\mathscr{U}}_{\psi(\bar{t},\cdot)} \backslash \mathscr{U}_{\psi(\bar{t},\cdot)}$. Since we are interested in preserving this connected component $\mathscr{U}_{\phi(\bar{t},\cdot)}$, we want to rule out the possibility of either a "merge in" or "spin off" occurring. In other words, there might be component of $\mathcal{M}_{\psi(\bar{t},\cdot)}$ connecting with $\bar{\mathcal{U}}_{\psi(\bar{t},\cdot)}$ through these isolated singular disks: Two components before $t = \bar{t}$ may merge locally into one smooth connected component after $t = \bar{t}$. The situation can also occur in the reverse order: an open set of the moduli $\mathcal{M}_{\psi(t,\cdot)}$ may pinch off of a "neck S^{2n-1}" at $t = \bar{t}$ and go on to become two separate components after $t = \bar{t}$, at least locally near this "neck." We call the first case "merge-in" and the second case "pinching-off". If either one occurs, this "good" component will change after singular disks. The deformation of almost super regular foliations is impossible if either of these phenomenon occur beyond the time when singular disks appear. We will deal only with the "merge-in" case here, since the other cases (like the "pinching-off" case) can be handled in a similar fashion.

Note that the "merge-in" of the *moduli* spaces occurs only at singular disks. Since there is only finite number of singular disks and "merge in" only occur locally near singular disk, we may assume without loss of generality, there is only one non-regular disk $\bar{f}$ in $\bar{\mathcal{U}}_{\psi(t)}$.

Without loss of generality, set $\bar{t} = \bar{t}_1$ and $\mathcal{M}_t = \mathcal{M}_{\psi(t,0)}$ is an almost super regular foliation for any $t \in (0, \bar{t}\,]$. Suppose $\bar{f}$ is the only isolated singular disk at $t = \bar{t}$. Then the metric ball $B_r(\bar{f})$ in $\mathcal{M}_{\bar{t}}$ can be represented by a cone in R^{2n+1}:

$$\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{2n+1} x_i^2 = 0$$

where

$$\sum_{i=k+1}^{2n+1} x_i^2 < r^2.$$

Here $(0, 0, ..., 0)$ represent $\bar{f}$. The "merge in" or "spin off" case corresponds to $k = 2n$ or $k = 1$. We only discuss the "merge in" case here. For $\bar{t} - t$ small, the corresponding metric ball in $\mathcal{M}_t$ is

$$\sum_{i=1}^{2n} x_i^2 - x_{2n+1}^2 = t - \bar{t}, \quad x_{2n+1}^2 < r^2 + \bar{t} - t, \ x_{2n+1} > 0.$$

For $t - \bar{t}$ small, the corresponding metric ball in $\mathcal{M}_t$ is

$$\sum_{i=1}^{2n} x_i^2 - x_{2n+1}^2 = t - \bar{t}, \quad \sum_{k=1}^{2n} x_k^2 \le r^2 + t - \bar{t}.$$

Choose a continuous path of disks $f(t) \in \mathcal{M}_{\psi(t,\cdot)}$ such that $f(t) = \bar{f}$ and $f(t)(t \neq \bar{t})$ is either super regular or almost super regular disk. For notational simplicity, we denote $f(t)$ by $\bar{f}$. Note that for $r > 0$ small enough, the intersection $B_r(\bar{f}) \cap \mathcal{M}_t$ consists of two disjoint disks for $t < \bar{t}$, but is cylinder-like for $t > \bar{t}$. We consider the intersection of this ball with the central fibre $\{0\} \times M$. In this proof, we use ev to denote the map $\pi \circ ev(0, \cdot)$. Set $ev(\bar{f}) = p$.

Note that for $t > \bar{t}$, the boundary of $B_r(\bar{f})$ consists of two components $N_1 \approx N_2 \approx S^{2n-1}$ ($\approx$ means diffeomorphic to) which are homotopic to each other in $B_r(\bar{f})$. These boundary spheres are perturbations of $\partial B_r(\bar{f}) \cap \mathscr{M}_{\psi(\bar{t},\cdot)}$. Let us pick up one of these spheres, say N_1 for $t > \bar{t}$. Note that each component of $ev(B_r(\bar{f}) \cap \mathscr{M}_{\psi(\bar{t},\cdot)})$ bounds a deformation retractable domain in the central fibre $\{0\} \times M$. By continuity, $ev(N_1)$ also bounds a domain Ω which is deformation contractible to an interior point $q \in \Omega$ for $t - \bar{t}$ sufficiently small. Let us denote this contraction by $F : [0, 1] \times ev(N_1) \mapsto \Omega$ such that $F(0, p) = p$ and $F(1, p) = q$ for any $p \in ev(N_1)$. Since the set of disks which are neither super regular nor almost super regular has codimension 2 or higher, there is an open subset $V \subset ev(N_1)$ such that $F([0, 1) \times V)$ does not intersect with the image of the set of disks which are neither super regular nor almost super regular. Now we can lift this $F([0, 1) \times V)$ to $\mathscr{M}_t$ since any point in the subset has its pre-image covered by either an super regular disks or an almost super regular disk. This implies that there is a subset N_3 such that $ev(N_3)$ is a single point, where N_3 consists of all limiting points of the lifting of $[0, 1) \times V$. Clearly, any disk in N_3 is neither super regular nor almost super regular. Observing that N_3 has codimension one, we get a contradiction to the fact that the set of all disks which are not almost super regular has codimension at least two.

By similar arguments, we can prove that there is no "pinching-off" at $t = \bar{t}$. □

4. Basic curvature equations along leaves

4.1. *Introduction*

In this section, we show some deformation results for the homogenous complex Monge–Ampere equations. In particular, we give a basic curvature formula for the restriction of involved metric to **leaves** (the integration curve of Kernel direction to the Levi form of the solution). This formula plays a crucial role in deriving key *a priori* estimates. Suppose that ϕ is a solution of (1.1). Suppose that the $\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\phi$, referred as the *Levi* form, has constant co-rank 1. This gives rise to a foliation of the domain by holomorphic disks. We further assume that:

> **At each point of the domain, the leaf vector is always transversal to M in $\Sigma \times M$.**

Under this assumption, the *Levi* form restricts to a Kähler metric in M for each $z \in \Sigma$. In this way, a solution of (1.1) can be alternatively viewed as a disk family of

Kähler metrics satisfying certain geometric conditions. We will study the restriction of the complex tangent bundle TM over this family of holomorphic disks. These bundles are equipped with natural Hermitian metrics (the varying Levi form of the underlying solution). Thus, we get a family of Hermitian bundle over disks. In this section, we will compute curvature of these Hermitian bundles. The main results are

1. The curvature of these Hermitian metrics is always non-positive (Theorem 4.2.8);
2. The foliation is holomorphic if and only if the "trace of the curvature" of these Hermitian metrics vanishes (Theorem 4.2.9);
3. The trace of the Hermitian curvature is always super harmonic (Corollary 4.2.11).

The results in this section lay foundation for global deformation of almost super regular foliations in this paper.

4.2. *Curvature formulas*

In local coordinate, we write

$$\omega_0 = \sqrt{-1}\sum_{\alpha,\beta=1}^{n} g_{0,\alpha\bar{\beta}}dw^{\alpha}\wedge dw^{\bar{\beta}}, \quad \omega_\phi = \sqrt{-1}\sum_{\alpha,\beta=1}^{n} g_{\phi,\alpha\bar{\beta}}dw^{\alpha}\wedge dw^{\bar{\beta}}$$

where

$$g_{\phi,\alpha\bar{\beta}} = g_{0,\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial w_\alpha \partial \bar{w}_\beta}, \quad \forall\, \alpha,\beta = 1,2,\ldots,n.$$

As before, z denotes the coordinate variable of Σ. Then (1.1) can be re-written as

$$\frac{\partial^2\phi}{\partial z\partial\bar{z}} - g_\phi^{\alpha\bar{\beta}}\frac{\partial^2\phi}{\partial z\partial\bar{w}_\beta}\frac{\partial^2\phi}{\partial\bar{z}\partial w_\alpha} = 0. \tag{4.1}$$

Here we are assuming that $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_0$ in M. In this section, we will simply write g for the metric g_ϕ if there is no confusion. Write the leaf vector field (cf. Definition 1.4.3) as

$$X = \sum_{\alpha=1}^{n}\eta^{\alpha}\frac{\partial}{\partial w_\alpha} = \sum_{\alpha=1}^{n} -g^{\alpha\bar{\beta}}\frac{\partial^2\phi}{\partial z\partial\bar{w}_\beta}\frac{\partial}{\partial w_\alpha}. \tag{4.2}$$

Denote the linearized operator by Δ_z. There is a natural splitting of this operator since all disks are holomorphic:

$$\Delta_z = \partial_z\bar{\partial}_z, \quad \text{where } \partial_z = \frac{\partial}{\partial z} + \eta^{\alpha}\frac{\partial}{\partial w_\alpha}. \tag{4.3}$$

Proposition **4.2.1.** — *The leaf vector field* X *(cf. Definition 1.3) is holomorphic in z. In other words*

$$[\partial_z, \bar{\partial}_z] = \partial_{\bar{z}} X = 0. \tag{4.4}$$

Proof. — Direct calculation. □

Proposition **4.2.2.** — *The commutator of local differential operator on* TM *and the leaf derivative ∂_z is*

$$\left[\partial_z, \frac{\partial}{\partial w_i}\right] = -\frac{\partial \eta^\alpha}{\partial w_i}\frac{\partial}{\partial w_\alpha}, \quad \left[\partial_z, \frac{\partial}{\partial \bar{w}_i}\right] = -\frac{\partial \eta^\alpha}{\partial \bar{w}_i}\frac{\partial}{\partial w_\alpha}. \tag{4.5}$$

Proof

$$\left[\partial_z, \frac{\partial}{\partial w_i}\right] = \left[\frac{\partial}{\partial z} + \eta^\alpha \frac{\partial}{\partial w_\alpha}, \frac{\partial}{\partial w_i}\right] = -\frac{\partial \eta^\alpha}{\partial w_i}\frac{\partial}{\partial w_\alpha},$$

and

$$\left[\partial_z, \frac{\partial}{\partial \bar{w}_i}\right] = \left[\frac{\partial}{\partial z} + \eta^\alpha \frac{\partial}{\partial w_\alpha}, \frac{\partial}{\partial w_i}\right] = -\frac{\partial \eta^\alpha}{\partial \bar{w}_i}\frac{\partial}{\partial w_\alpha}.$$

□

Remark **4.2.3.** — Note that $\frac{\partial \eta^\alpha}{\partial w_i}$ is not a globally well defined tensor, while $\frac{\partial \eta^\alpha}{\partial \bar{w}_i}$ is a globally well defined tensor since

$$\frac{\partial \eta^\alpha}{\partial \bar{w}_i} = -g^{\alpha \bar{s}} \left(\frac{\partial \phi}{\partial z}\right)_{,\bar{i}\bar{s}}.$$

In a local coordinate chart, suppose $g_{0,\alpha\bar{\beta}} = \frac{\partial^2 \rho}{\partial w_\alpha \partial \bar{w}_\beta}$ where ρ is independent of the z direction. Let Φ denote the local Kähler potential for g, then

$$\Phi = \phi + \rho.$$

Proposition **4.2.4.** — *For global Kähler distortion potentials, the following is true:*

$$\Delta_z \phi = \partial_z \bar{\partial}_z(\phi) = -|X|^2_{g_0}. \tag{4.6}$$

Proof. — By a straightforward calculation. □

Lemma **4.2.5** *(The bootstrapping lemma). — The following commutation formula for the third transversal derivatives holds*

$$\frac{\partial \eta^\alpha}{\partial w_i} = -g^{\alpha\bar\beta}\partial_z g_{i\bar\beta}, \quad \text{and} \quad \frac{\partial \eta^\alpha}{\partial \bar w_i} = -g^{\alpha\bar\beta}\partial_z \frac{\partial^2 \Phi}{\partial \bar w_i \partial \bar w_\beta}. \tag{4.7}$$

Proof

$$\begin{aligned}\partial_z g_{i\bar\beta} = \partial_z \frac{\partial^2 \Phi}{\partial w_i \partial \bar w_\beta} &= \frac{\partial}{\partial w_i}\partial_z \frac{\partial \Phi}{\partial \bar w_\beta} - \frac{\partial \eta^\alpha}{\partial w_i}\frac{\partial \Phi}{\partial \bar w_\beta \partial w_\alpha}\\ &= -\frac{\partial \eta^\alpha}{\partial w_i} g_{\alpha\bar\beta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\partial_z \frac{\partial^2 \Phi}{\partial \bar w_i \partial \bar w_\beta} &= \frac{\partial}{\partial \bar w_i}\partial_z \frac{\partial \Phi}{\partial \bar w_\beta} - \frac{\partial \eta^\alpha}{\partial \bar w_i}\frac{\partial \Phi}{\partial \bar w_\beta \partial w_\alpha}\\ &= -\frac{\partial \eta^\alpha}{\partial \bar w_i} g_{\alpha\bar\beta}.\end{aligned}$$

□

Remark **4.2.6.** — The significance of this equation (4.7) is that it changes the 1st derivatives on the transversal direction into the 1st derivatives along the disk of the 2nd order jet of local Kähler potentials.

Lemma **4.2.7** *(Regularity lemma). — The following equation holds along the disk (for any $\alpha, i = 1, 2, \ldots, n$):*

$$\bar\partial_z \frac{\partial \eta^\alpha}{\partial w_i} = -\overline{\left(\frac{\partial \eta^\beta}{\partial \bar w_i}\right)}\frac{\partial \eta^\alpha}{\partial \bar w_\beta}, \quad \text{and} \quad \bar\partial_z \frac{\partial \eta^\alpha}{\partial \bar w_i} = -\overline{\left(\frac{\partial \eta^\beta}{\partial w_i}\right)}\frac{\partial \eta^\alpha}{\partial \bar w_\beta}. \tag{4.8}$$

Proof. — This lemma follows from the commutation formula (4.5) and Lemma 4.2.8 immediately. □

The major obstacle in establishing an *a priori* estimate for (1.1) is that its linearized operator Δ_z only has rank 1. This is a severe restriction in deriving any meaningful estimate directly. However, if we restrict the $k-(k = 0, 1, 2, \ldots)$ jet of the potential function Φ along the disk first, we can study how these quantities changes along this disc – in other words, what kind of equation(s) these quantities must satisfy. In this section, we consider the second order jet of Φ over this holomorphic disk. Notice that the restriction of TM bundle to the disk is a trivial $\mathbf{C}^n$ bundle. At each point, g_ϕ is a Hermitian metric in this TM bundle over disk. Suppose that $(\mathrm{F}^\alpha_\beta)$ is the curvature of this metric. Then

Theorem **4.2.8.** — *The curvature of the bundle metric is always non-positive.*

Proof

$$\begin{aligned} F^r_\alpha &= -\partial_{\bar{z}}(g^{r\bar{\delta}}\partial_z g_{\phi,\alpha\bar{\delta}}) \\ &= \partial_{\bar{z}}\frac{\partial \eta^r}{\partial w_{\bar{\alpha}}} = -\frac{\partial \eta^r}{\partial w_{\bar{i}}}\cdot\frac{\partial \eta^{\bar{i}}}{\partial w_\alpha}. \end{aligned}$$

It is not difficult to see that the last term in the right hand side is a Hermitian symmetric non-positive 2-tensors. For any holomorphic section $s : \Sigma \to T^{1,0}M$, we have

$$\begin{aligned} F(s,s) &= F^\alpha_\beta s^\beta s^{\bar{\gamma}} g_{\phi\alpha\bar{\gamma}} \\ &= -\frac{\partial \eta^\alpha}{\partial w_{\bar{i}}}\cdot\frac{\partial \eta^{\bar{i}}}{\partial w_\beta} s^\beta s^{\bar{\gamma}} g_{\phi\alpha\bar{\gamma}} \leq 0. \end{aligned}$$

□

A quick corollary follows

Theorem **4.2.9.** — *This foliation by holomorphic disks is a holomorphic foliation if and only if the curvature of these Hermitian metrics vanishes.*

Proof. — This is quite evident from Theorem 4.2.8. □

Proposition **4.2.10.** — $g^{i\bar{j}}\frac{\partial \eta^\alpha}{\partial \bar{w}_j}$ *is holomorphic along the disk.*

Proof

$$\bar{\partial}_z g^{i\bar{j}} = -g^{i\bar{k}}\bar{\partial}_z g_{\bar{k}l} g^{l\bar{j}} = g^{i\bar{k}}\frac{\partial \eta^{\bar{\beta}}}{\partial \bar{w}_k} g_{l\bar{\beta}} g^{l\bar{j}} = g^{i\bar{k}}\frac{\partial \eta^{\bar{j}}}{\partial \bar{w}_k}.$$

Now, we have

$$\begin{aligned} \bar{\partial}_z\left(g^{i\bar{j}}\frac{\partial \eta^\alpha}{\partial \bar{w}_j}\right) &= \left(\bar{\partial}_z g^{i\bar{j}}\right)\frac{\partial \eta^\alpha}{\partial \bar{w}_j} + g^{i\bar{j}}\bar{\partial}_z\frac{\partial \eta^\alpha}{\partial \bar{w}_j} \\ &= g^{i\bar{k}}\frac{\partial \eta^{\bar{j}}}{\partial \bar{w}_k}\frac{\partial \eta^\alpha}{\partial \bar{w}_j} - g^{i\bar{j}}\overline{\left(\frac{\partial \eta^\beta}{\partial w_j}\right)}\frac{\partial \eta^\alpha}{\partial \bar{w}_\beta} = 0. \end{aligned}$$

□

Corollary **4.2.11**[19]**.** — *The anti-canonical line bundle in* M *equipped with* ω^n_ϕ *as Hermitian metric, restricted to a holomorphic disk, has non-positive curvature. More precisely, we have*

$$\Delta_z \log \omega^n_\phi = \partial_z\bar{\partial}_z \log \omega^n_\phi = \frac{\partial \eta^\alpha}{\partial \bar{w}_i}\overline{\left(\frac{\partial \eta^i}{\partial \bar{w}_\alpha}\right)}. \tag{4.9}$$

Note that the right hand side measures whether the given geodesic (or disk version) is holomorphic or not.

[19] This was first derived in 1996 by the first author and S. Donaldson using some different methods.

Proof. — In a local coordinate, we have

$$\partial_z \log \omega_\phi^n = g^{\alpha\bar\beta}\partial_z g_{\bar\beta\alpha} = -\frac{\partial \eta^\alpha}{\partial w_\alpha}.$$

Thus,

$$\begin{aligned}\bar\partial_z\partial_z \log \omega_\phi^n &= -\bar\partial_z \frac{\partial \eta^\alpha}{\partial w_\alpha} = -\left(\frac{\partial}{\partial w_\alpha}\bar\partial_z - \frac{\partial \eta^{\bar\beta}}{\partial w_\alpha}\frac{\partial}{\partial w_{\bar\beta}}\right)\eta^\alpha \\ &= \frac{\partial \eta^{\bar\beta}}{\partial w_\alpha}\frac{\partial \eta^\alpha}{\partial \bar w_\beta} \geq 0.\end{aligned}$$

□

Theorem **4.2.12.** — *Set* $-\mathrm{S} = \mathrm{F}_\alpha^\alpha = \frac{\partial \eta^{\bar\beta}}{\partial w_\alpha}\frac{\partial \eta^\alpha}{\partial \bar w_\beta} = g^{\alpha\bar p}\phi_{,z\bar i\bar p}g^{\bar i q}\phi_{,\bar z\alpha q} \geq 0$ *(this denotes the covariant derivatives w.r.t. g.) as the trace of the curvature of this* TM *bundle with Hermitian metric* g_ϕ. *Then, this trace of curvature is sub-harmonic. More specifically, we have*

$$(\mathbf{4.10})\qquad \begin{aligned}\partial_z\bar\partial_z \mathrm{S} &= \left(\partial_z\frac{\partial\eta^\alpha}{\partial\bar w_i} - \frac{\partial\eta^\beta}{\partial\bar w_i}\frac{\partial\eta^\alpha}{\partial w_\beta}\right)\left(\bar\partial_z\frac{\partial\eta^{\bar i}}{\partial w_\alpha} - \frac{\partial\eta^{\bar i}}{\partial\bar w_p}\frac{\partial\eta^{\bar p}}{\partial w_\alpha}\right) + 2\frac{\partial\eta^\alpha}{\partial\bar w_i}\frac{\partial\eta^{\bar p}}{\partial w_\alpha}\frac{\partial\eta^\beta}{\partial\bar w_p}\frac{\partial\eta^{\bar i}}{\partial w_\beta} \\ &= \frac{\partial}{\partial\bar w_i}\partial_z(\eta^\alpha)\frac{\partial}{\partial w_\alpha}\bar\partial_z(\eta^{\bar i}) + 2g^{\alpha\bar i}g^{\bar\gamma j}g^{\beta\bar k}g^{\bar\delta l}\phi_{,z\bar i\bar\gamma}\phi_{,\bar z j\beta}\phi_{,z\bar k\bar\delta}\phi_{,\bar z l\alpha} \geq \frac{2}{n}\mathrm{S}^2.\end{aligned}$$

Proof. — According to Lemma 4.2.7, we have

$$\bar\partial_z\frac{\partial\eta^\alpha}{\partial\bar w_i} = -\frac{\partial\eta^{\bar\beta}}{\partial\bar w_i}\frac{\partial\eta^\alpha}{\partial\bar w_\beta}.$$

Thus

$$\begin{aligned}\partial_z\bar\partial_z\frac{\partial\eta^\alpha}{\partial\bar w_i} &= -\partial_z\left(\frac{\partial\eta^{\bar\beta}}{\partial\bar w_i}\right)\frac{\partial\eta^\alpha}{\partial\bar w_\beta} - \frac{\partial\eta^{\bar\beta}}{\partial\bar w_i}\partial_z\frac{\partial\eta^\alpha}{\partial\bar w_\beta} \\ &= \frac{\partial\eta^p}{\partial\bar w_i}\frac{\partial\eta^{\bar\beta}}{\partial w_p}\frac{\partial\eta^\alpha}{\partial\bar w_\beta} - \frac{\partial\eta^{\bar\beta}}{\partial\bar w_i}\partial_z\frac{\partial\eta^\alpha}{\partial\bar w_\beta}.\end{aligned}$$

We have

$$\begin{aligned}&\Delta_z\left(\frac{\partial\eta^\alpha}{\partial\bar w_i}\frac{\partial\eta^{\bar i}}{\partial w_\alpha}\right) \\ &= \partial_z\left(\frac{\partial\eta^\alpha}{\partial\bar w_i}\right)\bar\partial_z\left(\frac{\partial\eta^{\bar i}}{\partial w_\alpha}\right) + \bar\partial_z\left(\frac{\partial\eta^\alpha}{\partial\bar w_i}\right)\partial_z\left(\frac{\partial\eta^{\bar i}}{\partial w_\alpha}\right) \\ &\quad + \mathrm{L}\left(\frac{\partial\eta^\alpha}{\partial\bar w_i}\right)\left(\frac{\partial\eta^{\bar i}}{\partial w_\alpha}\right) + \left(\frac{\partial\eta^\alpha}{\partial\bar w_i}\right)\mathrm{L}\left(\frac{\partial\eta^{\bar i}}{\partial w_\alpha}\right)\end{aligned}$$

$$
\begin{aligned}
&= \partial_z\left(\frac{\partial\eta^\alpha}{\partial\bar{w}_i}\right)\bar{\partial}_z\left(\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha}\right) + \frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^p}{\partial w_\alpha}\frac{\partial\eta^{\bar{i}}}{\partial w_p}\\
&\quad + \left(\frac{\partial\eta^p}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{\beta}}}{\partial w_p}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta} - \frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_i}\partial_z\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\right)\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha}\\
&\quad + \frac{\partial\eta^\alpha}{\partial\bar{w}_i}\overline{\left(\frac{\partial\eta^p}{\partial\bar{w}_\alpha}\frac{\partial\eta^{\bar{\beta}}}{\partial w_p}\frac{\partial\eta^i}{\partial\bar{w}_\beta} - \frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_\alpha}\partial_z\frac{\partial\eta^i}{\partial\bar{w}_\beta}\right)}\\
&= \partial_z\left(\frac{\partial\eta^\alpha}{\partial\bar{w}_i}\right)\bar{\partial}_z\left(\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha}\right) + \frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^p}{\partial w_\alpha}\frac{\partial\eta^{\bar{i}}}{\partial w_p} - \partial_z\left(\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\right)\frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha}\\
&\quad - \frac{\partial\eta^\alpha}{\partial\bar{w}_i}\frac{\partial\eta^\beta}{\partial w_\alpha}\bar{\partial}_z\frac{\partial\eta^{\bar{i}}}{\partial w_\beta} + \frac{\partial\eta^p}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{\beta}}}{\partial w_p}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} + \frac{\partial\eta^\alpha}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\frac{\partial\eta^\beta}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{i}}}{\partial w_\beta}\\
&= \partial_z\left(\frac{\partial\eta^\alpha}{\partial\bar{w}_i}\right)\bar{\partial}_z\left(\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha}\right) + \frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^p}{\partial w_\alpha}\frac{\partial\eta^{\bar{i}}}{\partial w_p} - \partial_z\left(\frac{\partial\eta^\alpha}{\partial\bar{w}_i}\right)\frac{\partial\eta^{\bar{i}}}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\\
&\quad - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta}\bar{\partial}_z\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} + \frac{\partial\eta^p}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{\beta}}}{\partial w_p}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} + \frac{\partial\eta^\alpha}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\frac{\partial\eta^\beta}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{i}}}{\partial w_\beta}\\
&= \left(\partial_z\frac{\partial\eta^\alpha}{\partial\bar{w}_i} - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta}\right)\left(\bar{\partial}_z\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} - \frac{\partial\eta^{\bar{i}}}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\right) - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta}\frac{\partial\eta^{\bar{i}}}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\\
&\quad + \frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^p}{\partial w_\alpha}\frac{\partial\eta^{\bar{i}}}{\partial w_p} + \frac{\partial\eta^p}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{\beta}}}{\partial w_p}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} + \frac{\partial\eta^\alpha}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\frac{\partial\eta^\beta}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{i}}}{\partial w_\beta}\\
&= \left(\partial_z\frac{\partial\eta^\alpha}{\partial\bar{w}_i} - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta}\right)\left(\bar{\partial}_z\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} - \frac{\partial\eta^{\bar{i}}}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\right) - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\frac{\partial\eta^{\bar{i}}}{\partial\bar{w}_p}\\
&\quad + \frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^p}{\partial w_\alpha}\frac{\partial\eta^{\bar{i}}}{\partial w_p}\frac{\partial\eta^{\bar{\beta}}}{\partial\bar{w}_i} + \frac{\partial\eta^p}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{\beta}}}{\partial w_p}\frac{\partial\eta^\alpha}{\partial\bar{w}_\beta}\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} + \frac{\partial\eta^\alpha}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\frac{\partial\eta^\beta}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{i}}}{\partial w_\beta}\\
&= \left(\partial_z\frac{\partial\eta^\alpha}{\partial\bar{w}_i} - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta}\right)\left(\bar{\partial}_z\frac{\partial\eta^{\bar{i}}}{\partial w_\alpha} - \frac{\partial\eta^{\bar{i}}}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\right) + 2\frac{\partial\eta^\alpha}{\partial\bar{w}_i}\frac{\partial\eta^{\bar{p}}}{\partial w_\alpha}\frac{\partial\eta^\beta}{\partial\bar{w}_p}\frac{\partial\eta^{\bar{i}}}{\partial w_\beta}.
\end{aligned}
$$

The last equality holds because in the line above the last equation, the second term and the 3rd cancel each other, while the 3rd and 4th term are the same.

Note that $\partial_z\frac{\partial\eta^\alpha}{\partial\bar{w}_i} - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta}$ is a tensor since

$$
\begin{aligned}
\partial_z\frac{\partial\eta^\alpha}{\partial\bar{w}_i} - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\frac{\partial\eta^\alpha}{\partial w_\beta} &= \frac{\partial}{\partial z}\frac{\partial\eta^\alpha}{\partial\bar{w}_i} + \eta^r\frac{\partial}{\partial w_r}\frac{\partial\eta^\alpha}{\partial\bar{w}_i} - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\left(\frac{\partial\eta^\alpha}{\partial w_\beta} + \eta^r\Gamma^\alpha_{r\beta} - \eta^r\Gamma^\alpha_{r\beta}\right)\\
&= \frac{\partial}{\partial z}\frac{\partial\eta^\alpha}{\partial\bar{w}_i} + \eta^r\left(\frac{\partial}{\partial w_r}\frac{\partial\eta^\alpha}{\partial\bar{w}_i} + \frac{\partial\eta^\beta}{\partial\bar{w}_i}\Gamma^\alpha_{r\beta}\right) - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\eta^\alpha{}_{,\beta}\\
&= \frac{\partial}{\partial z}\frac{\partial\eta^\alpha}{\partial\bar{w}_i} + \eta^r\eta^\alpha{}_{,\bar{i}r} - \frac{\partial\eta^\beta}{\partial\bar{w}_i}\eta^\alpha{}_{,\beta}.
\end{aligned}
$$

Moreover, this is a (0,2) symmetric tensor since

$$\begin{aligned}
\partial_z \frac{\partial \eta^\alpha}{\partial \bar{w}_i} - \frac{\partial \eta^\beta}{\partial \bar{w}_i}\frac{\partial \eta^\alpha}{\partial w_\beta} &= \partial_z\left(- g^{\alpha\bar\beta}\partial_z \frac{\partial^2 \Phi}{\partial \bar{w}_i \partial \bar{w}_\beta}\right) - \frac{\partial \eta^\beta}{\partial \bar{w}_i}\frac{\partial \eta^\alpha}{\partial w_\beta}\\
&= g^{\alpha\bar p}\partial_z g_{\bar p q} g^{q\bar\beta}\frac{\partial^2 \Phi}{\partial \bar{w}_i \partial \bar{w}_\beta} - g^{\alpha\bar\beta}(\partial_z)^2 \frac{\partial^2 \Phi}{\partial \bar{w}_i \partial \bar{w}_\beta} - \frac{\partial \eta^\beta}{\partial \bar{w}_i}\frac{\partial \eta^\alpha}{\partial w_\beta}\\
&= \frac{\partial \eta^q}{\partial \bar{w}_i}\frac{\partial \eta^\alpha}{\partial w_q} - g^{\alpha\bar\beta}(\partial_z)^2 \frac{\partial^2 \Phi}{\partial \bar{w}_i \partial \bar{w}_\beta} - \frac{\partial \eta^\beta}{\partial \bar{w}_i}\frac{\partial \eta^\alpha}{\partial w_\beta}\\
&= -g^{\alpha\bar\beta}(\partial_z)^2 \frac{\partial^2 \Phi}{\partial \bar{w}_i \partial \bar{w}_\beta}\\
&= -g^{\alpha\bar\beta}\partial_z \left(\frac{\partial \phi}{\partial z}\right)_{,\bar i \bar\beta}.
\end{aligned}$$

Thus, the first term in (4.10) can be changed into

$$\begin{aligned}
&\left(\partial_z \frac{\partial \eta^\alpha}{\partial \bar{w}_i} - \frac{\partial \eta^\beta}{\partial \bar{w}_i}\frac{\partial \eta^\alpha}{\partial w_\beta}\right)\left(\bar\partial_z \frac{\partial \eta^{\bar i}}{\partial w_\alpha} - \frac{\partial \eta^{\bar i}}{\partial \bar{w}_p}\frac{\partial \eta^{\bar p}}{\partial w_\alpha}\right)\\
&\quad = g^{\alpha\bar\beta}(\partial_z)^2 \frac{\partial^2 \Phi}{\partial \bar{w}_i \partial \bar{w}_\beta} g^{\bar i p}(\bar\partial_z)^2 \frac{\partial^2 \Phi}{\partial w_\alpha \partial w_p} \geq 0.
\end{aligned}$$

The lemma is then proved. □

This theorem should be compared with Calabi's third derivative estimate for the non-degenerate Monge–Ampere equation. Following a result of R. Osserman [23] (later generalized by E. Calabi [3]), we have

Proposition **4.2.13.** — *Let d denote the Euclidean distance to the boundary of* Σ, *then there exists a uniform constant* C *such that the trace of the curvature has the following interior estimate*

$$S(z, x) = -\sum_{\alpha=1}^{n} F_\alpha^\alpha \leq \frac{C}{d(z, \partial\Sigma)^2}.$$

5. Compactness of holomorphic disks

In this section, we continue our study of the HCMA equation from the point of view initiated in the previous section. Namely, in the foliation by holomorphic disks of $\Sigma \times M$, we study the family of restricted TM bundles equipped with the varying Hermitian metric g_ϕ. For any $\phi_0 : \partial\Sigma \to \mathcal{H}_\omega$ and any holomorphic disk $f \in \mathcal{M}_{\phi_0}$, we define its area as

$$\text{(5.1)} \qquad A(f) = \int_\Sigma \left(\frac{\sqrt{-1}}{2} dz \wedge d\bar{z} + f^* \omega_0 \right).$$

Note that this is the area of $\pi \circ f(\Sigma)$ in $\Sigma \times M$, not the area of $f(\Sigma) \in \Sigma \times \mathscr{W}_M$. When no confusion is possible, we will not distinguish between $f, \pi \circ f$, or even $\pi_2 \circ \pi \circ f$. Similarly, we also define the **Capacity** for any super regular disk f in $\mathscr{M}_{\phi_0}$ by

$$\text{(5.2)} \qquad \mathrm{Cap}(f) = \int_\Sigma f^* \left(\frac{\omega^n_{\phi_0(z_0)}}{\omega^n_\phi} \right) \frac{\sqrt{-1}}{2} dz d\bar{z}$$

for some $z_0 \in \partial\Sigma$. For simplicity, we fix z_0 in this section and assume without loss of generality that $\phi_0(z_0) = 0$. Under this assumption, $\omega_{\phi_0(z_0)} = \omega_0$. Obviously, a non-super regular disk has infinite capacity.

Let us set up some notations first. Fixing a positive number $\alpha \in (0, 1)$ in this section, for any $\epsilon, \delta, \Lambda$, we define $\mathscr{C}(\delta, \Lambda)$ as the space of all embedded holomorphic disks in $\Sigma \times \mathscr{W}_M$ with vanishing normal *Maslov* indice mapping $\partial\Sigma$ into $\bar{\Lambda}_{\phi_0}$, where ϕ_0 is a map from $\partial\Sigma$ to $\mathscr{H}_\omega$ which satisfies

$$\text{(5.3)} \qquad \omega_{\phi_0} \geq \delta\omega_0, \quad \|\phi_0\|_{C^{2,\alpha}(\partial\Sigma\times M)} \leq \Lambda.$$

The space $\mathscr{C}(\delta, \Lambda, L_0)$ is a subset of $\mathscr{C}(\delta, \lambda)$ such that for each disk $f \in \mathscr{C}(\delta, \Lambda, L_0)$, the corresponding germ ϕ of HCMA equation associated with $\mathscr{F}_{\phi_0}$ (cf. Section 2.2) is smooth locally and

$$\text{(5.4)} \qquad |\phi|_{C^{1,1}} \leq L_0$$

hold in a small tubular neighborhood of $\pi \circ f(\Sigma)$. Define

$$\mathscr{C}(\delta, \Lambda, L_0, L_1) = \{f \in \mathscr{C}(\delta, \Lambda, L_0) \mid A(f) \leq L_1\}$$

and

$$\mathscr{C}(\delta, \Lambda, L_0, L_1, L_2) = \{f \in \mathscr{C}(\delta, \Lambda, L_0) \mid A(f) \leq L_1, \mathrm{Cap}(f) \leq L_2\}.$$

In this section, we will prove

Theorem **5.0.14.** — *The space* $\mathscr{C}(\delta, \Lambda, L_0, L_1)$ *is compact in* $\mathscr{C}(\delta, \Lambda, L_0)$.

Theorem **5.0.15.** — *The space* $\mathscr{C}(\delta, \Lambda, L_0, L_1, L_2)$ *is compact.*

Suppose $\mathscr{F}_{\phi_0}$ is an almost super regular foliation, we want to identify $\mathscr{U}_{\phi_0}$ with an open and dense subset of M via evaluation map:

$$\begin{aligned} \sharp : \Sigma \times \mathscr{M}_{\phi_0} &\to \Sigma \times \mathrm{M} \\ (z, f) &\to \pi \circ ev(z, f). \end{aligned}$$

Then, $\sharp$ is invertible on $\Sigma \times \mathscr{U}_{\phi_0}$. Define $\mathrm{F} = \pi \circ ev \circ \sharp^{-1}$. We can identify $\mathscr{U}_{\phi_0}$ with some open dense subset $\pi \circ ev(z_0, \mathscr{U}_{\phi_0})$ of M. We will use this point of view from time to time in this section.

Theorem **5.0.16.** — *If ϕ_0 satisfies* (5.3) *and if $\mathscr{F}_{\phi_0}$ is an almost super regular foliation, then there exist two constants $\mathrm{L}_0, \mathrm{L}_1$ which depend only on δ, Λ such that*

$$\mathscr{U}_{\phi_0} \subset \mathscr{C}(\delta, \Lambda, \mathrm{L}_0, \mathrm{L}_1).$$

Moreover,

$$\int_{f \in \mathscr{U}_{\phi_0}} \mathrm{Cap}(f) \omega^n_{\phi_0(z_0)} = \int_{f \in \mathscr{U}_{\phi_0}} \mathrm{Cap}(f) \omega^n_0 \leq \mathrm{C}.$$

These three theorems can be used to derive the compactness of almost super regular foliations.

Theorem **5.0.17.** — *Suppose $\{{\phi_0}^{(m)}, m \in \mathscr{N}\}$ is a sequence of elements in $\mathrm{C}^\infty(\partial\Sigma, \mathscr{H}_\omega)$ which satisfies the uniform bound of* (5.3). *Suppose further that sequence converges to ${\phi_0}^{(\infty)} \in \mathrm{C}^\infty(\partial\Sigma, \mathscr{H}_\omega)$ in the $\mathrm{C}^{2,\alpha}(\partial\Sigma \times \mathrm{M})$ norm. Suppose that $\{\mathscr{F}_{\phi_0^{(m)}}, m \in \mathscr{N}\}$ is a sequence of almost super regular foliations, while $\{\phi^{(m)}\}$ is a corresponding sequence of almost smooth solution with Dirichlet boundary value $\{{\phi_0}^{(m)}\}$. Passing to a subsequence if necessary, $\mathscr{F}_{{\phi_0}^{(m)}}$ converges to a partially smooth foliation $\mathscr{F}_{{\phi_0}^{(\infty)}}$. In particular, at least one component of $\mathscr{M}_{{\phi_0}^{(\infty)}}$ contains at least one super regular disk. Moreover, $\phi^{(m)}$ converges in the weak $\mathrm{C}^{1,1}$ norm to $\phi^{(\infty)}$ such that $\omega^n_{\phi^{(\infty)}}$ is a continuous volume form on $\Sigma^0 \times \mathrm{M}$.*

5.1. *Proof of Theorem 5.0.16*

Proof. — Suppose that ϕ is the corresponding almost smooth solution of the HCMA equation (1.1) with the prescribed boundary map $\phi_0 : \partial\Sigma \to \mathscr{H}_\omega$. By Theorem 1.2 [9], there is a uniform constant $\mathrm{C}(\delta, \Lambda)$ such that

$$|\partial\bar{\partial}\phi| \leq \mathrm{C}(\delta, \Lambda).$$

It is clear that for any super regular disk $f \in \mathscr{U}_{\phi_0}$, any germ of HCMA equation (1.1) in a small tubular neighborhood of $\pi \circ ev(f(\Sigma))$, associated with $\mathscr{F}_{\phi_0}$, must agree with ϕ in any small open and saturated neighborhood of $\pi \circ ev(f(\Sigma)) \subset \Sigma \times \mathrm{M}$. Next, we want to show that each disk must also have a uniform upper bound on its area.

Lemma **5.1.1.** — *For any regular disk* $f : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathscr{W}_M, \bar{\Lambda}_{\phi_0})$ *where* ϕ *is the corresponding "almost smooth" solution of* (1.1). *There is a uniform constant* L_1 *which depends only on* δ, Λ *such that* $A(f) \leq L_1$ *holds uniformly.*

Proof. — Recalled that the leaf vector field X (cf. (4.2)) along this disk in $\Sigma \times M$ can be expressed as

$$X = \sum_{\alpha=1}^{n} \eta^\alpha \frac{\partial}{\partial w_\alpha} = -\sum_{\alpha=1}^{n} g^{\alpha\bar{\beta}} \frac{\partial^2 \phi}{\partial z \partial w_{\bar{\beta}}} \frac{\partial}{\partial w_\alpha}. \tag{5.5}$$

According to Corollary 4.2.4, we have

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}(\phi \circ f) = -g_{0\alpha\bar{\beta}} \eta^\alpha \eta^{\bar{\beta}}.$$

Using this holomorphic map f, the pull back of the fixed product metric metric on $\Sigma \times M$ to Σ is:

$$\begin{aligned} f^*\left(g_0 + |dz|^2\right) &= \left| \frac{\partial}{\partial z} + X \right|_{g_0}^2 |dz|^2 \\ &= \left(1 + g_{0\alpha\bar{\beta}} \eta^\alpha \eta^{\bar{\beta}}\right) |dz|^2. \end{aligned}$$

Thus, the area of any disk is:

$$\begin{aligned} \int_{g_x(\Sigma)} 1 &= \int_\Sigma f^*\left(g_0 + |dz|^2\right) \\ &= \int_\Sigma \left(1 + g_{0\alpha\bar{\beta}} \eta^\alpha \eta^{\bar{\beta}}\right) |dz|^2 \\ &= |\Sigma| - \int_\Sigma \partial_z \partial_{\bar{z}} \phi |dz|^2 \\ &= |\Sigma| - \int_\Sigma \frac{\partial^2}{\partial z \partial \bar{z}} (\phi \circ f) |dz|^2 \\ &= |\Sigma| + \int_{\partial\Sigma} \frac{\partial}{\partial z} (\phi \circ f) \cdot \mathbf{n}_\Sigma \\ &= |\Sigma| + \int_{\partial\Sigma} \partial_z \phi|_{\partial\Sigma} \cdot \mathbf{n}_\Sigma |dz|^2 \\ &= |\tilde{\Sigma}| + \int_{\partial\Sigma} \left(\frac{\partial}{\partial z} + \eta^\alpha \frac{\partial}{\partial w_\alpha} \right) \phi|_{\partial\Sigma} \cdot \mathbf{n}_\Sigma |dz|^2 \\ &= |\Sigma| + \int_{\partial\Sigma} \left(\frac{\partial \phi}{\partial z} - g_\phi^{\alpha\bar{\beta}} \frac{\partial^2 \phi}{\partial z \partial w_{\bar{\beta}}} \frac{\partial \phi}{\partial w_\alpha} \right) \Bigg|_{\partial\Sigma} \cdot \mathbf{n}_\Sigma |dz|^2, \end{aligned}$$

where $\mathbf{n}_\Sigma$ represents the normal direction on the boundary of Σ. On $\partial\Sigma$, we have

$$\begin{aligned} g_{\phi,\alpha\bar\beta}(z,\cdot) &= g_{0\alpha\bar\beta}(z,\cdot) + \sqrt{-1}\partial\bar\partial\phi(z,\cdot) \\ &= g_{0\alpha\bar\beta}(z,\cdot) + \sqrt{-1}\partial\bar\partial\phi_0(z,\cdot) \geq \delta g_{0\alpha\bar\beta}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_\Sigma 1 &= |\Sigma| + \int_{\partial\Sigma} \left(\frac{\partial\phi}{\partial z} - g_\phi^{\alpha\bar\beta} \frac{\partial^2\phi}{\partial z\partial\bar\beta} \frac{\partial\phi}{\partial w_\alpha} \right) \Bigg|_{\partial\Sigma} \cdot \mathbf{n}_\Sigma |dz|^2 \\ &\leq L_1(\delta, \Lambda). \end{aligned}$$

□

Now we return to the proof of Theorem 5.0.16. Note that

$$F^*\omega_\phi^n = \omega_{\phi_0(z_0)}^n = \omega_0^n,$$

and

$$\begin{aligned} F^*\omega_{\phi_0(z_0)}^n &= F^*\left(\frac{\omega_0^n}{\omega_\phi^n}\omega_\phi^n\right) \\ &= \left(\frac{\omega_0^n}{\omega_\phi^n}\right) \circ F \cdot \omega_0^n. \end{aligned}$$

Thus

$$\begin{aligned} \int_M \int_\Sigma \left(\frac{\omega_0^n}{\omega_\phi^n}\right) \circ F \frac{\sqrt{-1}}{2} dz \wedge d\bar z z\omega_0^n &= \int_M \int_\Sigma \left(\frac{\omega_0^n}{\omega_{\phi^{(m)}}^n}\right) \circ F\omega_0^n \ \frac{\sqrt{-1}}{2} dz \wedge d\bar z \\ &= \int_M \int_\Sigma F^*\omega_0^n \frac{\sqrt{-1}}{2} dz \wedge d\bar z \\ &= \int_\Sigma \int_M \omega_0^n \frac{\sqrt{-1}}{2} dz \wedge d\bar z = C. \end{aligned}$$

In other words, we have

$$\int_M \mathrm{Cap}(\Sigma_x) dx = \int_M \int_\Sigma \left(\frac{\omega_0^n}{\omega_\phi^n}\right) \circ F \frac{\sqrt{-1}}{2} dzd \wedge \bar z dx \leq C,$$

where C is a topological constant. This concludes our proof of Theorem 5.0.16. □

5.2. *No vertical bubble – Proof of Theorem 5.0.14*

We want to re-phrase Theorem 5.0.14.

Theorem **5.2.1.** — *For any sequence of super regular disks (in a sequence of almost super regular foliations* $({\phi_0}^{(m)}, \mathscr{U}_{{\phi_0}^{(m)}})$*), if* ${\phi_0}^{(m)}$ *converges in the* $C^{2,\alpha}(\partial\Sigma \times M)$ *norm, then there is no vertical bubble in the limit.*

Proof. — Suppose

$$f^{(m)} : \Sigma \mapsto \Sigma \times \mathscr{W}_{\mathrm{M}}$$
$$z \mapsto (z, f^{(m)}(z), \zeta^{(m)}(f^{(m)}(z))), \quad m = 1, 2, ..., \infty$$

is a family of super regular disks in $\mathscr{C}(\delta, \Lambda, \mathrm{L}_0, \mathrm{L}_1)$ where

$$\frac{\partial f^{(m)\alpha}}{\partial z} = -g^{(m)\alpha\bar{\beta}} \frac{\partial^2 \phi^{(m)}}{\partial z \partial \bar{w}_\beta}.$$

Here ζ is the corresponding fibre component of $f^{(m)}(\Sigma)$ in $\mathscr{W}_{\mathrm{M}}$. In a local coordinate, we write

$$\zeta^{(m)i}(z, x) = \frac{\partial(\phi^{(m)} + \rho)}{\partial w_i}(z, x), \quad \forall\, i = 1, 2, ..., n$$

where ρ is a local Kähler potential for the given form ω. Note that in a uniform size neighborhood of $\pi \circ f^{(m)}(\Sigma) \subset \Sigma \times \mathrm{M}$, we have

$$|\phi^{(m)}|_{\mathrm{C}^{1,1}} \leq \mathrm{L}_0.$$

In particular, there exists a uniform constant $\mathrm{C}(\delta, \Lambda)$ such that

$$\max_{\partial\Sigma} \left| \frac{\partial f^{(m)}}{\partial z} \right|_{g_0} \leq \mathrm{C}. \tag{5.6}$$

Consequently, all bubble points[20] that occur must occur in the interior of $\Sigma \times \mathrm{M}$ (although the bubble may travel to some boundary point in the limit.). We want to show that no such bubble point exists; which in turns implies the present theorem.

Suppose bubbling does occur and there is a sequence of points (z_m, x_m) such that

$$\epsilon_m^{-1} = \max_{\Sigma\times\mathrm{M}} \left| \frac{\partial f^{(m)}}{\partial z} \right|_{g_0} = \left| \frac{\partial f^{(m)}}{\partial z}(z_m, x_m) \right|_{g_0} \to \infty.$$

W.l.o.g., we may assume that $\lim_{m\to\infty} z_m = z_\infty \in \Sigma$ and $\lim_{m\to\infty} x_m = x_\infty \in \mathrm{M}$. We want to argue that there exists a point $z'_m \in \mathrm{B}_{2\sqrt{\epsilon_m}}(z_m) \cap \Sigma$ such that the area of $\mathrm{B}_{\epsilon_m}(z'_m) \geq c_0$ for some uniform constant $c_0 > 0$. If the area functional were the area of a holomorphic disk in $\Sigma \times \mathscr{W}_{\mathrm{M}}$, then this follows from standard literatures in this direction. In our setting, this is still true which in turn implies that there are at most finite number of bubbles. We give a brief explanation here and leave interested readers to fill in the details.

[20] A point $\{(z_m, x_m), m \in \mathbf{N}\}$ is called bubble point if a) $|\frac{\partial f^{(m)}}{\partial z}|_{g_0} \to \infty$ and b) it is a global maximal of $|\frac{\partial f^{(m)}}{\partial z}|_{g_0}$.

Set $d_m = d(z_m, \partial\Sigma)$. If $d_m > \frac{1}{2}\epsilon_m$, then an easy calculation implies

$$\text{(5.7)} \qquad \mathrm{A}\big(f^{(m)}, \mathrm{B}_{\epsilon_m}(z_m) \cap \Sigma\big) \geq c_0.$$

However, we need to establish inequality (5.7) even when $\frac{d_m}{\epsilon_m} \to 0$. In such a case, there must exists another point $z_m' \in \partial \mathrm{B}_{1-d_m-\epsilon_m}(\mathrm{O}) \cap \mathrm{B}_{\sqrt{\epsilon_m}}(z_m)$ such that

$$\left| \frac{\partial f^{(m)}}{\partial z}(z_m', x_m') \right|_{g_0} > \frac{\epsilon_m}{2}, \qquad x_m' = f^{(m)}(z_m').$$

This can be proved by using inequality (5.6) and the maximum principle for holomorphic function along long strip. The main point is that, for this point z_m', the inequality (5.7) holds. Consequently, there exists at most a finite number of bubble points. Next we want to argue that there is no bubble point at all. For this purpose, we consider two cases: $z_\infty \in \partial\Sigma$ or $z_\infty \in \Sigma^0$. In both cases, we want to show that the existence of a non trivial bubble must lead to contradiction.

Part 1: No bubble in the boundary. — We choose a number $\varsigma \in \mathrm{B}_1$, to be fixed throughout the following argument. Set

$$\phi^{(m)}(\zeta) = z_m + \zeta \cdot \epsilon_m, \qquad \forall\, \zeta \in \mathbf{C}$$

and

$$l^{(m)} = f^{(m)} \circ \phi^{(m)}, \qquad \text{and} \qquad \tilde{z}_m = \phi^{(m)}(\varsigma).$$

By definition, we have

$$|\tilde{z}_m - z_m| \leq \epsilon_m \to 0, \qquad \forall\, |\varsigma| \leq 1.$$

Thus $\lim_{m\to\infty} \tilde{z}_m = z_\infty$. First set

$$\tilde{x}_m = \pi \circ f^{(m)}(\phi^{(m)}(\varsigma)).$$

Note that

$$\begin{aligned} l^{(m)}(\varsigma) &= (\tilde{z}_m, \tilde{x}_m, \zeta^{(m)}(\tilde{z}_m, \tilde{x}_m)) \\ &= (\phi^{(m)}(\varsigma), \pi \circ f^{(m)}(\phi^{(m)}(\varsigma)), \zeta^{(m)}(\phi^{(m)}(\varsigma), \pi \circ f^{(m)}(\phi^{(m)}(\varsigma)))). \end{aligned}$$

Since $\lim_{m\to\infty} l^{(m)} = l^\infty$, we may assume

$$\lim_{m\to\infty} \tilde{x}_m = \pi \circ l^\infty(\varsigma).$$

Set

$$\lim_{m\to\infty} \zeta^{(m)}(\tilde{z}_m, \tilde{x}_m) = \zeta^{\infty}(\varsigma)$$

for some function ζ^{∞}. We want to show

$$\zeta^{\infty}(\varsigma) = \bar{\partial}\phi^{\infty}(z_{\infty}, \pi \circ l^{\infty}(\varsigma)).$$

Now for any $0 < \alpha < 1$, there exists a uniform constant C such that

$$\frac{|\zeta^{(m)}(\tilde{z}_m, \tilde{x}_m) - \zeta^{(m)}(z_{\infty}, \pi \circ l^{\infty}(\varsigma))|}{(|\tilde{z}_m - z_{\infty}| + |\tilde{x}_m - \pi \circ l^{\infty}(\varsigma)|)^{\alpha}} < C.$$

Since $\lim_{m\to\infty}(|\tilde{z}_m - z_{\infty}| + |\tilde{x}_m - \pi \circ l^{\infty}(\varsigma)|) = 0$, we have

$$\lim_{m\to\infty} (\zeta^{(m)}(\tilde{z}_m, \tilde{x}_m) - \zeta^{(m)}(z_{\infty}, \pi \circ l^{\infty}(\varsigma))) = 0.$$

On the other hand, we have

$$\begin{aligned} &\lim_{m\to\infty} (\zeta^{(m)}(z_{\infty}, \pi \circ l^{\infty}(\varsigma)) - \bar{\partial}\phi^{\infty}(z_{\infty}, \pi \circ l^{\infty}(\varsigma))) \\ &\quad = \lim_{m\to\infty} (\bar{\partial}\phi^{(m)}(z_{\infty}, \pi \circ l^{\infty}(\varsigma)) - \bar{\partial}\phi^{\infty}(z_{\infty}, \pi \circ l^{\infty}(\varsigma))) \\ &\quad = 0. \end{aligned}$$

Thus,

$$\lim_{m\to\infty} (\zeta^{(m)}(\tilde{z}_m, \tilde{x}_m) - \bar{\partial}\phi^{\infty}(z_{\infty}, \pi \circ l^{\infty}(\varsigma))) = 0.$$

Consequently,

$$\zeta^{\infty}(\varsigma) = \bar{\partial}\phi^{\infty}(z_{\infty}, \pi \circ l^{\infty}(\varsigma)).$$

Thus

$$l^{\infty}(\varsigma) = (z_{\infty}, \pi \circ l^{\infty}(\varsigma), \bar{\partial}\phi^{\infty}(z_{\infty}, \pi \circ l^{\infty}(\varsigma))) \in \{z_{\infty}\} \times \Lambda_{z_{\infty}, \phi^{\infty}(z_{\infty})}.$$

Since $\varsigma \in B_1$ is chosen randomly, we have

$$l^{\infty}(B_1) \subset \{z_{\infty}\} \times \Lambda_{z_{\infty}, \phi^{\infty}(z_{\infty})} \subset \{z_{\infty}\} \times \mathscr{W}_M.$$

Next note that $z_{\infty} \in \partial\Sigma$, we have

$$\phi^{\infty}(z_{\infty}, \cdot) = {\phi_0}^{\infty}(z_{\infty}, \cdot).$$

Thus

$$\Lambda_{z_{\infty}, \phi^{\infty}(z_{\infty})} = \Lambda_{z_{\infty}, {\phi_0}^{\infty}(z_{\infty})} \subset \{z_{\infty}\} \times \mathscr{W}_M$$

is a totally real submanifold. This contradicts to the fact that $l^{\infty}(B_1) \subset \{z_{\infty}\} \times \mathscr{W}_M$ is a holomorphic disk! Consequently, there is no bubble sphere/disk developed along the boundary of Σ.

Now we proceed to Part 2.

Part 2: Non existence of interior bubbles. — Suppose that $z_\infty \in \Sigma^0 = \Sigma \setminus \partial\Sigma$. Set

$$\phi^{(m)}(\varsigma) = z_m + \frac{\epsilon_m}{\delta_m}\varsigma$$

and $l^{(m)} = f^{(m)} \circ \phi^{(m)}$ for $\epsilon_m \ll \delta_m \to 0$. For any k fixed, the following map has a non-trivial limit:

$$l^\infty = \lim_{m\to\infty} l^{(m)} : \mathrm{B}_k \subset \bigcup_{m=1}^{\infty} \mathrm{B}_{\delta_m} \to \Sigma \times \mathscr{W}_{\mathrm{M}}.$$

Let $k \to \infty$. Then l^∞ defines a holomorphic map from R^2 to $\mathscr{W}_{\mathrm{M}}$ with bounded area in $\Sigma \times \mathrm{M}$. Therefore, the image must be a holomorphic S^2.

Here

$$\frac{\partial l^{(m)\alpha}}{\partial \varsigma} = \frac{\partial f^{(m)\alpha}}{\partial z}\frac{\epsilon_m}{\delta_m} = -g^{(m)\alpha\bar\beta}\frac{\partial^2\phi^{(m)}}{\partial z \partial \bar w_\beta}\frac{\epsilon_m}{\delta_m}.$$

Set

$$\begin{cases} \eta^{(m)\alpha} &= -g^{(m)\alpha\bar\beta}\frac{\partial^2\phi^{(m)}}{\partial z\partial \bar w_\beta}, \\ \tilde\eta^{(m)\alpha} &= -g^{(m)\alpha\bar\beta}\frac{\partial^2\phi^{(m)}}{\partial z\partial \bar w_\beta}\frac{\epsilon_m}{\delta_m}. \end{cases}$$

By assumption, $\tilde\eta^{(m)}$ has a non-trivial limit η^∞ such that

$$\frac{\partial l^{(m)\alpha}}{\partial \varsigma} = \eta^{(m)\alpha}, \quad \text{and} \quad \frac{\partial l^{\infty\alpha}}{\partial \varsigma} = \eta^{\infty\alpha}.$$

The above equations imply

$$\begin{aligned} \partial_\varsigma\left(\frac{\partial\phi^{(m)}}{\partial w_{\bar\beta}} \circ l^{(m)}\right) &= \left(\frac{\partial}{\partial\varsigma} + \tilde\eta^{(m)\alpha}\frac{\partial}{\partial w_\alpha}\right)\frac{\partial\phi^{(m)}}{\partial w_{\bar\beta}} \\ &= \frac{\epsilon_n}{\delta_n}\frac{\partial^2\phi^{(m)}}{\partial z\partial w_{\bar\beta}} + \tilde\eta^{(m)\alpha}\frac{\partial^2\phi^{(m)}}{\partial w_\alpha \partial w_{\bar\beta}} \\ &= \frac{\epsilon_n}{\delta_n}\frac{\partial^2\phi^{(m)}}{\partial z\partial w_{\bar\beta}} + \tilde\eta^{(m)\alpha}\big(g^{(m)}_{\alpha\bar\beta} - g_{0,\alpha\bar\beta}\big) \\ &= \frac{\epsilon_n}{\delta_n}\frac{\partial^2\phi^{(m)}}{\partial z\partial w_{\bar\beta}} + \left(-g^{(m)\alpha\bar r}\frac{\partial^2\phi^{(m)}}{\partial z\partial\bar w_r}\frac{\epsilon_m}{\delta_m}\right)g^{(m)}_{\alpha\bar\beta} - \tilde\eta^{(m)\alpha}g_{0,\alpha\bar\beta} \\ &= -g_{0,\alpha\bar\beta}\tilde\eta^{(m)\alpha}. \end{aligned}$$

Thus

$$\begin{aligned}-g_{0,\alpha\bar\beta}\tilde\eta^{(m)\alpha}\tilde\eta^{(m)\bar\beta} &= \partial_\varsigma\left(\frac{\partial\phi^{(m)}}{\partial w_{\bar\beta}}\circ l^{(m)}\right)\tilde\eta^{(m)\bar\beta}\\ &= \partial_\varsigma\left(\tilde\eta^{(m)\bar\beta}\frac{\partial\phi^{(m)}}{\partial w_{\bar\beta}}\circ l^{(m)}\right)\\ &= \partial_\varsigma\left(\partial_{\bar\varsigma}(\phi^{(m)}\circ l^{(m)}) - \frac{\epsilon_n}{\delta_n}\frac{\partial\phi^{(m)}}{\partial\bar z}\circ l^{(m)}\right)\\ &= \partial_\varsigma\partial_{\bar\varsigma}(\phi^{(m)}\circ l^{(m)}) - \frac{\epsilon_n}{\delta_n}\partial_\varsigma\left(\frac{\partial\phi^{(m)}}{\partial\bar z}\circ l^{(m)}\right).\end{aligned}$$

Let $\chi(\varsigma)$ be any smooth test function which vanish outside a compact domain of R^2. For m large enough, the domain of χ is contained inside in the domain of $l^{(m)}$. Then,

$$\begin{aligned}&-\int_{R^2}\chi g_{0,\alpha\bar\beta}\tilde\eta^{(m)\alpha}\tilde\eta^{(m)\bar\beta}|d\varsigma|^2\\ &\quad= \int_{R^2}\chi\partial_\varsigma\partial_{\bar\varsigma}(\phi^{(m)}\circ l^{(m)})|d\varsigma|^2 - \int_{R^2}\chi\frac{\epsilon_m}{\delta_m}\partial_\varsigma\left(\frac{\partial\phi^{(m)}}{\partial\bar z}\circ l^{(m)}\right)|d\varsigma|^2\\ &\quad= -\int_{R^2}\frac{\partial\chi}{\partial\varsigma}\partial_{\bar\varsigma}(\phi^{(m)}\circ l^{(m)})|d\varsigma|^2 + \frac{\epsilon_m}{\delta_m}\int_{R^2}\frac{\partial\chi}{\partial\varsigma}\frac{\partial\phi^{(m)}}{\partial\bar z}\circ l^{(m)}|d\varsigma|^2.\end{aligned}$$

Taking limit as $m\to\infty$, we have

$$-\int_{R^2}\chi g_{0,\alpha\bar\beta}\tilde\eta^{\infty\alpha}\tilde\eta^{\infty\bar\beta}|d\varsigma|^2 = -\int_{R^2}\frac{\partial\chi}{\partial\varsigma}\partial_{\bar\varsigma}(\phi^\infty\circ l^\infty)|d\varsigma|^2.$$

This holds for any test function in $\mathbf{R}^2$. Now the image of l^∞ is a smooth S^2 in $\{z_\infty\}\times M$. Therefore, $l^{\infty *}\omega_0$ is an induced (smooth) Kähler form in this class. We may as well assume $l^{\infty *}\omega_0$ is cohomologous to the standard Kähler form in S^2. Then there exists a bounded smooth function λ in this S^2 such that

$$\begin{aligned}l^{\infty *}\omega_0 &= g_{0,\alpha\bar\beta}\tilde\eta^{\infty\alpha}\tilde\eta^{\infty\bar\beta}|d\varsigma|^2\\ &= \frac{\partial^2}{\partial\varsigma\partial\bar\varsigma}(-2\ln(1+|\varsigma|^2)+\lambda\circ l^\infty)d\varsigma d\bar\varsigma.\end{aligned}$$

Then

$$\int_{R^2}\frac{\partial\chi(\varsigma)}{\partial\varsigma}\cdot\frac{\partial}{\bar\partial\varsigma}(-2\ln(1+|\varsigma|^2)+\lambda\circ l^\infty+\phi^\infty\circ l^\infty)|d\varsigma|^2 = 0.$$

Set

$$\Phi = (-2\ln(1+|\varsigma|^2)+\lambda\circ l^\infty+\phi^\infty\circ l^\infty).$$

Since χ is an arbitrary test function, this implies that Φ is a weakly $C^{1,1}$, one side bounded harmonic function in $\mathbf{R}^2$. Then Φ must be a constant function c. Then

$$\phi^\infty \circ l^\infty = c + 2\ln(1+|\varsigma|^2) + \lambda \circ l^\infty$$

is not a bounded function as $\varsigma \to \infty$. This contradicts with the fact that ϕ^∞ is uniformly bounded, which in turn implies that there is no bubble in the interior.

The proof of this theorem is then completed. □

5.3. *Proof of Theorem 5.0.15*

5.3.1. *Uniform* C^1 *transversal derivatives of almost super regular foliations.* — In this subsection, we continue to derive the *a priori* regularity estimate for the disk with uniform upper bound on area and capacity.

Theorem **5.3.1.** — *Let* Ω *be any compact sub-domain in* Σ^0. *For any holomorphic disk* $f \in \mathscr{C}(\delta, \Lambda, L_0, L_1, L_2)$, *there exists a constant* $C > 1$ *such that:*

$$\frac{1}{C} \leq \left(\frac{\omega_0^n}{\omega_\phi^n}\right) \circ f \leq C.$$

Here C *depends on* $\delta, \Lambda, L_0, L_2$ *and* $d(\partial\Omega, \partial\Sigma)$. *Moreover, this constant approaches to* ∞ *if* $d(\partial\Omega, \partial\Sigma) \to 0$.

Proof. — Proposition 4.2.13 implies that the trace of curvature of the TM bundle over the disk Σ has interior estimates:

$$0 \leq S \circ f(z, \cdot) \leq \frac{C_1}{d(z, \partial\Sigma)^2}.$$

Corollary 4.2.11 then implies

$$0 \leq \partial_z \bar{\partial}_z \left(\ln \frac{\omega_\phi^n}{\omega_0^n}\right) \leq C_2, \quad \forall z \in \Omega,$$

for some constants C_2 which depends on C_1. On the other hand, finite capacity implies

$$\int_\Omega \left(\frac{\omega_0^n}{\omega_\phi^n}\right) \circ f \frac{\sqrt{-1}}{2} dz d\bar{z} \leq L_2.$$

This in turn implies that in a slightly smaller sub-domain $\Omega_1 \subsetneq \Omega$, we have

$$\left|\ln \frac{\omega_\phi^n}{\omega_0^n} - C_3\right| \leq C_4$$

for some constant C_3 which may depend on ϕ. The key observation is that C_4 depends only on C_2, C_3 and L_2. Consequently, there exists a constant C_5 such that

$$\frac{1}{C_5} \le \left(\frac{\omega_0^n}{\omega_\phi^n}\right) \circ f \le C_5.$$

Here C_5 depends only on C_1, L_2. □

Theorem **5.3.2.** — *For any super regular disk $f \in \mathscr{C}(\delta, \Lambda, L_0, L_1, L_2)$, the $T^{*(1,0)}M$ component of the first transversal derivatives of the leaf vector field is bounded in any compact subdomain $\Omega \subset \Sigma$. Namely, there exists a constant C depends on $\delta, \Lambda, L_0, L_1, L_2$ and $d(\partial\Omega, \partial\Sigma)$, such that (cf.* (5.5)*)*

$$\|\nabla_\partial X\|_{g_0} < C, \quad \text{in } \Omega.$$

Here X is the leaf vector field and $\nabla_\partial X$ is the covariant derivative of X in the $T^{1,0}M$ direction with respect to background metric g_0. Locally, we have

$$\|\nabla_\partial X\|^2_{g_0} = \sum_{i,j,\alpha,\beta=1}^{n} g_{0,\alpha\bar\beta} g_0^{i\bar j} \eta^\alpha_{,i} \eta^{\bar\beta}_{,\bar j}$$

where

$$\eta^\alpha_{,i} = \frac{\partial \eta^\alpha}{\partial w_i} + \eta^\beta \Gamma^\alpha_{\beta i}(g_0), \quad \forall\, \alpha, i = 1, 2, ..., n.$$

Moreover, this constant C blows up if $d(\partial\Omega, \partial\Sigma) \to 0$.

Proof. — For any $z \in \Omega$, consider function $d(z, \partial\Omega) \cdot \|\nabla_\partial X\|_{g_0}(z)$. This is a non-negative function in Ω which vanishes on $\partial\Omega$. The maximum value must be attained in Ω^0. If this theorem is false, then there exists a sequence of super regular holomorphic disks $\{f^{(m)}\} \subset \mathscr{C}(\delta, \Lambda, L_0, L_1, L_2)$ such that

$$\lim_{m\to\infty} \max_{\Omega} d(z, \partial\Omega) \cdot \|\nabla_\partial X^{(m)}\|_{g_0}(z) = \infty.$$

Without loss of generality, one may assume that the maximum is attained at the point z_m. Set

$$\lim_{m\to\infty} z_m = z_\infty \in \bar\Omega.$$

On the other hand, Theorem 5.0.14 implies that there exists a subsequence of $f^{(m)}$ which converges in $\mathscr{C}(\delta, \Lambda, L_0, L_1, L_2)$ as an embedded holomorphic disk. Without loss of generality, we may assume that $f^{(m)}$ is fixed but the restricted TM bundle varies. Denote the sequence of re-scaling factors as

$$\frac{1}{\epsilon_m} = \|\nabla_\partial X\|_{g_0}(z_m) \to \infty.$$

Write this sequence of disks as

$$\begin{aligned} f^{(m)} : \Sigma \mapsto \Sigma \times M &\hookrightarrow \Sigma \times \mathscr{W}_M \\ z \mapsto (z, f^{(m)}(z)) &\hookrightarrow (z, f^{(m)}(z), \xi^{(m)}) \end{aligned}$$

where

$$\xi^{(m)\alpha}(z) = \frac{\partial(\rho + \phi)}{\partial w_\alpha} \circ f^{(m)}(z), \quad \forall \alpha = 1, 2, ..., n.$$

Moreover

$$X^{(m)} = \sum_{i=1}^{n} \eta^{(m)i} \frac{\partial}{\partial w_i}$$

and

$$\eta^{(m)\alpha} = \frac{\partial f^{(m)\alpha}}{\partial z} = -g^{(m)\alpha\bar{\beta}} \frac{\partial^2 \phi^{(m)}}{\partial z \partial \bar{w}_\beta}.$$

Theorem 5.3.1 implies that there is a positive number $C_3 > 0$ such that

$$\frac{1}{C} \leq \left(\frac{\omega_0^n}{\omega_{\phi^{(m)}}^n} \right) \circ f^{(m)} \leq C_3$$

hold uniformly in Ω since $d(\Omega, \partial\Sigma) > 0$. Combining this with the $C^{1,1}$ estimate in [9], there exists a small positive constant $\epsilon_0 0$, such that

$$\epsilon_0 (g_{0,i\bar{j}})_{n\times n} \leq \left(g_{i\bar{j}}^{(m)}\right)_{n\times n} \leq C(g_{0,i\bar{j}})_{n\times n}$$

holds for these disks on Ω. Set

$$\tilde{\phi}^{(m)}(z, w) = \phi^{(m)}(z_m + \epsilon_m \cdot z, w), \quad \tilde{X}^{(m)} = \sum_{\alpha=1}^{n} \tilde{\eta}^{(m)\alpha} \frac{\partial}{\partial w_\alpha}$$

where

$$\begin{aligned} \tilde{\eta}^{(m)\alpha}(z, w) &= g^{(m)\alpha\bar{\beta}} \frac{\partial^2 \tilde{\phi}^{(m)}}{\partial z \partial w_{\bar{\beta}}} \\ &= \epsilon_m \eta^{(m)\alpha}(\epsilon_m \cdot z + z_m, w). \end{aligned}$$

Moreover,

$$\|\nabla_\partial \tilde{X}^{(m)}\|_{g_0}(0) = \epsilon_m \cdot \|\nabla_\partial X^{(m)}\|_{g_0}(z_m) = 1.$$

Set

$$\Omega^{(m)} = \{z \mid \epsilon_m \cdot z + z_m \in \Omega\}.$$

Clearly

$$\lim_{m\to\infty} d(0, \partial\Omega^{(m)}) = \lim_{m\to\infty} d(0, \partial\Omega^{(m)}) \cdot |\nabla_\partial \tilde{X}^{(m)}|_{g_0}(0)$$
$$= \lim_{m\to\infty} d(z_m, \partial\Omega) \cdot \|\nabla_\partial X^{(m)}\|_{g_0}(z_m) = \infty.$$

In other words,

$$\Omega^{(m)} \to \mathbf{R}^2.$$

Here

$$\begin{aligned}\tilde{S}^{(m)}(z, w) &= \frac{\partial \tilde{\eta}^{(m)\bar{\beta}}}{\partial w_\alpha}\frac{\partial \tilde{\eta}^{(m)\alpha}}{\partial \bar{w}_\beta}\\ &= \epsilon_m^2 \frac{\partial \eta^{(m)\bar{\beta}}}{\partial w_\alpha}\frac{\partial \eta^{(m)\alpha}}{\partial \bar{w}_\beta}\\ &= \epsilon_m^2 \cdot S^{(m)}(z_m + \epsilon_m z, w).\end{aligned}$$

Then $\tilde{S}^{(m)}$ still satisfies the inequality

$$\partial_z \partial_{\bar{z}} \tilde{S}^{(m)} \geq \frac{2}{n} \tilde{S}^{(m)2}$$

in $\Omega^{(m)}$. Consequently, we have (cf. Proposition 4.2.13)

$$0 \leq \tilde{S}^{(m)}(z) \leq \frac{C}{d(z, \partial\Omega^{(m)})^2} \to 0, \quad \forall\, z \in \Omega^{(k)}$$

for any fixed k and $m \to \infty$. Recall that

$$\begin{aligned}\tilde{S}^{(m)} &= \epsilon_m^2 \frac{\partial \eta^{(m)\alpha}}{\partial w_{\bar{\beta}}}\frac{\partial \eta^{(m)\bar{\beta}}}{\partial w_\alpha}\\ &= \epsilon_m^2 g^{(m)\alpha\bar{a}} g^{(m)\bar{\beta}b} \left(\frac{\partial \phi^{(m)}}{\partial z}\right)_{,\bar{a}\bar{\beta}} \left(\frac{\partial \phi^{(m)}}{\partial \bar{z}}\right)_{,\alpha a}\\ &\geq C^{-2} \epsilon_m^2 g_0^{\alpha\bar{a}} g_0^{\bar{\beta}b} \left(\frac{\partial \phi^{(m)}}{\partial z}\right)_{,\bar{a}\bar{\beta}} \left(\frac{\partial \phi^{(m)}}{\partial \bar{z}}\right)_{,\alpha a}.\end{aligned}$$

The last inequality holds because $g^{(m)}_{\alpha\bar\beta}$ has a uniform upper bound. Thus,

$$\lim_{m\to\infty} \epsilon_m \cdot \left(\frac{\partial\phi^{(m)}}{\partial\bar z}\right)_{,\alpha a} = 0.$$

Consequently, we have

$$\begin{aligned}\lim_{m\to\infty} \frac{\partial\tilde\eta^{(m)\alpha}}{\partial w_{\bar\beta}} &= \lim_{m\to\infty} \epsilon_m \cdot \frac{\partial\eta^{(m)\alpha}}{\partial w_{\bar\beta}} \\ &= \lim_{m\to\infty} \epsilon_m \cdot g^{(m)\alpha\bar a}\left(\frac{\partial\phi^{(m)}}{\partial z}\right)_{,\bar\beta\bar a} = 0.\end{aligned}$$

The last inequality holds since $g^{(m)}_{\alpha\bar\beta}$ has a uniform positive lower bound on Ω.

Moreover, for any fixed z, we have

$$\|\nabla_\partial \tilde X^{(m)}\|_{g_0}(z) d(z, \partial\Omega^{(m)}) \le \|\nabla_\partial X^{(m)}\|_{g_0}(0) d(0, \partial\Omega^{(m)}).$$

Therefore[21]

$$\begin{aligned}\|\nabla_\partial \tilde X^{(m)}\|_{g_0}(z) &\le \|\nabla_\partial \tilde X^{(m)}\|_{g_0}(0)\frac{d(0,\partial\Omega^{(m)})}{d(z\,\partial\Omega^{(m)})} \\ &\le 2\cdot \|\nabla_\partial \tilde X^{(m)}\|_{g_0}(0) \\ &\le 4.\end{aligned}$$

Consequently, $\nabla_\partial \tilde X^{(m)}$ [22] is uniformly bounded and $|\nabla_\partial \tilde X^{(m)}|_{g_0} \approx 1$ at the origin. By Lemma 4.2.7, both $\frac{\partial\tilde\eta^{(m)}}{\partial w_i}$ and $\frac{\partial\tilde\eta^{(m)}}{\bar\partial w_i}$ are uniformly C^α $(\forall\alpha < 1)$ bounded. Since $g^{(m)}_{i\bar j}$ has a uniform upper and lower bound in Ω, we have

$$\lim_{m\to\infty} \frac{\partial\tilde\eta^{(m)}}{\bar\partial w_i}(z,\cdot) = 0.$$

On the other hand, $\frac{\partial\tilde\eta^{(m)}}{\partial w_i}$ is a bounded holomorphic function in the limit since $\frac{\partial\tilde\eta^{(m)}}{\partial\bar w_i} = 0$ in the limit (cf. Lemma 4.2.7). Therefore, $\frac{\partial\tilde\eta^{(m)}}{\partial w_i}$ is a constant matrix everywhere in the

[21] The second inequality holds since $\lim_{m\to\infty} d(0,\partial\Omega^{(m)}) = \infty$ while $d(0,z) = |z|$ is fixed.

[22] Note that

$$\tilde\eta^{(m)\alpha}_{,i} = \frac{\partial\tilde\eta^{(m)\alpha}}{\partial w_i} + \tilde\eta^{(m)\beta}\Gamma^\alpha_{\beta i}(g_0).$$

where

$$\lim_{m\to\infty} \tilde\eta^\beta\Gamma^\alpha_{\beta i}(g_0) = \lim_{m\to\infty} \epsilon_m\cdot\eta^{(m)\beta}\Gamma^\alpha_{\beta i}(g_0) = 0.$$

This is because the disk, when restricted in Ω, uniformly converge to a smooth limit surface. Hence $\Gamma^\alpha_{\beta i}(g_0)$ is uniformly bounded. Thus,

$$\frac{\partial\tilde\eta^{(m)}}{\partial w_i} \approx \tilde\eta^{(m)\alpha}_{,i}.$$

limit! Set

$$g^{\infty}_{i\bar{j}} = \lim_{m\to\infty} g^{(m)}_{i\bar{j}}, \quad \tilde{\eta}^{\infty} = \lim_{m\to\infty} \tilde{\eta}^{(m)}.$$

Then

$$\frac{1}{C} I_{n\times n} \le \left(g^{(m)}_{i\bar{j}}\right) \le C I_{n\times n}.$$

Theorem 4.2.8 takes the form

$$\partial_z \bar{\partial}_z g^{\infty}_{,i\bar{j}} = a^{\alpha}{}_i \overline{a^{\beta}}{}_j g^{\infty}_{,\bar{\beta}\alpha}$$

where

$$\frac{\partial \tilde{\eta}^{\infty\alpha}}{\partial w_i} = a^{\alpha}{}_i, \quad \text{and} \quad \frac{\partial \tilde{\eta}^{\infty\bar{\beta}}}{\partial \bar{w}_j} = \overline{a^{\beta}{}_j}$$

are constant matrices. This clearly violates the maximum principle. Note that $g^{\infty}_{i\bar{j}}$ and its derivatives are uniformly bounded in the entire plan. The contradiction implies the constant matrix $(a^{\alpha}{}_i)$ must vanish identically which contradicts to the blow-up assumption. □

Combining this with Lemma 4.2.5 and Lemma 4.2.7, we have shown that

Corollary **5.3.3.** — *For any super regular disk* $f \in \mathscr{C}(\delta, \Lambda, L_0, L_1, L_2)$, *the* $T^*(0,1)M$ *component of the first transversal derivatives of the leaf vector field is bounded in any compact subdomain* $\Omega \subset \Sigma$. *Namely, there exists a constant* C *depending only on* $\delta, \Lambda, L_0, L_1, L_2$ *and* $d(\partial\Omega, \partial\Sigma)$, *such that (cf.* (5.5)*)*

$$\|\nabla_{\bar{\partial}} X\|_{g_0} < C, \quad \text{in } \Omega.$$

Here X *is the leaf vector field and* $\nabla_{\partial} X$ *is the covariant derivative of* X *in the* $T^{0,1}M$ *direction with respect to the background metric* g_0. *Locally, we have*

$$\|\nabla_{\bar{\partial}} X\|^2_{g_0} = \sum_{i,j,\alpha,\beta=1}^{n} g_{0,\alpha\beta} g_0^{\bar{i}\bar{j}} \eta^{\alpha}_{,\bar{i}} \eta^{\beta}_{,\bar{j}}$$

where

$$\eta^{\alpha}_{,\bar{i}} = \frac{\partial \eta^{\alpha}}{\partial \bar{w}_i}, \quad \forall \alpha, i = 1, 2, \ldots, n.$$

Moreover, this constant C *blows up if* $d(\partial\Omega, \partial\Sigma) \to 0$.

5.3.2. *The limit of super-regular disks with finite capacity is super regular*

Theorem **5.3.4.** — *Suppose that* $\{\phi_0{}^{(m)}(m = 1, 2, ..., \infty)\}$ *is a sequence of loops in* $C^{2,\alpha}(\partial\Sigma, \mathcal{H}_\omega)$ *which satisfies the uniform bound* (5.3). *Suppose that* $f^{(m)} : (\Sigma, \partial\Sigma) \to (\Sigma \times \mathcal{W}_M, \bar{\Lambda}_{\phi_0{}^{(m)}})$ *is a sequence of super-regular holomorphic disks in* $\mathcal{C}(\delta, \Lambda, L_0, L_1, L_2)$ *with vanishing normal Maslov index. There exists a subsequence of* $\{f_m\}$ *which converges to an embedded, super-regular holomorphic disk.*

Note that this in fact implies Theorem 5.0.15.

Proof. — We use $1 \leq i, j, k, \alpha, \beta, \gamma \leq n$ to denote the labels on Kähler manifold M, and use $1 \leq p, q, ... \leq 2n$ to denote the labels in the *moduli* space $\mathcal{M}_{\phi_0}$. Recall

$$\begin{aligned} \sharp : \Sigma \times \mathcal{M}_{\phi_0} &\to \Sigma \times M \\ (z, f) &\to \pi \circ ev(z, f). \end{aligned}$$

Then, $\sharp$ is invertible on set of super regular discs. As before, set $F = \pi \circ ev \circ \sharp^{-1}$. We can identify the domain of super regular discs in the moduli space with some open subset (image of super regular discs) of M. We will use this point of view from time to time in this section. Note that

$$\begin{aligned} F^{(m)} : \Sigma \times \mathcal{M}_{\phi_0} &\to \Sigma \times M \\ (z, f) &\to \pi \circ ev(z, f). \end{aligned}$$

By definition, we have

$$\frac{\partial F^{(m)i}}{\partial \bar{z}} = 0, \quad \forall\, i = 1, 2, \dots .$$

Let $\nu_p^{(m)} = \|\frac{\partial F^{(m)}}{\partial t_p}\|_{W^{1,2}(\Sigma)}$. Following the standard theory on elliptic problems, there exists a uniform constant $C(\delta, \Lambda)$ (which depends on the totally real boundary submanifold) such that

$$\left\| \frac{\partial F^{(m)}}{\partial t_p} \right\|_{L^\infty} < C \cdot \nu_p^{(m)}. \tag{5.8}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial}{\partial z} \frac{\partial F^{(m)i}}{\partial t_p} &= \frac{\partial \eta^{(m)i}}{\partial w_\alpha} \frac{\partial F^{(m)\alpha}}{\partial t_p} + \frac{\partial \eta^{(m)i}}{\partial w_{\bar{\beta}}} \frac{\partial F^{(m)\bar{\beta}}}{\partial t_p} \\ \frac{\partial}{\partial \bar{z}} \frac{\partial F^{(m)i}}{\partial t_p} &= 0. \end{aligned} \tag{5.9}$$

By Theorem 5.3.1, we have

$$\left|\frac{\partial \eta^{(m)i}}{\partial w_\alpha}\right|_{g_0} + \left|\frac{\partial \eta^{(m)i}}{\partial w_{\bar\beta}}\right|_{g_0} \le C \tag{5.10}$$

uniformly in any sub-domain $\Omega \subsetneq \Sigma$ such that $d(\partial\Omega, \partial\Sigma) > 0$. Fix Ω now.

For any $z_0 \in \Omega$, and we re-parametrize the family of disks such that for

$$F^{(m)}(z_0, t) = (z_0, x), \quad \text{where} \quad x = t$$

for any (z_0, t) in the domain of $F^{(m)}$. This is possible since $f^{(m)}$ is a super regular disk. By definition, we have

$$\left(\frac{\partial F^{(m)i}}{\partial t_p}\frac{\partial F^{(m)\bar i}}{\partial t_p}\right)_{2n\times(2n)}$$

is non-singular at $z = z_0$. From (5.9) and *a priori* estimate (5.10), we have the following important Harnack-type inequality

$$C^{-1}\cdot\left|\frac{\partial F^{(m)}}{\partial t_p}\right|_{g_0}(z_1) \le \left|\frac{\partial F^{(m)}}{\partial t_p}\right|_{g_0}(z_2) \le C\cdot\left|\frac{\partial F^{(m)}}{\partial t_p}\right|_{g_0}(z_1), \quad \forall\, z_1, z_2 \in \Omega.$$

Here C depends on Ω only. Consequently, we have shown that

$$\left(\frac{\partial F^{(m)i}}{\partial t_p}\frac{\partial F^{(m)\bar i}}{\partial t_p}\right)_{2n\times(2n)}$$

is a bounded matrix and non-singular in Ω. In particular, there exists a small constant c such that

$$\left|\det\left(\frac{\partial F^{(m)i}}{\partial t_p}\frac{\partial F^{(m)\bar i}}{\partial t_p}\right)_{2n\times(2n)}\right| > c. \tag{5.11}$$

Here c depend only on (5.9). This implies that $v_p (1 \le p \le 2n)$ has a uniform positive low bound. Now we claim that they all have uniform upper bound:

$$\sup_{m\to\infty} v_p^{(m)} < \infty. \tag{5.12}$$

Otherwise, there exists a subsequence (use the same notation for convenience) such that

$$\lim_{m\to\infty} v_p^{(m)} = \infty.$$

The matrix

$$\left(\frac{\partial F^{(m)i}}{\partial t_p}\frac{\partial F^{(m)\bar{i}}}{\partial t_p}\right)_{2n\times(2n)}$$

is uniformly bounded from above and below on Ω. Set

$$u_p^{(m)} = \left(u_p^{(m)1}, u_p^{(m)2}, \ldots, u_p^{(m)n}\right)$$

where

$$u_p^{(m)i} = \frac{\frac{\partial F^{(m)i}}{\partial t_p}}{\nu_p^{(m)}}.$$

Then, $u_p^{(m)} \to 0$ in Ω uniformly. Since $u_p^{(m)}$ is holomorphic on Σ, then $u_p^{(m)}$ converges to 0 at least in Σ^0. This contradicts to the fact that $u_p^{(m)}$ has a non-zero limit (cf. Lemma 5.3.5 below). Consequently, our claim holds and ν_p has a uniform upper bound and

$$\left(\frac{\partial F^{(m)i}}{\partial t_p}\frac{\partial F^{(m)\bar{i}}}{\partial t_p}\right)_{2n\times(2n)}$$

is uniformly bound from above on Σ. Recall that

$$g_{pq}^{(m)}(z_0) = g_{i\bar{j}}^{(m)}\left(\frac{\partial F^{(m)i}}{\partial t_p}\frac{\partial F^{(m)\bar{j}}}{\partial t_q} - \frac{\partial F^{(m)i}}{\partial t_q}\frac{\partial F^{(m)\bar{j}}}{\partial t_p}\right)(z), \quad \forall\, z \in \Sigma.$$

Or

$$\begin{aligned}\det\left(g_{pq}^{(m)}\right)(z_0) &= \det{}^2\left(g_{i\bar{j}}^{(m)}\right)(z)\det{}^2\left(\frac{\partial F^{(m)i}}{\partial t_p}\frac{\partial F^{(m)\bar{j}}}{\partial t_p}\right)(z)\\ &\leq C\det{}^2\left(g_{i\bar{j}}^{(m)}\right)(z), \quad \forall\, z \in \Sigma.\end{aligned}$$

Thus $g_{i\bar{j}}^{(m)}$ has a uniformly positive lower bound on Σ. According to Donaldson [12], the limiting disk must be super-regular. □

Lemma **5.3.5** *(No bubble in the kernel level). — For a sequence of boundary value* $S_z^{(m)}$, $A^{(m)}(z)$, $z \in \partial\Sigma$, *we consider a sequence of pairs of* $\mathbf{C}^n$*-valued functions* (u^m, v^m) *over the disk satisfying the linear boundary condition*

$$v^{(m)}(z) = S_z^{(m)}(u^{(m)}(z)) + A_z^{(m)}(\bar{u}(z)), \tag{5.13}$$

for $z \in \partial\Sigma$. *Here* $S^{(m)}$ *is a positive definite Hermitian matrix which has a uniform positive lower bound; while both* $S^{(m)}$ *and* $A^{(m)}$ *are uniformly bounded from above and below. If*

$$\lim_{m\to\infty} S^{(m)} = S^{\infty}, \quad \text{and} \quad \lim_{m\to\infty} A^{(m)} = A^{\infty},$$

in $C^{1,\alpha}$ *norm for some* $0 < \alpha < 1$ *and*

$$\|u^{(m)}\|^2_{W^{1,2}(\Sigma)} + \|v^{(m)}\|^2_{W^{1,2}(\Sigma)} = 1,$$

then $u^{(m)}$ *and* $v^{(m)}$ *has non-zero limit (away from possible bubble points provided* S^∞ *and* A^∞ *are continuous on* $\partial\Sigma$*).*

5.4. *Proof of Theorem 5.0.17: Compactness of almost super regular foliations*

We give a proof directly based on our work in the previous two subsections.

Proof. — As before, identify $\mathscr{U}_{\phi_0^{(m)}}$ with an open dense set $\tilde{M}$ of M. For every point x in $\tilde{M}$, consider g_x as the holomorphic disk in $\mathscr{U}_{\phi_0^{(m)}}$ which passes through the point (z_0, x). Set

$$h^{(m)}(x) = \mathrm{Cap}\big(g_x^{(m)}(\Sigma)\big).$$

Theorem 5.0.16 then implies that

$$\int_{\tilde{M}} h^{(m)}(x)dx = \int_{\tilde{M}} \mathrm{Cap}\left(g_x^{(m)}(\Sigma)\right)dx \le C.$$

Therefore, for generic points $x \in \tilde{M}$, there exists a subsequence of $g_x^{(m)}$ such that

$$h^{(m)}(x) = \mathrm{Cap}\big(g_x^{(m)}\big) \le C(x), \quad \forall\, m = 1, 2, ..., \infty. \tag{\textbf{5.14}}$$

According to Theorem 5.0.15, after passing to a subsequence if necessary, this sequence $\{g_x^{(m)}, m \in \mathbf{N}\}$ of disks has a uniform limit g_x^∞ which is super-regular again. In particular, it is a regular disc. Therefore there exists a small open set $B_{r(x)}(x) \subset \tilde{M}$ such that the evaluation map $(\pi \circ ev)$ of this sequence of almost super regular foliations $\{\mathscr{F}_{\phi_0^{(m)}}, m \in \mathbf{N}\}$ have a uniform parametrization near g_x^∞. In other words, $F^{(m)}|_{\Sigma\times B_{r(x)}(x)} : \Sigma \times B_{r(x)}(x) \to \Sigma \times \mathscr{W}_M$ has a unique smooth limiting evaluation map $F^{(\infty)} : \Sigma \times B_{r(x)}(x) \to \Sigma \times \mathscr{W}_M$. Consequently, $h^{(m)}$ is uniformly continuous in $B_{r(x)}(x) \subset \tilde{M}$. Following Lemma 5.4.1, there exists a set E of measure 0, and a subsequence of $h^{(m)}$ (ultimately of $g_x^{(m)}$) such that $h^{(m)}$ is a uniformly continuous function in any compact subset of $\tilde{M} \setminus E$. Theorem 5.0.15 again implies that the foliation has a uniform limit in this compact subset. We define $\mathscr{U}_{\phi_0^{(\infty)}} = \lim_{m\to\infty} F^{(m)\infty} : \Sigma \times \tilde{M} \setminus E \to \Sigma \times \mathscr{W}_M$.

For every $x \in E$, the disks $\{g_x^{(m)}, m \in \mathbf{N}\}$ have a uniform upper bound on their areas. Following Theorem 5.0.14, there exists a subsequence of disks $\{g_x^{(m)}, m \in \mathbf{N}\}$ which converges to an embedded holomorphic disk S_x^∞. This limit may not be unique. However, the image of each limiting disk $S_x^{(\infty)}$ does not intersect the image of any disk of $\mathscr{U}_{\phi_0^\infty}$ on $\Sigma^0 \times M$. Otherwise, for m large enough, there will exist two super regular discs which intersect at some interior point of $\Sigma^0 \times M$. That is an contradiction for an almost super regular foliation.

Now expanding the foliation $F^{(\infty)}$ by including $E_\infty = \bigcup_{x\in E} S_x^{(\infty)}$. Then, $F^{(\infty)}$ is a partially smooth foliation detailed in Definition 3.4.7, where $\mathscr{U}_{\phi_0^{(\infty)}}$ is the set of super regular disks among them.

The only remaining item is to show that $\omega^n_{\phi_0^{(\infty)}}$ is a continuous form on $\Sigma^0 \times M$. As before, set the image of $\tilde{M} \setminus E$ under the evaluation map $\pi \circ ev$ as $\mathscr{V}_{\phi_0}$. Let $E_1 = \Sigma \times M \setminus \mathscr{V}_{\phi_0}$ be the union of all of the singular points. Clearly, $\omega^n_{\phi_0^{(\infty)}}$ is a smooth (n, n) form in $\mathscr{V}_{\phi_0}$ and vanishes completely in E_1. To show that this n-form is continuous, we just need to justify that for any sequence $(z_i, x_i) \in \mathscr{V}_{\phi_0}$, which converges to $(\bar{z}, \bar{x}) \in E_1$, we have

$$\lim_{i\to\infty} \omega^n_{\phi_0^{(\infty)}}(z_i, x_i) = 0.$$

Set f_i to be the unique super regular disk in $\mathscr{U}_{\phi_0^{(\infty)}}$ passing through the point (z_i, x_i). Without loss of generality, we may assume that this sequence of disks converges to some other disk f.

By definition, $\lim_{i\to\infty} \operatorname{Cap}(f_i) = \infty$. In other words,

$$\lim_{i\to\infty} \int_\Sigma \left(\frac{\omega^n_{\phi_0^{(\infty)}}}{\omega^n} \right)^{-1} |dz|^2 = \infty. \tag{5.15}$$

However, $\log\left(\frac{\omega^n_{\phi_0^{(\infty)}}}{\omega^n}\right)$ is a sub-harmonic function with a uniform upper bound. Theorem 4.2.4 and Proposition 4.2.13 imply that, for any compact sub-domain $\Omega \subset \Sigma^0$, we have

$$\left| \Delta_z \log\left(\frac{\omega^n_{\phi_0^{(\infty)}}}{\omega^n}\right) \right| \le C_\Omega,$$

where C_Ω depends on $d(\partial\Omega, \partial\Sigma)$. Choose Ω so that $\{z_i, i \in \mathbf{N}\} \subset \Omega$ and $z \in \Omega$. The Harnack inequality for negative harmonic functions implies that, either $\log\left(\frac{\omega^n_{\phi_0^{(\infty)}}}{\omega^n}\right)$ tends to $-\infty$ simultaneously in Ω or $\log\left(\frac{\omega^n_{\phi_0^{(\infty)}}}{\omega^n}\right)$ are uniformly bounded from above and below. This dichotomy holds for any compact sub-domain of Σ^0. Thus, if (passing to a subsequence if necessary), we have

$$\lim_{i\to\infty} \left(\frac{\omega^n_{\phi_0^{(\infty)}}}{\omega^n}\right)(z_i, x_i) = c > 0.$$

Then for any $z \in \Sigma^0$ fixed, there is a unique point w_i such that (z, w_i) lies in the image of $\pi \circ f$, we have

$$\lim_{i\to\infty} \left(\frac{\omega^n_{\phi_0^{(\infty)}}}{\omega^n}\right)(z, w_i(z)) > 0.$$

Following the proof of Theorem 5.3.1 and Proposition 4.2.13, we can show that in fact,

$$\lim_{i\to\infty}\left(\frac{\omega^n_{\phi_0(\infty)}}{\omega^n}\right)(z, w_i(z)) > 0, \quad \forall\, z \in \Sigma.$$

The fact contradicts to the assumption that the capacity of this sequence of disks blows up. Consequently, the varying volume form $\left(\frac{\omega^n_{\phi_0(\infty)}}{\omega^n}\right)$ must converges to 0. In other words, the volume form must be continuous in the interior of $\Sigma \times \mathrm{M}$. □

Lemma **5.4.1.** — *Suppose that $h^{(m)}$ is a sequence of continuous, positive functions defined in a fixed domain Ω satisfying the following two conditions*

1. *The L^1 norm of $h^{(m)}$ is uniformly bounded;*
2. *For any $x \in \Omega$, if $\sup_{1\leq m\leq\infty} h^{(m)}(x) < \infty$, there exists a small neighborhood $\mathscr{O}_x$ of x such that this sequence (pass to a a subsequence if necessary) of functions $\{h^{(m)}\}_{m=1}^{\infty}$ is uniformly continuous on $\mathscr{O}_x$.*

Then there exists a set E of measure at most 0 and a subsequence of $\{h^{(m)}\}_{m=1}^{\infty}$ such that this subsequence is uniformly continuous on any compact subset of $\Omega \setminus \mathrm{E}$. Moreover, there exists a limit function h^{∞} such that $\lim_{m\to\infty} h^{(m)} = h^{\infty}$ on $\mathrm{M} \setminus \mathrm{E}$. Moreover, $\frac{1}{1+h^{\infty}}$ is a continuous function.

Proof. — By an elementary and straightforward argument. □

6. The (modified) K energy along almost solutions

In this section, we want to prove first that the K energy function is sub-harmonic along any almost smooth solution of HCMA equation (1.1). One can view this as a generalization of the fact that the K energy functional is convex along a smooth geodesic. Secondly, we want to use this property of subharmonicity to prove that the (modified) K energy has a lower bound in any Kähler class, provided that there exists a constant scalar curvature metric (or an extremal Kähler metric) in this class.

6.1. *The sub-harmonicity of the K energy*

Suppose that $\mathscr{F}_{\phi_0}$ is an almost super regular foliation; and suppose $\phi : \Sigma \to \overline{\mathscr{H}_\omega}$ is an almost smooth solution of the HCMA equation (1.1) corresponding to it. Note that the evaluation map $ev : \Sigma \times \mathscr{M}_{\phi_0} \to \Sigma \times \mathscr{W}_{\mathrm{M}}$ is smooth everywhere. The set of holomorphic disks which are not super regular has codimension at least 1 in the *moduli* space. Recall that $\mathscr{V}_{\phi_0} = \pi \circ ev(\Sigma \times \mathscr{U}_{\phi_0})$. As before, set $\mathscr{S}_{\phi_0} = \Sigma \times \mathrm{M} \setminus \mathscr{V}_{\phi_0}$. Clearly, $\mathscr{S}_{\phi_0}$

is a smooth sub-manifold and $\mathscr{S}_{\phi_0} \cap (\partial\Sigma \times \mathrm{M})$ has codimension at least 1 at $\partial\Sigma \times \mathrm{M}$. We follow notations in Section 4 in general. For convenience of the readers, let us re-state Theorem 1.3.5 here

Theorem **6.1.1.** — *Suppose that $\phi : \Sigma \to \overline{\mathscr{H}_\omega}$ is an almost smooth solution described as in Definition 1.3.3. Then the induced K energy function $\mathbf{E}_\omega : \Sigma \to \mathbf{R}$ (by $\mathbf{E}_\omega(z) = \mathbf{E}_\omega(\phi(z,\cdot))$) is weakly sub-harmonic and C^1 continuous (up to the boundary). More precisely,*

$$\frac{\partial^2}{\partial z \partial \bar z}\mathbf{E}_\omega(\phi(z,\cdot)) = \int_{\pi \circ ev(z, \mathscr{U}_{\phi_0})} \left|\mathscr{D}\frac{\partial \phi}{\partial \bar z}\right|^2_{\omega_\phi} {\omega_\phi}^n \geq 0, \quad \forall z \in \Sigma^0$$

holds in Σ^0 in the weak sense. On $\partial\Sigma$, we have

$$\int_{\partial\Sigma} \frac{\partial \mathbf{E}_\omega}{\partial \mathbf{n}}(\phi) ds = \int_{\pi \circ ev(z, \mathscr{U}_{\phi_0})} \left|\mathscr{D}\frac{\partial \phi}{\partial \bar z}\right|^2_{\omega_\phi} {\omega_\phi}^n \, ds,$$

where ds is the length element of $\partial\Sigma$, and $\mathbf{n}$ is the outward pointing unit normal direction at $\partial\Sigma$.

To help readers to understand its proof better, we will first present a proof of this theorem in the case that the disc version geodesic ϕ is smooth. Note that for any smooth path $\phi(t)$, we have

$$\frac{d^2\mathbf{E}_\omega}{dt^2}(\phi(t)) = -\int_{\mathrm{M}} \left(\frac{\partial^2\phi}{\partial t^2} - \left|\nabla \frac{\partial \phi}{\partial t}\right|_\phi\right)(\mathrm{S}_\varphi - \mu)\omega^n_{\phi(t)} + \int_{\mathrm{M}} \left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|^2_\phi \omega^n_\phi.$$

Note that for a disc version geodesic, we have

$$z = t + \sqrt{-1}s, \quad \text{and} \quad \triangle_\Sigma = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}.$$

Thus,

$$\begin{aligned}
\triangle_\Sigma \mathbf{E}_\omega &= \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}\right)\mathbf{E}_\omega \\
&= -\int_{\mathrm{M}} \left(\frac{\partial^2\phi}{\partial t^2} - \left|\nabla \frac{\partial \phi}{\partial t}\right|^2_\phi\right)(\mathrm{S}_\varphi - \mu)\omega^n_{\phi(t,s)} \\
&\quad -\int_{\mathrm{M}} \left(\frac{\partial^2\phi}{\partial s^2} - \left|\nabla \frac{\partial \phi}{\partial s}\right|^2_\phi\right)(\mathrm{S}_\varphi - \mu)\omega^n_{\phi(t,s)} \\
&\quad + \int_{\mathrm{M}} \left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|^2_\phi \omega^n_\phi + \int_{\mathrm{M}} \left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|^2_\phi \omega^n_\phi.
\end{aligned}$$

Using the equation for disc version geodesics, we have

$$\begin{aligned}\Delta_\Sigma \phi &= g_\phi^{\alpha\bar\beta}\left(\frac{\partial\phi}{\partial\bar z}\right)_\alpha\left(\frac{\partial\phi}{\partial z}\right)_{\bar\beta} = g_\phi^{\alpha\bar\beta}\left(\frac{\partial\phi}{\partial t}-\sqrt{-1}\frac{\partial\phi}{\partial s}\right)_\alpha\left(\frac{\partial\phi}{\partial t}+\sqrt{-1}\frac{\partial\phi}{\partial s}\right)_{\bar\beta}\\
&= \left|\nabla\frac{\partial\phi}{\partial t}\right|_\phi^2+\left|\nabla\frac{\partial\phi}{\partial s}\right|_\phi^2\\
&\quad+\sqrt{-1}g_\phi^{\alpha\bar\beta}\left(\left(\frac{\partial\phi}{\partial t}\right)_\alpha\left(\frac{\partial\phi}{\partial s}\right)_{\bar\beta}-\left(\frac{\partial\phi}{\partial s}\right)_\alpha\left(\frac{\partial\phi}{\partial t}\right)_{\bar\beta}\right)\\
&= \left|\nabla\frac{\partial\phi}{\partial t}\right|_\phi^2+\left|\nabla\frac{\partial\phi}{\partial s}\right|_\phi^2+\left\{\frac{\partial\phi}{\partial t},\frac{\partial\phi}{\partial s}\right\}_\phi.\end{aligned}$$

The last term gives the Poisson bracket with respect to the symplectic form ω_ϕ. Thus, we have

$$\begin{aligned}\Delta_\Sigma\mathbf{E}_\omega &= -\int_M\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial s^2}-\left|\nabla\frac{\partial\phi}{\partial t}\right|_\phi^2-\left|\nabla\frac{\partial\phi}{\partial t}\right|_\phi^2\right)(S_\varphi-\mu)\omega^n_{\phi(t,s)}\\
&\quad+\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|_\phi^2\omega_\phi^n+\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|_\phi^2\omega_\phi^n\\
&= -\left(\left\{\frac{\partial\phi}{\partial t},\frac{\partial\phi}{\partial s}\right\}_\phi, S_\varphi-\mu\right)_\phi+\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|_\phi^2\omega_\phi^n+\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|_\phi^2\omega_\phi^n.\end{aligned}$$

The first term in the last line may give us some trouble. However, one notices that

$$\begin{aligned}\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|_\phi^2\omega_\phi^n &= \left(\mathscr{D}\left(\frac{\partial\phi}{\partial t}-\sqrt{-1}\frac{\partial\phi}{\partial s}\right),\mathscr{D}\left(\frac{\partial\phi}{\partial t}-\sqrt{-1}\frac{\partial\phi}{\partial s}\right)\right)_\phi\\
&= \int_M\left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|_\phi^2\omega_\phi^n+\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|_\phi^2\omega_\phi^n\\
&\quad+\sqrt{-1}\left(\mathscr{D}\frac{\partial\phi}{\partial t},\mathscr{D}\frac{\partial\phi}{\partial s}\right)_\phi-\sqrt{-1}\left(\mathscr{D}\frac{\partial\phi}{\partial s},\mathscr{D}\frac{\partial\phi}{\partial t}\right)_\phi\\
&= \int_M\left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|_\phi^2\omega_\phi^n+\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|_\phi^2\omega_\phi^n\\
&\quad+\sqrt{-1}\left((\bar{\mathscr{D}}\mathscr{D}-\mathscr{D}\bar{D})\frac{\partial\phi}{\partial t},\frac{\partial\phi}{\partial s}\right)_\phi\\
&= \int_M\left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|_\phi^2\omega_\phi^n+\int_M\left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|_\phi^2\omega_\phi^n\\
&\quad+\sqrt{-1}\int_M g_\phi^{\alpha\bar\beta}\left(S_\alpha\left(\frac{\partial\phi}{\partial t}\right)_{\bar\beta}-S_{\bar\beta}\left(\frac{\partial\phi}{\partial t}\right)_\alpha\right)\frac{\partial\phi}{\partial s}\omega_\phi^n\end{aligned}$$

$$
\begin{aligned}
&= \int_M \left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|_\phi^2 \omega_\phi^n + \int_M \left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|_\phi^2 \omega_\phi^n + \left(\left\{S, \frac{\partial\phi}{\partial t}\right\}_\phi, \frac{\partial\phi}{\partial s}\right)_\phi \\
&= \int_M \left|\mathscr{D}\frac{\partial\phi}{\partial t}\right|_\phi^2 \omega_\phi^n + \int_M \left|\mathscr{D}\frac{\partial\phi}{\partial s}\right|_\phi^2 \omega_\phi^n \\
&\quad + \left(\left\{\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial s}\right\}_\phi, S - \mu\right)_\phi.
\end{aligned}
$$

Now, plugging this into $\triangle_\Sigma \mathbf{E}_\omega$, we have

$$
\triangle_\Sigma \mathbf{E}_\omega = \int_M \left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|_\phi^2 \omega_\phi^n \geq 0.
$$

Thus, we prove subharmonicity for any smooth disc version geodesic.

Unfortunately, our solution is not smooth and the proof is much more involved. Before we proceed to the proof, let us note the following calculation scheme. For any function $f \in C^\infty(\Sigma \times M)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial z}\int_M f\omega_\phi^n &= \int_M \frac{\partial f}{\partial z}\omega_\phi^n + \int_M f\Delta_\phi \frac{\partial\phi}{\partial z}\omega_\phi^n \\
&= \int_M \frac{\partial f}{\partial z}\omega_\phi^n - \int_M \frac{\partial f}{\partial w_\alpha} g_\phi^{\alpha\bar\beta} \frac{\partial^2\phi}{\partial z \partial w_{\bar\beta}}\omega_\phi^n \\
&= \int_M \frac{\partial f}{\partial z}\omega_\phi^n + \int_M \frac{\partial f}{\partial w_\alpha}\eta^\alpha \omega_\phi^n \\
&= \int_M \partial_z(f)\omega_\phi^n.
\end{aligned}
$$

Similarly, we have

$$
\frac{\partial}{\partial \bar z}\int_M f\omega_\phi^n = \int_M \partial_{\bar z}(f)\omega_\phi^n.
$$

We will use these schemes throughout the proof below. We also need to use the decomposition formula of the K energy given in [8] (cf. [28]). For any $\phi \in \mathscr{H}_\omega$, we have

$$
\mathbf{E}_\omega(\phi) = \int_M \ln\frac{\omega_\phi^n}{\omega_0^n}\omega^n + J(\phi) + \mu I(\phi), \tag{6.1}
$$

$$
I(\phi) = \sum_{p=0}^{n} \frac{1}{p+1}\int_M \phi\,\omega_0^{[n-p]} \wedge (\sqrt{-1}\partial\bar\partial\phi)^{[p]}, \tag{6.2}
$$

$$
J(\phi) = -\sum_{p=0}^{n-1} \frac{1}{p+1}\int_M \phi\,\mathrm{Ric}(\omega_0) \wedge \omega_0^{[n-p-1]} \wedge (\sqrt{-1}\partial\bar\partial\phi)^{[p]}. \tag{6.3}
$$

Here μ is the average of the scalar curvature function of any Kähler metric in $[\omega]$. By the definition of the functional I, we have

$$\frac{\partial \mathrm{I}}{\partial z} = \int_{\mathrm{M}} \frac{\partial \phi}{\partial z} \omega_\phi^n.$$

Thus,

$$\begin{aligned}\frac{\partial^2 \mathrm{I}}{\partial z \partial \bar z} &= \int_{\mathrm{M}} \partial_{\bar z} \frac{\partial \phi}{\partial z} \omega_\phi^n \\ &= \int_{\mathrm{M}} \left(\frac{\partial^2 \phi}{\partial z \partial \bar z} - \eta^{\bar\beta} \frac{\partial}{\partial \bar w^\beta} \frac{\partial \phi}{\partial \bar z} \right) \omega_\phi^n = 0.\end{aligned}$$

Thus, the component I makes no contribution to the calculation of the 2nd mixed derivatives of the K energy $\mathbf{E}_\omega$. Thus, we can basically leave it aside as we calculate the second mixed derivatives of the K energy.

Proof. — Let χ be any non-negative function whose support lies inside of the set $\Sigma \times \mathrm{M} \setminus \mathscr{S}_{\phi_0}$. Set

$$\mathrm{K}_\chi = \int_{\mathrm{M}} \chi \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n.$$

Then

$$\begin{aligned}\frac{\partial^2 \mathrm{K}_\chi}{\partial z \partial \bar z} &= \int_{\mathrm{M}} \partial_{\bar z} \partial_z \left(\chi \log \frac{\omega_\phi^n}{\omega_0^n} \right) \omega_\phi^n \\ &= \int_{\mathrm{M}} \chi \left(\Delta_z \log \frac{\omega_\phi^n}{\omega_0^n} + \partial_{\bar z} \left((\partial_z \chi) \log \frac{\omega_\phi^n}{\omega^n} \right) \right) \omega_\phi^n \\ &\quad + \int_{\mathrm{M}} (\partial_z \chi) \left(\log \frac{\omega_\phi^n}{\omega_0^n} \right) \omega_\phi^n \\ &= \int_{\mathrm{M}} \chi \left| \mathscr{D} \frac{\partial \phi}{\partial \bar z} \right|_\phi^2 \omega_\phi^n + \int_{\mathrm{M}} \chi \,\mathrm{Ric}(\omega_0)_{\alpha\bar\beta} \eta^\alpha \eta^{\bar\beta} \omega_\phi^n \\ &\quad + \frac{\partial}{\partial \bar z} \int_{\mathrm{M}} (\partial_z \chi) \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n + \int_{\mathrm{M}} (\partial_z \chi) \left(\partial_{\bar z} \log \frac{\omega_\phi^n}{\omega_0^n} \right) \omega_\phi^n.\end{aligned}$$

Let $v(z)$ be any non-negative cut off function in Σ^0.

$$\begin{aligned}\int_\Sigma \mathrm{K}_\chi \Delta_z v &= \int_\Sigma v(z) \Delta_z \mathrm{K}_\chi \\ &= \int_\Sigma v(z) \int_{\mathrm{M}} \chi \left| \mathscr{D} \frac{\partial \phi}{\partial z} \right|_\phi^2 \omega_\phi^n + \int_\Sigma v(z) \int_{\mathrm{M}} \chi \,\mathrm{Ric}(\omega_0)_{\alpha\bar\beta} \eta^\alpha \eta^{\bar\beta} \omega_\phi^n \\ &\quad - \int_\Sigma \frac{\partial v}{\partial \bar z} \cdot \int_{\mathrm{M}} (\partial_z \chi) \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n + \int_\Sigma v \int_{\mathrm{M}} (\partial_z \chi) \partial_{\bar z} \left(\frac{\omega_\phi^n}{\omega_0^n} \right) \omega_0^n.\end{aligned}$$

Consider the evaluation map:

$$\begin{aligned} \sharp : \Sigma \times \mathscr{M}_{\phi_0} &\to \Sigma \times \mathrm{M} \\ (z, f) &\to (z, \pi(f(z))). \end{aligned}$$

Then, $\sharp$ is invertible on $\Sigma \times \mathscr{U}_{\phi_0}$. Consider any C^∞ function $\phi \in \mathrm{C}_0^\infty(\mathscr{U}_{\phi_0}) \subset \mathrm{C}^\infty(\mathscr{M}_{\phi_0})$ which vanishes on the boundary of $\mathscr{U}_{\phi_0}$. Set

$$\chi(z, w) = \phi(\sharp^{-1}(z, w)), \quad \forall\, (z, w) \in \Sigma \times \mathrm{M}. \tag{6.4}$$

Then $\chi(z, w)$ is a smooth function in $\Sigma \times \mathrm{M}$ whose support lies completely inside $\Sigma \times \mathrm{M} \setminus \mathscr{S}_{\phi_0}$. By definition, the disk derivative $\partial_z \chi$ vanishes completely along super regular disks. Consequently, for any cut off function defined via formula (6.4), we have

$$\int_\Sigma \mathrm{K}_\chi \Delta_z v = \int_\Sigma v(z) \int_\mathrm{M} \chi \left| \mathscr{D} \frac{\partial \phi}{\partial \bar z} \right|_\phi^2 \omega_\phi^n + \int_\Sigma v(z) \int_\mathrm{M} \chi \,\mathrm{Ric}(\omega)_{\alpha\bar\beta} \eta^\alpha \eta^{\bar\beta} \omega_\phi^n.$$

Now let ϕ tend to characteristic function of $\mathscr{U}_{\phi_0}$ inside $\mathscr{M}_{\phi_0}$. Then, we have:

$$\begin{aligned} \int_\Sigma \Delta_z v(z) \int_\mathrm{M} \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n &= \int_{\Sigma\times\mathrm{M}\setminus\mathscr{S}_{\phi_0}} \Delta_z v(z) \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n \\ &= \int_{\Sigma\times\mathrm{M}\setminus\mathscr{S}_{\phi_0}} v(z) \left| \mathscr{D} \frac{\partial \phi}{\partial \bar z} \right|_\phi^2 \omega_\phi^n \\ &\quad + \int_{\Sigma\times\mathrm{M}\setminus\mathscr{S}_{\phi_0}} v(z)\, \mathrm{Ric}(\omega)_{\alpha\bar\beta} \eta^\alpha \eta^{\bar\beta} \omega_\phi^n \\ &= \int_{\Sigma\times\mathrm{M}\setminus\mathscr{S}_{\phi_0}} v(z) \left| \mathscr{D} \frac{\partial \phi}{\partial \bar z} \right|_\phi^2 \omega_\phi^n \\ &\quad + \int_{\Sigma\times\mathrm{M}} v(z)\, \mathrm{Ric}(\omega)_{\alpha\bar\beta} \eta^\alpha \eta^{\bar\beta} \omega_\phi^n. \end{aligned}$$

The first and the last equality holds because that

$$\log \frac{\omega_\phi^n}{\omega_0^n} \frac{\omega_\phi^n}{\omega_0^n} \quad \text{and} \quad \mathrm{Ric}(\omega_0)_{\alpha\bar\beta} \eta^\alpha \eta^{\bar\beta} \frac{\omega_\phi^n}{\omega_0^n}$$

both vanish on $\mathscr{S}_{\phi_0}$. On the other hand,

$$\begin{aligned} \frac{\partial^2 \mathrm{J}(\phi)}{\partial z \partial \bar z} &= - \int_\mathrm{M} \frac{\partial^2 \phi}{\partial z \partial \bar z} \mathrm{Ric}(\omega) \wedge \omega_\phi^{n-1} \\ &\quad - \int_\mathrm{M} \frac{\partial \phi}{\partial z} \mathrm{Ric}(\omega_0) \wedge \sqrt{-1} \partial \bar\partial \frac{\partial \phi}{\partial \bar z} \wedge \omega_\phi^{[n-2]} \\ &= - \int_\mathrm{M} \frac{\partial^2 \phi}{\partial z \partial \bar z} \mathrm{Ric}(\omega_0) \wedge \omega_\phi^{n-1} \\ &\quad + \int_\mathrm{M} \mathrm{Ric}(\omega_0) \wedge \sqrt{-1} \partial \frac{\partial \phi}{\partial \bar z} \wedge \bar\partial \frac{\partial \phi}{\partial z} \wedge \omega_\phi^{[n-2]} \end{aligned}$$

$$\begin{aligned}
&= -\int_M \frac{\partial^2 \phi}{\partial z \partial \bar{z}} \operatorname{Ric}(\omega_0) \wedge \omega_\phi^{n-1} \\
&\quad + \int_M \left(g_\phi^{\alpha\bar{\beta}} \operatorname{Ric}(\omega_0)_{\alpha\bar{\beta}} \cdot g_\phi^{r\bar{\delta}} \frac{\partial^2 \phi}{\partial \bar{z} \partial w_r} \frac{\partial^2 \phi}{\partial z \partial w_{\bar{\delta}}} - \operatorname{Ric}(\omega_0)_{\alpha\bar{\beta}} \eta^\alpha \eta^{\bar{\beta}} \right) \omega^n \\
&= -\int_M \left(\frac{\partial^2 \phi}{\partial z \partial \bar{z}} - g_\phi^{r\bar{\delta}} \eta^\alpha \eta^{\bar{\beta}} \right) \operatorname{Ric}(\omega_0) \wedge \omega_\phi^{n-1} \\
&\quad - \int_M \operatorname{Ric}(\omega_0)_{\alpha\bar{\beta}} \eta^\alpha \eta^{\bar{\beta}} \omega^n \\
&= -\int_M \operatorname{Ric}(\omega_0)_{\alpha\bar{\beta}} \eta^\alpha \eta^{\bar{\beta}} \omega^n.
\end{aligned}$$

The last equality holds since ϕ is a solution to the homogenous complex Monge–Ampere equation. Therefore,

$$\int_\Sigma \Delta_z v \mathrm{J}(\phi) = -\int_\Sigma \Delta_z v \int_M \operatorname{Ric}(\omega_0)_{\alpha\bar{\beta}} \eta^\alpha \eta^{\bar{\beta}} \omega^n.$$

Using the decomposition formula for K energy (6.1), we have

$$\int_\Sigma (\Delta_z v(z)) \mathbf{E}_\omega(\phi(z, \cdot)) = \int_{\Sigma \times M \setminus \mathscr{S}_{\phi_0}} v(z) \left| \mathscr{D} \frac{\partial \phi}{\partial \bar{z}} \right|_\phi^2 \omega_\phi^n \geq 0. \tag{6.5}$$

This implies that the K energy functional is sub-harmonic in Σ^0. Next we want to derive a formula for the first derivative of the K energy. For any $v \in C_0^\infty(\Sigma)$, we have

$$\begin{aligned}
\int_\Sigma \frac{\partial v(z)}{\partial z} \cdot \mathrm{K}_\chi &= \int_\Sigma v(z) \cdot \frac{\partial}{\partial z} \mathrm{K}_\chi \\
&= \int_\Sigma v(z) \left(\int_M \partial_z(\chi) \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n + \int_M \chi \partial_z \left(\frac{\omega_\phi^n}{\omega_0^n} \right) \omega^n \right).
\end{aligned}$$

For any small $\delta > 0$, let $\Sigma_\delta = \{z \in \Sigma : |z| \leq 1 - \delta\}$. Since $v(z)$ is an arbitrary compactly supported function in Σ^0, we obtain

$$\int_{\partial \Sigma_\delta} \zeta(z) \frac{\partial}{\partial z} \mathrm{K}_\chi = \int_{\partial \Sigma_\delta} \zeta(z) \left(\int_M \partial_z(\chi) \log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^n + \int_M \chi \partial_z \left(\frac{\omega_\phi^n}{\omega_0^n} \right) \omega_0^n \right),$$

where ζ is any smooth function.

Now let χ tend to the characteristic function of $\Sigma \times M \setminus \mathscr{S}_{\phi_0}$. As before, the first term in the right hand side vanishes, we have

$$\int_{\partial \Sigma_\delta} \zeta(z) \frac{\partial}{\partial z} \int_M \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n = \int_{\partial \Sigma_\delta} \zeta(z) \int_{M \setminus \mathscr{S}_{\phi_0}} \partial_z \left(\frac{\omega_\phi^n}{\omega_0^n} \right) \omega_0^n.$$

For the K energy, we have

$$\int_{\partial\Sigma_\delta} \zeta(z)\frac{\partial}{\partial z}\mathbf{E}_\omega(\phi(z,\cdot)) = \int_{\partial\Sigma_\delta}\zeta(z)\int_{M\setminus\mathscr{S}_{\phi_0}} \partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n - \int_{\partial\Sigma_\delta}\zeta(z)\int_M \frac{\partial\phi}{\partial z}(\mathrm{Ric}(\omega_0)-\mu\omega_\phi)\wedge\omega_\phi^{n-1}.$$

Since ζ is an arbitrary test function, we have

$$\frac{\partial}{\partial z}\mathbf{E}_\omega(\phi(z,\cdot)) = \int_{M\setminus\mathscr{S}_{\phi_0}} \partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n - \int_M \frac{\partial\phi}{\partial z}(\mathrm{Ric}(\omega_0)-\mu\omega_\phi)\wedge\omega_\phi^{n-1}.$$

To show the first derivative of the K energy is continuous up to the boundary, we just need to show the first term on the (RHS) is continuous up to boundary.

Integrating by parts on the left hand side of (6.5) and letting v approach the characteristic function of Σ_δ, we obtain

$$\begin{aligned}\int_{\Sigma_\delta\times M\setminus\mathscr{S}_{\phi_0}}\left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|_\phi^2\omega_\phi^n &= \int_{\partial\Sigma_\delta}\frac{\bar z}{|z|}\frac{\partial}{\partial z}\mathbf{E}_\omega(\phi(z,\cdot))\\ &= \int_{\partial\Sigma_\delta}\frac{\bar z}{|z|}\int_{M\setminus\mathscr{S}}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n\\ &\quad - \int_{\Sigma_\delta}\frac{\bar z}{|z|}\int_M\frac{\partial\phi}{\partial z}(\mathrm{Ric}(\omega_0)-\mu\omega_\phi)\wedge\omega_\phi^{n-1}.\end{aligned}$$

The first and the third terms in the formula above are both integration on $\pi\circ ev(\Sigma\times\mathscr{U}_{\phi_0})$, where Kähler metric is smooth. Therefore, taking limit as $\delta\to 0$, we arrive at

$$\int_{\Sigma\times M\setminus\mathscr{S}_{\phi_0}}\left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|_\phi^2\omega_\phi^n = \int_{\partial\Sigma}\frac{\bar z}{|z|}\int_{M\setminus\mathscr{S}_{\phi_0}}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n - \int_{\partial\Sigma}\frac{\bar z}{|z|}\int_M\frac{\partial\phi}{\partial z}(\mathrm{Ric}(\omega_0)-\mu\omega_\phi)\wedge\omega_\phi^{n-1}.$$

In the above process of taking the limit, the only term which needs special attentions is:

$$\lim_{z\to z_0\in\partial\Sigma}\int_{\{z\}\times M\setminus\mathscr{S}_{\phi_0}}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n = \int_{\{z_0\}\times M\setminus\mathscr{S}_{\phi_0}}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n.$$

This is equivalent to say that the z-derivative of the K energy is continuous as $z\to z_0\in\partial\Sigma(|z|<1=|z_0|,\forall z_0\in\partial\Sigma)$. For any $\delta>0$ fixed, choose any δ neighborhood of the set of non-super regular disks (denoted by $E_\delta\subset\mathscr{M}_{\phi_0}$) such that

$$\lim_{\delta\to 0} mes(E_\delta) = 0.$$

Set

$$\mathscr{S}_\delta = \sharp(\Sigma \times \mathrm{E}_\delta).$$

Let $(t_1, t_2, \ldots, t_{2n})$ be the coordinate variables in $\mathscr{M}_{\phi_0}$ and $w_1, w_2, \ldots, w_n$ be the complex coordinate variables in M. Set

$$\mathrm{J} = \left(\frac{\partial w_\alpha}{\partial t_i}\frac{\partial w_{\bar\beta}}{\partial t_i}\right)$$

to be the Jacobian matrix. Then, J is a smooth complex matrix valued function in $\mathscr{M}_{\phi_0}$, and invertible at $\mathscr{U}_{\phi_0}$. Denote by $\Gamma(\omega_0)$ the canonical connection form of the Kähler metric ω_0. In the following calculation, we take covariant derivatives with respect to ω_0. Clearly,

$$\lim_{z\to z_0\in\partial\Sigma}\int_{\{z\}\times \mathrm{M}\setminus\mathscr{S}_\delta}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n = \int_{\{z_0\}\times \mathrm{M}\setminus\mathscr{S}_\delta}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n.$$

Now we need to show that the remaining portion is $o(\delta)$.

$$\begin{aligned}
&\lim_{z\to z_0\in\partial\Sigma}\int_{\{z\}\times \mathscr{S}_\delta\setminus\mathscr{S}_{\phi_0}}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n\\
&= \int_{\{z_0\}\times\mathscr{S}_\delta\setminus\mathscr{S}}\eta^\alpha{}_{,w_\alpha}(\omega_0)\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n\\
&= \int_{\{z_0\}\times\mathscr{S}_\delta\setminus\mathscr{S}}\left(\frac{\partial\eta^\alpha}{\partial w_\alpha} - \Gamma^\alpha_{\beta\alpha}(\omega_0)\eta^\beta\right)\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega^n\\
&= \int_{\{z_0\}\times(\mathrm{E}_\delta\cap\mathscr{U}_{\phi_0})}\left(\frac{\partial\eta^\alpha}{\partial w_\alpha} - \Gamma^\alpha_{\beta\alpha}(\omega_0)\eta^\beta\right)\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\det g_{\alpha\bar\beta}\det(\mathrm{J})dt\\
&= \int_{\{z_0\}\times(\mathrm{E}_\delta\cap\mathscr{U}_{\phi_0})}\frac{\partial\eta^\alpha}{\partial x_k}\left(\frac{\partial x_k}{\partial w_\alpha}\det(\mathrm{J})\right)\det g_{\alpha\bar\beta}dt\\
&\quad - \int_{\{z_0\}\times(\mathrm{E}_\delta\cap\mathscr{U}_{\phi_0})}\Gamma^\alpha_{\beta\alpha}(\omega_0)\eta^\beta\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\det g_{\alpha\bar\beta}\det(\mathrm{J})dt \to 0
\end{aligned}$$

as $\delta \to 0$. This is because all terms in the last formula are uniformly bounded and the measure of E_δ tends to 0 as $\delta \to 0$. Here

$$dt = dt^1 dt^2 \cdots dt^{2n}.$$

Consequently, we have shown that

$$\begin{aligned}
\int_{\Sigma\times\mathrm{M}\setminus\mathscr{S}_{\phi_0}}\left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|^2_\phi\omega_\phi^n &= \int_{\partial\Sigma}\frac{\bar z}{|z|}\int_{\mathrm{M}\setminus\mathscr{S}_{\phi_0}}\partial_z\left(\frac{\omega_\phi^n}{\omega_0^n}\right)\omega_0^n\\
&\quad - \int_{\partial\Sigma}\frac{\bar z}{|z|}\int_{\mathrm{M}}\frac{\partial\phi}{\partial z}(\mathrm{Ric}(\omega_0) - \mu\omega_\phi)\wedge\omega_\phi^{n-1}.
\end{aligned}$$

In other words, we have

$$\int_{\Sigma\times M\setminus\mathscr{S}_{\phi_0}} \left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|_{\phi}^2 \omega_\phi^n = \int_{\partial\Sigma} \frac{\bar z}{|z|}\frac{\partial}{\partial z}\mathbf{E}_\omega(\phi(z,\cdot))$$
$$= \int_{\partial\Sigma}\frac{\partial \mathrm{E}}{\partial \mathbf{n}}.$$

The theorem is then proved. □

If we replace the almost smooth solution by a partially smooth solution, then

Corollary **6.1.2.** — *Suppose that* $\phi : \Sigma \to \overline{\mathscr{H}_\omega}$ *is a partially smooth solution described as in Definition 1.3.1. Then the induced K energy function* $\mathbf{E}_\omega : \Sigma \to \mathbf{R}$ *(by* $\mathbf{E}_\omega(z) = \mathbf{E}_\omega(\phi(z,\cdot))$*) is a bounded weakly sub-harmonic function in* Σ *such that*

$$\frac{\partial^2}{\partial z\partial\bar z}\mathbf{E}_\omega(\phi(z,\cdot)) \geq \int_{M\setminus\mathscr{S}_{\phi_0}} \left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|_{\omega_\phi}^2 {\omega_\phi}^n \geq 0$$

holds in Σ *in the weak sense. Moreover,*

$$\int_{\partial\Sigma}\frac{\partial\mathbf{E}_\omega}{\partial\mathbf{n}}(\phi(z,\cdot))ds \geq \int_{\Sigma\times M\setminus\mathscr{S}_{\phi_0}} \left|\mathscr{D}\frac{\partial\phi}{\partial\bar z}\right|_{\omega_\phi}^2 {\omega_\phi}^n ds,$$

where ds is the length element of $\partial\Sigma$*, and* $\mathbf{n}$ *is the outward pointing unit normal direction at* $\partial\Sigma$*.*

Next, let us discuss the general case of **extremal Kähler metrics**. In [5], E. Calabi showed the following structure theorem for extremal Kähler metrics:

Proposition **6.1.3.** — *Let* M *be a connected, compact Kähler manifold with an extremal Kähler metric* g*. Then the identity component of the isometry group* $\mathrm{Isom}_0(M, g)$ *is a maximal compact, connected subgroup of the automorphism group* $\mathrm{Aut}_0(M, J)$*. Moreover, any extremal Kähler metric must be invariant under one of the maximal compact subgroup of the automorphism group.*

Following this theorem of Calabi, we may consider only those metrics which are invariant under the same maximal connected compact subgroup of the automorphism group. According to [15], there exists a unique holomorphic vector field

$$Y = Y^\alpha\frac{\partial}{\partial w_\alpha},$$

which is the gradient vector field of the scalar curvature function if the metric is extremal. Note that this vector filed is unique up to holomorphic conjugation in each

Kähler class. From this point on, we will use Y to denote the extremal vector field and we use it to modified K energy functional. Consider

$$\mathscr{L}_{Y}\omega_\phi = \sqrt{-1}\partial\bar{\partial}\theta(\phi).$$

Here $\theta(\phi)$ is a real valued potential function for ϕ if and only if Im(Y) is a Killing vector field of ω_ϕ. This is true for those Kähler potentials which are invariant under the maximal compact subgroup.

It is well known that one can modify the definition of the K energy by this potential function such that the critical point of the new functional is the extremal Kähler metric. Set

$$\frac{d\tilde{\mathbf{E}}_\omega}{dt}(\phi(t)) = -\int_M (S(\phi) - \mu - \theta(\phi))\frac{\partial\phi}{\partial t}\omega_\phi^n \tag{6.6}$$

$$= \frac{d\mathbf{E}_\omega}{dt}(\phi(t)) + \int_M \theta(\phi)\frac{\partial\phi}{\partial t}\omega_\phi^n. \tag{6.7}$$

To prove that this can be integrated into a well-defined functional $\tilde{\mathbf{E}}_\omega$, we let $\phi(t, s)$ be a family of Kähler potentials parametrized by t and s. We compute

$$\frac{\partial\theta(\phi)}{\partial w_{\bar\beta}} = Y^\alpha g_{\phi,\alpha\bar\beta}.$$

It is easy to see that

$$\frac{\partial\theta}{\partial s} = Y\left(\frac{\partial\phi}{\partial s}\right) = g_\phi^{\alpha\bar\beta}\frac{\partial\theta}{\partial w_{\bar\beta}}\frac{\partial^2\phi}{\partial s\partial w_\alpha}.$$

Then,

$$\begin{aligned}
&\frac{d}{ds}\int_M \theta(\phi)\frac{\partial\phi}{\partial t}\omega_\phi^n\\
&= \int_M \left(\frac{\partial\theta}{\partial s}\frac{\partial\phi}{\partial t} + \theta\frac{\partial^2\phi}{\partial t\partial s} + \theta\frac{\partial\phi}{\partial t}\Delta_\phi\left(\frac{\partial\phi}{\partial s}\right)\right)\omega_\phi^n\\
&= \int_M \left(g_\phi^{\alpha\bar\beta}\frac{\partial\theta}{\partial w_{\bar\beta}}\frac{\partial^2\phi}{\partial s\partial w_\alpha}\frac{\partial\phi}{\partial t} + \theta\frac{\partial^2\phi}{\partial t\partial s} + \theta\frac{\partial\phi}{\partial t}\Delta_\phi\left(\frac{\partial\phi}{\partial t}\right)\right)\omega_\phi^n\\
&= \int_M \theta(\phi)\left(\frac{\partial^2\phi}{\partial t\partial s} - \frac{1}{2}g_\phi\left(\nabla\frac{\partial\phi}{\partial t}, \nabla\frac{\partial\phi}{\partial s}\right)\right)\omega_\phi^n\\
&= \frac{d}{dt}\int_M \theta(\phi)\frac{\partial\phi}{\partial s}\omega_\phi^n.
\end{aligned}$$

This means our definition of modified K energy is well defined. It follows that by integrating $\frac{d\tilde{\mathbf{E}}_\omega}{dt}$ along paths, we can get a modified K energy $\tilde{\mathbf{E}}_\omega$ which may depend on a given Kähler metric as $\mathbf{E}_\omega$ does.

By a similar calculation, for Kähler metrics which are invariant under Im(Y), we obtain

$$\frac{\partial^2 \tilde{\mathbf{E}}_\omega}{\partial z\, \partial \bar{z}} = \frac{\partial^2\, \mathbf{E}_\omega}{\partial z\, \partial \bar{z}} + \int_{\mathrm{M}} \left(\frac{\partial^2 \phi}{\partial z \partial \bar{z}} - \frac{1}{2} \left| \nabla \frac{\partial \phi}{\partial \bar{z}} \right|^2_\phi \right) \theta(\phi(\cdot, z)) \omega^n_\phi.$$

Note that θ_{Y} is real for Kähler metric which is invariant under action of Im(Y). For an almost smooth solution to HCMA equation (1.1), the second term vanishes completely. Technically, if ω_ϕ is uniformly bounded from above, then $\theta(\phi)$ is uniformly Lipschitz. This is a key technical step in generalizing Theorem 6.1.1 to the case of extremal Kähler metrics. Similarly, the same approximating proof will work in this case as well. Thus [23].

Corollary **6.1.4.** — *For any partially smooth solution* $\phi \in \overline{\mathcal{H}_\omega}$ *(cf. Definition 1.3.1), we have*

$$\int_{\mathrm{M}\times\Sigma\setminus\mathcal{S}_{\phi_0}} \left| \mathcal{D} \frac{\partial \phi}{\partial \bar{z}} \right|^2_{\omega_\phi} \omega^n_\phi d\tau d\bar{z} \leq \int_{\partial \Sigma} \frac{\partial \tilde{\mathbf{E}}_\omega}{\partial\, \mathbf{n}} (\phi) ds,$$

where the left hand side is evaluated at points where Kähler metric is smooth. Equality holds for any partially smooth solution. Moreover, $\tilde{\mathrm{E}}$ *is a bounded weakly sub-harmonic function on* Σ.

6.2. *The lower bound of the (modified) K energy*

In this subsection, we want to use Theorems 6.1.1 and Corollary 6.1.2 to establish a lower bound for the (modified) K energy. Here we first set up some notations.

For any two Kähler potentials $\phi_0, \phi_1 \in \mathcal{H}_\omega$, we want to use almost smooth solutions to approximate the $C^{1,1}$ geodesic between ϕ_0 and ϕ_1. For any integer l, consider the Dirichlet problem for HCMA equation (1.1) on the rectangle domain $\Sigma_l = [-l, l] \times [0, 1]$ with boundary value as

$$\phi(s, 0) = \phi_0, \quad \phi(s, 1) = \phi_1; \qquad (\mathbf{6.8})$$
$$\phi(\pm l, t) = (1-t)\phi_0 + (1-t)\phi_1, \quad (s, t) \in \Sigma_l.$$

We may modify this boundary map in the four corners so that the domain is smooth without corner. Here we use rectangle just for notation convenience. In fact, Σ_l is a long oval consists of two lines $(-l, l) \times \{0\}, (-l, l) \times \{1\}$ and two semi circles of

[23] The crucial point is that $\theta(\phi(z, \cdot))$ is uniformly Lipschitz.

radius $\frac{1}{2}$ to form a long oval. For the boundary map, we always assign it to be ϕ_0, ϕ_1 in the two long lines and smoothly interpolates between the two Kähler potentials in both semi-circles (cf. (6.8)). When we change l, we will not change the boundary map at two semi-circles at all. Denote the almost smooth solution by $\phi^{(l)} : \Sigma_l \to \mathscr{H}_\omega$ which corresponds to this boundary map[24]. Following the proof in [7] carefully, we can prove that the $C^{1,1}$ bound of the sequence of functional $\phi^{(l)}$ in $\Sigma_l \times M$ is uniform (independent of l). It follows immediately from the decomposition formulas (6.1)–(6.3) that the K energy functional evaluated at this family of Kähler potentials is uniformly bounded. Set

$$\mathbf{E}_\omega^{(l)}(s,t) = \mathbf{E}_\omega(\phi^{(l)}(s,t)), \quad \forall\,(s,t) \in \Sigma_l. \tag{6.9}$$

Then, $\mathbf{E}_\omega^{(l)}$ is a sequence of weakly sub-harmonic function with uniform bound such that

$$\mathbf{E}_\omega^{(l)}(s,0) = \mathbf{E}_\omega(\phi_0) = A, \quad \text{and} \quad \mathbf{E}_\omega^{(l)}(s,1) = \mathbf{E}_\omega(\phi_1) = B. \tag{6.10}$$

Now we are ready to prove (cf. Theorem 1.1.2)

Theorem **6.2.1.** — *For Kähler metrics which are invariant under imaginary part of extremal vector field, the modified K energy is uniformly bounded from below when the Kähler class admit an extremal metric. In particular, the minimum is achieved at an extremal Kähler metric when* $\nabla^{(1,0)}S$ *coincides with the extremal vector field (used to modify the K energy). Furthermore, if the underlying Kähler class is rational, then the Kähler class is K-semistable if it admits a constant scalar curvature metric.*

Proof. — We give a detailed proof in the case of constant scalar curvature Kähler metrics. The proof in the case general extremal Kähler metrics is similar and we leave it to interested readers. Suppose that ϕ_0 is a constant scalar curvature metric. Then $\frac{\partial \mathbf{E}_\omega^{(l)}}{\partial t} = \frac{\partial \mathbf{E}_\omega^{(l)}}{\partial s} = 0$ when $t = 0$. Our theorem is reduced to the following claim:

Claim. — $B = \mathbf{E}_\omega(\phi_1) \geq \mathbf{E}_\omega(\phi_0) = A.$

Let $\kappa : (-\infty, \infty) \to \mathbf{R}$ be a smooth non-negative function such that $\kappa \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and vanishes outside of $[-\frac{3}{4}, \frac{3}{4}]$. Set

$$\kappa^{(l)}(s) = \frac{1}{v}\kappa\left(\frac{s}{l}\right), \quad \text{where } v = \int_{-\infty}^{\infty} \kappa(s)ds.$$

Set

$$f^{(l)}(t) = \int_{-\infty}^{\infty} \kappa^{(l)}(s)\mathbf{E}_\omega^{(l)}(s,t)ds.$$

[24] We may need to alter the boundary value slightly.

Then

$$
\begin{aligned}
f^{(l)}(0) &= \int_{-\infty}^{\infty} \kappa^{(l)}(s)\mathbf{E}_{\omega}^{(l)}(s,0)ds = \int_{-\infty}^{\infty} \kappa^{(l)}(s)\mathrm{A}ds = \mathrm{A},\\
f^{(l)}(1) &= \int_{-\infty}^{\infty} \kappa^{(l)}(s)\mathbf{E}_{\omega}^{(l)}(s,1)ds = \int_{-\infty}^{\infty} \kappa^{(l)}(s)\mathrm{B}ds = \mathrm{B},\\
\left.\frac{df^{(l)}}{dt}\right|_{t=0} &= 0.
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbf{E}_{\omega}(\phi_1) - \mathbf{E}_{\omega}(\phi_0) &= \mathrm{B} - \mathrm{A} = f^{(l)}(t)|_0^1\\
&= \int_0^1 \int_0^{\theta} \frac{d^2 f^{(l)}}{dt^2} dtd\theta\\
&= \int_0^1 \int_0^{\theta} \int_{-\infty}^{\infty} \kappa^{(l)}(s) \frac{\partial^2 \mathbf{E}_{\omega}^{(l)}}{\partial t^2} dsdtd\theta\\
&= \int_0^1 \int_0^{\theta} \int_{-\infty}^{\infty} \kappa^{(l)}(s) \Delta_{s,t}\mathbf{E}_{\omega}^{(l)} dsdtd\theta\\
&\quad - \int_0^1 \int_0^{\theta} \int_{-\infty}^{\infty} \kappa^{(l)}(s) \frac{\partial^2 \mathbf{E}_{\omega}^{(l)}}{\partial s^2} dsdtd\theta\\
&\geq - \int_0^1 \int_0^{\theta} \int_{-\infty}^{\infty} \kappa^{(l)}(s) \frac{\partial^2 \mathbf{E}_{\omega}^{(l)}}{\partial s^2} dsdtd\theta\\
&= - \int_0^1 \int_0^{\theta} \int_{-\infty}^{\infty} \frac{d^2 \kappa^{(l)}(s)}{ds^2} \mathbf{E}_{\omega}^{(l)}(s,t) dsdtd\theta\\
&= - \frac{1}{l^2}\frac{1}{v} \int_0^1 \int_0^{\theta} \int_{-\infty}^{\infty} \left.\frac{d^2\kappa}{ds^2}\right|_{\frac{s}{l}} \mathbf{E}_{\omega}^{(l)}(s,t) dsdtd\theta.
\end{aligned}
$$

Note that $|\mathbf{E}_{\omega}^{(l)}(s,t)|$ has a uniform bound C. Thus,

$$
\begin{aligned}
\frac{1}{l^2}\frac{1}{v}\left| \int_{-\infty}^{\infty} \left.\frac{d^2\kappa}{ds^2}\right|_{\frac{s}{l}} \mathbf{E}_{\omega}^{(l)}(s,t)ds \right| &\leq \frac{1}{l^2}\frac{1}{v} \int_{-\infty}^{\infty} \left|\frac{d^2\kappa}{ds^2}\right|_{\frac{s}{l}} \mathbf{E}_{\omega}^{(l)}(s,t)ds\\
&\leq \frac{\mathrm{C}}{vl^2} \int_{-\infty}^{\infty} \left|\frac{d^2\kappa}{ds^2}\right|_{\frac{s}{l}} ds\\
&= \frac{\mathrm{C}}{vl} \int_{-\infty}^{\infty} \left|\frac{d^2\kappa}{ds^2}\right|_{s} ds\\
&\leq \frac{\mathrm{C}}{l}
\end{aligned}
$$

for some uniform constant C. Therefore, we have

$$\begin{aligned}\mathbf{E}_\omega(\phi_1) - \mathbf{E}_\omega(\phi_0) &\geq -\frac{1}{l^2}\frac{1}{v}\int_0^1\int_0^\theta\int_{-\infty}^{\infty}\frac{d^2\kappa}{ds^2}\Big|_{\frac{s}{l}}\mathbf{E}_\omega^{(l)}(s,t)\,dsdtd\theta \\ &\geq -\int_0^1\int_0^\theta\frac{1}{l^2}\frac{1}{v}\left|\int_{-\infty}^{\infty}\frac{d^2\kappa}{ds^2}\Big|_{\frac{s}{l}}\mathbf{E}_\omega^{(l)}(s,t)\,dsdtd\theta\right. \\ &\geq -\int_0^1\int_0^\theta\frac{\mathrm{C}}{l}\,dtd\theta = -\frac{\mathrm{C}}{2l}.\end{aligned}$$

As $l \to \infty$, we have

$$\mathbf{E}_\omega(\phi_1) \geq \mathbf{E}_\omega(\phi_0).$$

Since ϕ_1 is an arbitrary Kähler potential, the lower bound part is then proved.

To derive K or CM semi-stability from the existence of cscK metric in polarized algebraic class, we take k sufficiently large so that any basis of $\mathrm{H}^0(\mathrm{M}, \mathrm{L}^k)$ embeds M into some projective space $\mathbf{C}\mathrm{P}^{\mathrm{N}}$ and consider asymptotic behavior of the K-energy along one-parameter subgroups of $\mathrm{SL}(\mathrm{N}+1, \mathbf{C})$ which acts on $\mathbf{C}\mathrm{P}^{\mathrm{N}}$ by automorphisms. Here we adopt notations in Corollary 1.1.3. Let $\sigma(t)$ ($t \in \mathbf{C}^*$) be an one-parameter algebraic subgroup of $\mathrm{SL}(\mathrm{N}+1, \mathbf{C})$ and ω_{FS} be the Fubini–Study metric on $\mathbf{C}\mathrm{P}^{\mathrm{N}}$. Then $\frac{1}{k}\sigma(t)^*\omega_{\mathrm{FS}}$ restricts to a family of Kähler metrics $\omega_{\sigma(t)}$ on M with Kähler class equal to $c_1(\mathrm{L})$. Since we assume that there is a cscK metric in the Kähler class $c_1(\mathrm{L})$, $\mathbf{E}_\omega(\phi_{\sigma(t)})$ is uniformly bounded from below, where $\phi_{\sigma(t)}$ denotes the Kähler potential of $\omega_{\sigma(t)}$ with respect to ω. It follows from [24] that if $\mathbf{w}(\sigma)$ denotes either the K-stability or the CM-stability weigh of this one-parameter subgroup $\{\sigma(t)\}$, then as $t \to \infty$, we have

$$\mathbf{w}(\sigma)\, t \geq c(n)\mathbf{E}_\omega(\phi_{\sigma(t)}) - \mathrm{C},$$

where $c(n)$ is a universal constant and C is a constant which may depend on σ, but not on t. Therefore, $\mathbf{w}(\sigma) \geq 0$ and consequently, (M, L) is asymptotically K-semistable or CM-semistable in the sense of [27] (also see [29]). □

7. Partial regularity of the K energy minimizer

7.1. *Strong convergence lemma for volume form*

In this subsection, we want to prove

Theorem **7.1.1.** — *Suppose that* $\{\phi_m, m \in \mathbf{N}\}$ *is a sequence of Kähler potentials in* $\mathcal{H}_\omega$ *with uniform* $\mathrm{C}^{1,1}$ *bound and suppose that* $\phi_m \to \phi_0 \in \overline{\mathcal{H}_\omega}$ *strongly in* $\mathrm{C}^{1,\alpha}(\forall\, \alpha < 1)$ *and* *weakly in* $W^{2,p}$ *for* p *large enough. If the corresponding sequence of K energies* $\{\mathbf{E}_\omega(\phi_m), m \in \mathbf{N}\}$ *is a Cauchy sequence and*

$$\lim_{l\to\infty} \mathbf{E}_\omega(\phi_l) \le \mathbf{E}_\omega(\phi_0), \tag{7.1}$$

then $\frac{\omega^n_{\phi_m}}{\omega^n}$ *converges strongly to* $\frac{\omega^n_{\phi_0}}{\omega^n}$ *in* $L^2(M, \omega)$.

Proof. — Set $f_m = \frac{\omega^n_{\phi_m}}{\omega^n} \le C$ and $g = \frac{\omega^n_{\phi_0}}{\omega^n}$. Applying the decomposition formula of the K energy (6.1), we see that $\{\int_M f_m \log f_m, m \in \mathbf{N}\}$ is a Cauchy sequence.

Since $\{\phi_l, l \in \mathbf{N}\}$ converges to ϕ_0 weakly in the $W^{2,p}$ norm for p large enough, the lower order part of the K energy converges to the corresponding lower order part of the K energy of ϕ_0. Thus

$$\int_M f_l \log f_l \omega^n - \int_M g \log g = \mathbf{E}_\omega(\phi_l) - \mathbf{E}_\omega(\phi_0) + o\left(\frac{1}{l}\right) \le o\left(\frac{1}{l}\right). \tag{7.2}$$

Define $F(u) = u \log u$. For any l large enough and for any $\epsilon > 0$, set

$$F(t) = F(tf_l + (1-t)(g+\epsilon)) = F(at+b),$$

where

$$a = f_l - g - \epsilon, \quad \text{and} \quad b = g + \epsilon.$$

Note that a, b are both functions in M. Clearly, we have

$$|a| + |b| \le C.$$

Note that

$$F'(t) = a \log(at+b) + a,$$

and

$$F''(t) = \frac{a^2}{at+b} \ge \frac{a^2}{C}, \quad \forall\, t \in [0, 1].$$

Thus,

$$\begin{aligned}
\int_M F'(0)\omega^n &= \int_M (a \log b + a)\omega^n \\
&= \int_M (f_l - g - \epsilon) \log(g_m + \epsilon)\omega^n + \int_M (f_l - g - \epsilon)\omega^n \\
&= \int_M (f_l - g - \epsilon) \log(g + \epsilon)\omega^n + \int_M \omega^n_{\phi_l} - \int_M \omega^n_{\phi_0} - \epsilon\, vol(M) \\
&= \int_M (f_l - g - \epsilon) \log(g + \epsilon)\omega^n - o(\epsilon).
\end{aligned}$$

Taking the following double limits

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{l\to\infty}\int_M F'(0)\omega^n &= \lim_{\epsilon\to 0}\lim_{l\to\infty}\left(\int_M (f_l - g - \epsilon)\log(g+\epsilon)\omega^n - o(\epsilon)\right)\\ &= \lim_{\epsilon\to 0}\left(\int_M (g - g - \epsilon)\log(g+\epsilon)\omega^n - o(\epsilon)\right)\\ &= \lim_{\epsilon\to 0} o(\epsilon) = 0.\end{aligned}$$

The second equality used the fact that $f_l \rightharpoonup g$ weakly in $L^p(M,\omega)$ and the 3rd equality used the fact that $|g|$ is bounded. Thus,

$$\begin{aligned}F(1) - F(0) &= F(f_l) - F(g+\epsilon)\\ &= \int_0^1 F'(t)dt = F'(0) + \int_0^1\int_0^t F''(s)ds\,dt\\ &= F'(0) + \int_0^1\int_0^t \frac{a^2}{as+b}ds\,dt\\ &\geq F'(0) + \int_0^1\int_0^t \frac{a^2}{C}ds\,dt = F'(0) + \frac{a^2}{2C}.\end{aligned}$$

Integrating this over M, we have,

$$\begin{aligned}\int_M (f_l - g - \epsilon)^2\omega^n &= \int_M a^2\omega^n\\ &\leq 2C\int_M (F(1) - F(0))\omega^n - 2C\int_M F'(0)\omega^n\\ &= 2C\left(\int_M f_l\log f_l\omega^n - \int_M (g+\epsilon)\log(g+\epsilon)\omega^n\right)\\ &\quad - 2C\int_M F'(0)\omega^n.\end{aligned}$$

Using inequality (7.2), we have

$$\begin{aligned}&\int_M (f_l - g - \epsilon)^2\omega^n\\ &\leq C\left(\int_M g\log g\omega^n - \int_M (g+\epsilon)\log(g+\epsilon)\omega^n + o\left(\frac{1}{l}\right)\right)\\ &\quad - C\int_M F'(0)\omega^n\\ &\leq o\left(\epsilon + \frac{1}{l}\right) - C\int_M F'(0)\omega^n.\end{aligned}$$

Consequently, we have

$$\begin{aligned}
2\lim_{l\to 0}\int_M (f_l-g)^2\omega^n &= 2\lim_{\epsilon\to 0}\lim_{l\to 0}\int_M (f_l-g-\epsilon+\epsilon)^2\omega^n\\
&\leq \lim_{\epsilon\to 0}\lim_{l\to\infty}\int_M (f_l-g-\epsilon)^2\omega^n+\lim_{\epsilon\to 0}\lim_{l\to\infty}\int_M \epsilon^2\omega^n\\
&= \lim_{\epsilon\to 0}\lim_{l\to\infty}\int_M (f_l-g-\epsilon)^2\omega^n\\
&\leq \lim_{\epsilon\to 0}\lim_{l\to\infty} o\left(\frac{1}{l}+\epsilon\right)+C\lim_{\epsilon\to 0}\lim_{l\to\infty}\int_M F'(0)\omega^n\\
&= 0.
\end{aligned}$$

Therefore, $\frac{\omega_{\phi_l}^n}{\omega^n}$ converges strongly to $\frac{\omega_{\phi_0}^n}{\omega^n}$ in $L^2(M,\omega)$. □

7.2. *Special properties of the K energy minimizer*

We follow the setup in Subsection 6.2. Passing to a subsequence if necessary, there exists a $C^{1,1}$ map $\underline{\phi_0}:\Sigma_1\to\mathcal{H}_\omega$ such that

1. $\phi^{(l)}$ converges to $\underline{\phi_0}$ weakly in $W^{2,p}(\Sigma^1\times M)$ for p sufficiently large, with respect to a fixed Kähler metric $\pi_2^*\omega+|dz|^2$.
2. $\phi^{(l)}$ converges to $\underline{\phi_0}$ strongly in $C^{1,\alpha}$ for any $0<\alpha<1$.
3. $\underline{\phi_0}(s,0)=\phi_0$ and $\underline{\phi_0}(s,1)=\phi_1$.

The first key step in this subsection is to improve weak $L^p(p>1)$ convergence to a strong L^2 convergence for the volume ratio $\frac{\omega^n_{\phi^{(l)}}}{\omega^n}$.

Recall the notation (6.9)

$$\mathbf{E}_\omega^{(l)}(s,t)=\mathbf{E}_\omega(\phi^{(l)}(s,t)),\quad \forall\,(s,t)\in\Sigma_l.$$

As before, $E^{(l)}$ is a uniformly bounded, weakly sub-harmonic function in $\Sigma^{(l)}$ with boundary condition (6.10).

In this subsection, we assume that both ϕ_0 and ϕ_1 are Kähler metrics with constant scalar curvature in the fixed Kähler class. Then $\frac{\partial\mathbf{E}_\omega^{(l)}}{\partial t}=\frac{\partial\mathbf{E}_\omega^{(l)}}{\partial s}=0$ when $t=0,1$. Theorem 6.2.1 implies the following:

$$A=B=\inf_{\phi\in\mathcal{H}_\omega}\mathbf{E}_\omega(\phi)\tag{7.3}$$

and

$$\mathbf{E}_\omega^{(l)}(s,t)\geq A,\quad\forall\,(s,t)\in\Sigma_l.\tag{7.4}$$

Lemma **7.2.1.** — *As $l \to \infty$, the* L^1 *measure of the Laplacian* $\Delta_{s,t}\mathbf{E}_\omega^{(l)}$ *tends to* 0 *in any fixed compact sub-domain.*

When there is no confusion, we will drop the superscript (l).

Proof. — Let $\xi : (-\infty, \infty) \to \mathbf{R}$ be a smooth non-negative cut-off function such that $\xi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and vanishes outside $[-\frac{3}{4}, \frac{3}{4}]$.

$$
\begin{aligned}
&\int_{t=0}^{1}\int_{s=-\frac{l}{2}}^{\frac{l}{2}} |\Delta_{s,t}\mathbf{E}_\omega(s,t)|dsdt \\
&\qquad \le \int_{t=0}^{1}\int_{s=-l}^{l} \xi\left(\frac{s}{l}\right)\Delta_{s,t}\mathbf{E}_\omega(s,t)dsdt \\
&\qquad = \int_{s=-l}^{l} \frac{\partial \mathbf{E}_\omega(s,t)}{\partial t}\bigg|_0^1 \xi\left(\frac{s}{l}\right)ds - \frac{1}{l}\int_{t=0}^{1}\int_{s=-l}^{l}\xi'\left(\frac{s}{l}\right)\frac{\partial \mathbf{E}_\omega(s,t)}{\partial s}dsdt \\
&\qquad = 0 + \frac{1}{l^2}\int_{t=0}^{1}\int_{s=-l}^{l}\xi''\left(\frac{s}{l}\right)\mathbf{E}_\omega(s,t)dsdt \\
&\qquad \le \frac{1}{l^2}\int_{t=0}^{1}\int_{s=-l}^{l}\left|\xi''\left(\frac{s}{l}\right)\right| \cdot |\mathbf{E}_\omega(s,t)|dsdt \le \frac{1}{l^2}\int_{t=0}^{1}\int_{s=-l}^{l} C dsdt \\
&\qquad = \frac{C}{l} \to 0.
\end{aligned}
$$

□

Lemma **7.2.2.** — *For any point* (s, t) *in a fixed compact domain in* $\Sigma^{(l)}$*, except perhaps a set of measure* 0*, we have* $\lim_{l\to\infty} \mathbf{E}_\omega(s,t) = \lim_{l\to\infty} \mathbf{E}_\omega(\phi^{(l)}(s,t)) = A$.

Proof. — Set $f^{(l)} = \Delta_{s,t}\mathbf{E}_\omega^{(l)}(s,t) \ge 0$. In $\Sigma_{\frac{l}{2}} \subset \Sigma_l$, we have $\lim_{l\to\infty}\int_{\Sigma_{\frac{l}{2}}} f^{(l)} = 0$. Next, we decompose $\mathbf{E}_\omega^{(l)}$ into two parts:

$$\mathbf{E}_\omega^{(l)} = u^{(l)} + v^{(l)}, \quad \text{in} \quad \Sigma_{\frac{l}{2}}$$

such that

$$
\begin{cases}
\Delta_{s,t}u^{(l)} = 0, & \text{where } u^{(l)}|_{\partial\Sigma_{\frac{l}{2}}} = E^{(l)}, \\
\Delta_{s,t}v^{(l)} = f^{(l)} \ge 0, & \text{where } v^{(l)}|_{\partial\Sigma_{\frac{l}{2}}} = 0.
\end{cases}
$$

It is clear that $v^{(l)} \le 0$. Since $E^{(l)}$ is uniformly bounded, then $u^{(l)}$ is a bounded harmonic function in $\Sigma_{\frac{l}{2}}$ such that

$$u^{(l)}(s,0) = u^{(l)}(s,1) = A, \quad \forall s \in \left[-\frac{l}{2}, \frac{l}{2}\right].$$

Taking limit as $l \to \infty$, in any compact sub-domain Ω, we have $\lim_{l\to\infty} u^{(l)} = A$. Consequently,

$$\begin{aligned} A &\le \limsup_{l\to\infty} \mathbf{E}_\omega^{(l)} \\ &= \limsup_{l\to\infty}(u^{(l)} + v^{(l)}) \\ &\le \lim_{l\to\infty} u^{(l)} = A. \end{aligned}$$

Therefore, for every point in Ω (fixed), we have

$$\lim_{l\to\infty} \mathbf{E}_\omega^{(l)} = \limsup_{l\to\infty} \mathbf{E}_\omega^{(l)} = A.$$

□

Combining this with Theorem 7.1.1, we have

Corollary **7.2.3.** — *For any* $(s, t) \in \Sigma_1$, *except perhaps a set of measure* 0 *in* Σ_1, *the volume ratio* $\frac{\omega^n_{\phi^{(l)}}}{\omega^n}$ *converges strongly in* $L^2(M, \omega)$ *sense.*

This corollary is crucial in the following arguments.

For notational simplicity, set $\phi_l = \phi^{(l)} \in \mathcal{H}_\omega$. Let $\zeta(s) = \frac{1}{1+s^2}$. Then

$$\int_{z\in\Sigma_l}\int_M \omega^n \zeta(z) dz \wedge d\bar{z} \le C, \quad \forall\, l \in (1, \infty).$$

Lemma 7.2.1 and Theorem 6.1.1 imply

$$\int_{\Sigma_{\frac{l}{2}}}\int_M \left|\frac{\partial \eta^{l,\alpha}}{\partial w_{\bar\beta}}\right|^2_{\phi_l} \omega^n_{\phi_l} dz \wedge d\bar{z} \le \int_{z\in\Sigma_{\frac{l}{2}}} \Delta_z E(z, \cdot) dz \wedge d\bar{z} \to 0$$

where

$$\eta^{l,\alpha} = -g^{\alpha\bar\beta}_{\phi_l}\frac{\partial^2\phi_l}{\partial z \partial w_{\bar\beta}}, \quad \text{and} \quad g_{\phi_l,\alpha\bar\beta} = g_{\alpha\bar\beta} + \frac{\partial^2\phi_l}{\partial w_\alpha \partial w_{\bar\beta}}.$$

When there are no confusions which may arise, we will drop the dependence on l.

Now we adopt a symplectic point of view now: For any $l > 1$, the product manifold $\Sigma_l \times M$ is foliated by smooth holomorphic discs which are transversal to M direction, dictated by the structure of almost smooth solutions ϕ_l of HCMA equation (1.1). In other words, for any $z_0 = t_0 + \sqrt{-1}s_0 \in \Sigma_1$ fixed, we may use $\{z_0\} \times M$ as the parametrization space of the set of holomorphic discs which are transversal to $\{z_0\} \times M$. Note that this parametrization is effective except a set of codimension 2.

Along each holomorphic disc, the (n, n) form ω_ϕ^n is invariant. The above two inequalities can be re-stated as:

$$\int_M \int_{\Sigma_l} \frac{\omega^n}{\omega_{\phi_l}^n} \zeta(z) dz \wedge d\bar{z} \wedge \omega_{\phi_l}^n \le C, \tag{7.5}$$

and

$$\int_M \int_{\Sigma_{\frac{l}{2}}} \left| \frac{\partial \eta^{l,\alpha}}{\partial w_{\bar{\beta}}} \right|_{\phi_l}^2 dz \wedge d\bar{z} \wedge \omega_{\phi_l}^n \to 0. \tag{7.6}$$

Choosing any point z_0 in the interior of Σ_1 such that $\frac{\omega^n_{\phi_l(z_0,\cdot)}}{\omega^n}$ converges strongly to $\frac{\omega^n_{\phi_0(z_0,\cdot)}}{\omega^n}$ in $L^2(M, \omega)$. For any L^2 function h in M, we can normalize it by the following

$$h(x) = \begin{cases} \lim_{\epsilon \to 0} \oint_{B_\epsilon(x)} h, & \text{if limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $h(x)$ differs from the original function at most at a set of measure 0. Now, we decompose $\{z_0\} \times M$ into the union of two subsets F_1 and F_2 such that

$$\frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, x) > 0, \quad \forall\, x \in F_1$$

and

$$\frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, x) = 0, \quad \forall\, x \in F_2.$$

Clearly, $mes(F_1) > 0$ since

$$\begin{aligned} mes(F_1) &= \int_{F_1} \omega^n \\ &\ge c \int_{F_1} \frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, x)\omega^n \\ &= c \int_M \frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, x)\omega^n = c\, vol(M) > 0, \end{aligned}$$

where $\frac{1}{c}$ is the upper bound of the volume form ratio $\frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, \cdot)$ in M.

Ultimately, we want to show that $mes(F_2) = 0$. This is not attainable at this point. However, we can prove the following strong statement about the volume form ratio in the limit.

Theorem **7.2.4.** — *There exists a uniform constant ϵ_0, which depends only on $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$ (in particular, independent of x), such that, excluding at most a set of measure 0 from F_1, we have*

$$\frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, x) > \epsilon_0, \quad \forall\, x \in F_1.$$

Proof. — By our choice of z_0, $\frac{{\omega_{\phi_l}}^n}{\omega^n}(z_0, x) \to \frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, x)$ strongly in $L^2(M, \omega)$. For any $\delta > 0$, there exists an open set E_δ with measure $E_\delta < \frac{\delta}{2}$ such that $\frac{{\omega_{\phi_l}}^n}{\omega^n}(z_0, x) \to \frac{{\omega_{\phi_0}}^n}{\omega^n}(z_0, x)$ pointwisely in $\{z_0\} \times (M \setminus E_\delta)$. Set $\ell_l(z) : \{z_o\} \times M \to \{z\} \times M$ as the syndrome map such that (z_0, x) and $(z, \ell_l(z)(x))$ lies in the same holomorphic disc. Then $\ell_l(z)$ is well defined for generic $x \in M$. Now set

$$S_l(z, x) = \left|\frac{\partial \eta^{l,\alpha}}{\partial w_{\bar\beta}}\right|^2_{\phi_l}(z, \ell_l(z)(x)) \quad \text{and} \quad f_l(z, x) = \frac{\omega^n}{\omega^n_{\phi_l}}(z, \ell_l(z)(x))\zeta(z).$$

Denote

$$S_l(x) = \int_{\Sigma_{\frac{l}{2}}} S_l(z, x) dz \wedge d\bar z \quad \text{and} \quad f_l(x) = \int_\Omega f_l(z, x) dz \wedge d\bar z.$$

Then, (7.5) and (7.6) imply

$$\int_M S_l(x) \cdot \frac{\omega^n_{\phi_l}}{\omega^n}\omega^n \to 0, \quad \text{and} \quad \int_M f_l(x) \cdot \frac{\omega^n_{\phi_l}}{\omega^n}\omega^n \le C.$$

The first assertion implies that $\sqrt{S_l(x)} \cdot \sqrt{\frac{\omega^n_{\phi_l}}{\omega^n}} \to 0$ strongly in $L^2(M)$. Consequently, $\{S_l(x)\}$ uniformly converges to 0 in $(M \setminus E_\delta) \cap F_1$. On the other hand, there exists a set E'_δ of measure at most $\frac{\delta}{2}$ such that

$$\liminf_{l\to\infty} \left(f_l(x) \cdot \frac{\omega^n_{\phi_l}}{\omega^n} \right) < C(\delta), \quad \text{whenever } x \in M \setminus E'_\delta.$$

Let $F_\delta = F_1 \setminus (E_\delta \cup E'_\delta)$. Then

$$mes(F_\delta) \ge mes(F) - \delta.$$

We proceed to prove that our theorem holds in F_δ.

Let us pick any point $x_0 \in F_\delta$ and fix it for now. Passing to a subsequence if necessary, with loss of generality, we may assume that

$$\frac{\omega^n_{\phi_l}}{\omega^n}(z_0, x_0) = \frac{\omega^n_{\phi_0}}{\omega^n}(z_0, x_0) = \epsilon.$$

Here $\epsilon > 0$ may be very small. The goal is to show that a uniform positive lower bound of the volume form ratio exists. Clearly, $S_l(x_0) \to 0$ for any fixed compact subset[25] Ω. Next,

$$\liminf_{l\to\infty}\left(f_l(x_0)\cdot\frac{\omega^n_{\phi_l}}{\omega^n}(z_0,x_0)\right) < C(\delta).$$

Passing to a subsequence if necessary, we have

$$f_l(x_0) \le C(\delta,\epsilon,x_0),$$

in this subsequence. The main point is that it has a uniform bound in terms of this subsequence.

For any l, consider any holomorphic disc which passes through (z_0, x_0) and denote it by

$$\ell_l : (\Sigma_{(l)}, \partial\Sigma) \to (\Sigma_{(l)} \times M, (\partial\Sigma_{(l)}) \times M)$$

such that $l(z_0) = x_0$. It is easy to see that this holomorphic disc has uniformly bounded area in any fixed compact sub-domain (cf. Section 4). Let

$$g_l(z) = \log\frac{\omega^n_{\phi_l}}{\omega^n}(z,\ell_l(z)), \quad \forall\, z \in \Sigma_{(l)}.$$

By definition, g_l is uniformly bounded in the boundary $\partial\Sigma_l$. Then

$$g_l(z_0) = \log\frac{\omega^n_{\phi_l}}{\omega^n}(z_0,\ell_l(z_0)) = \log\frac{\omega^n_{\phi_l}}{\omega^n}(z_0,x_0) = \ln\epsilon, \quad \forall\, l.$$

Moreover, there exists a uniform constant C such that $|g_l(z)| \le C$ in the boundary $\partial\Sigma^{(l)}$. Since $\ell_l(\Sigma)$ is a measure 0 set in $\Sigma \times M$, the limit of $g_l(z)(z \neq 0)$ most likely have no bearing on $\log\frac{\omega^n_{\phi_0}}{\omega^n}$ in $\Sigma \times M$. However, we are interested in obtaining a uniform positive lower bound on ϵ through this procedure.

Recall that Corollary 4.9 implies

$$\Delta_z g_l(z) = S(z,\ell_l(z)) + R_{0,\alpha\bar\beta}\eta^{l,\alpha}\eta^{l,\bar\beta}|_{(z,\ell_l(z))}.$$

Split $g_l = u_l + v_l$ such that

$$\Delta_z v_l = R_{0,\alpha\bar\beta}\eta^{l,\alpha}\eta^{l,\bar\beta}|_{(z,\ell_l(z))}$$

and

$$v_l(z) = g_l(z), \quad \forall\, z \in \partial\Sigma.$$

---

[25] As a matter of fact, we may choose $\Omega = [-\frac{l}{2}, \frac{l}{2}]$ and our claim still holds.

Claim. — There exists a uniform constant C such that $|v_l| < C$.

Recall that Corollary 4.2.4 implies that

$$-\Delta_z \phi_l(z, \ell_l(z)) = g_{0,\alpha\bar\beta}\eta^{l,\alpha}\eta^{l,\bar\beta}, \quad \forall\, z \in \Sigma_l.$$

Note that $\phi_l(z, \ell_l(z))$ has a uniform bound in Σ_l. There exists a constant C such that

$$-Cg_{0,\alpha\bar\beta} < R_{0,\alpha\bar\beta} < Cg_{0,\alpha\bar\beta}.$$

Consequently,

$$\Delta_z(v_l - C\phi_l) < 0 < \Delta_z(v_l + C\phi_l).$$

By maximum principle, we have $|v_l| \leq C$ and our earlier claim holds. Next,

$$\Delta_z u_l = S_l(z, \ell(z)) > 0,$$

and $u_l|_{\partial\Sigma_l} = 0$. Obviously $u_l \leq 0$ (maximum principle). Moreover

$$\begin{aligned}\int_{\Sigma_l} e^{-u_l}(z)\cdot\zeta(z)dz\wedge d\bar z &\leq C\int_{\Sigma_l} e^{-\log\frac{\omega_{\phi_l}^n}{\omega^n}}(z)\cdot\zeta(z)dz\wedge d\bar z\\ &= \int_{\Sigma_l}\left(\frac{\omega^n}{\omega_{\phi_l}^n}\right)(z, \ell_l(z))\cdot\zeta(z)dz\wedge d\bar z\\ &= \int_{\Sigma_l} f_l(z, \ell_l(z))\cdot\zeta(z)dz\wedge d\bar z\\ &= f_l(x_0) \leq C(z_0, \delta, \epsilon).\end{aligned}$$

For any small positive number $\delta_1 > 0$, Theorem 4.2.13 implies

$$0 \leq \Delta_z u_l = S(z, \ell_l(z)) < \frac{C}{\delta_1^2}, \quad \forall\, z \in [-l + \delta_1, l - \delta_1] \times [\delta_1, 1 - \delta_1].$$

These two conditions imply that u_l converges strongly in $W^{1,\alpha}$ in any fixed compact sub-domain of $(-l + \delta_1, l - \delta_1) \times (\delta_1, 1 - \delta_1)$. Moreover,

$$\int_{\Sigma_{\frac{l}{2}}} |\Delta_z u_l| = \int_{\Sigma_{\frac{l}{2}}} S(z, \ell(z)) = S_l(x_0) \to 0.$$

Passing to a subsequence if necessary, there exists a non-negative harmonic function u_∞ in $(-\infty, \infty) \times [0, 1]$ such that for any fixed compact subset Ω, we have

$$u_l \rightharpoonup u_\infty$$

weakly in $L^p(\Omega)$ for any $p > 1$; and it converges strongly to u_∞ in $C^{1,\alpha}(0 < \alpha < 1)$ in any fixed compact sub-domain. Moreover, $u_\infty = 0$ in the boundary of this infinite long strip $\Sigma_\infty = (-\infty, \infty) \times [0, 1]$ and

$$\int_{\Sigma_\infty} e^{-u_l}(z) \cdot \zeta(z) dz \wedge d\bar{z} < C(\delta, \epsilon, x_0).$$

The only harmonic function with this growth condition in Σ_∞ is a constant function. Thus, $u_\infty \equiv 0$. In particular, for any fixed sub-domain $\Omega \subset (-l + \delta_1, l - \delta_1) \times (\delta_1, 1 - \delta_1)$, we have $u_l \to 0$ strongly. In particular,

$$\lim_{l \to \infty} u_l(z_0) = 0.$$

It follows

$$\begin{aligned} \log \epsilon &= g_l(z_0) \\ &= u_l(z_0) + v_l(z_0) > -C. \end{aligned}$$

Consequently, we have

$$\frac{\omega_{\phi_0}^n}{\omega^n}(z_0, x_0) > e^{-C} = \epsilon_0.$$

Note that we chose x_0 arbitrarily in F_δ, thus the lower bound $e^{-C} = \epsilon_0$ holds for every point in F_δ. Since $mes(F \setminus F_\delta) < \delta$, our theorem is proved. □

7.3. *A regularity theorem for a* $C^{1,1}$ *minimizer of the K energy functional and the weak "Kähler–Ricci" flow*

In this subsection, we want to prove a regularity lemma for any $C^{1,1}$ minimizer of the K energy in an arbitrary Kähler class.

Theorem **7.3.1.** — *Suppose* $\underline{\phi_0} \in C^{1,1}(M)$ *is in the closure of* $\mathcal{H}_\omega$ *under weak* $C^{1,1}$ *topology. If the K energy functional has a uniform lower bound in this Kähler class and* $E(\underline{\phi_0})$ *realizes the infimum of the K energy functional in this Kähler class, then* $\left(\frac{\omega_{\underline{\phi_0}}^n}{\omega^n}\right)^{\frac{1}{2}}$ *is in* $W^{1,2}(M, \omega)$.

Proof. — We will prove this by using the "Kähler–Ricci" flow. Let $\phi_0(s)$ $(0 \leq s \leq 1)$ be an one-parameter family of Kähler potentials such that the followings hold

1. $\phi_0(0) = \underline{\phi_0}$ and $\phi_0(s)$ $(0 < s \leq 1) \in \mathcal{H}_\omega$.
2. $\phi_0(s)$ has uniform $C^{1,1}$ upper bound and $\phi_0(s)(s > 0) \to \underline{\phi_0}$ strongly in $W^{2,p}(M, \omega)$ for p large enough.

We apply the "Kähler–Ricci flow" to this one-parameter family of Kähler potentials $\phi_0(s)$ $(0 < s \leq 1)$:

$$\frac{\partial\phi(s,t)}{\partial t} = \log\frac{\omega_\phi^n}{\omega^n} \tag{7.7}$$

$$\phi(s,0) = \phi_0(s). \tag{7.8}$$

Clearly, for each $s > 0$ and some fixed $T > 0$, there exists a uniform $C^{2,\alpha}$ bound for $\phi(s,t)$ $(s > 0, 0 \leq t \leq T)$. However, the upper-bound may depend on s and in particular, it may blow up when t, s are both approaching 0.

Claim **1.** — There exists a uniform constant C which is independent of the parameters s and t such that

$$n + \Delta_0\phi \leq C, \quad \forall\,(s,t) \in (0,1] \times [0,T].$$

Here we used Δ_ϕ, Δ_0 to denote the Laplacian operators of the Kähler metrics ω_ϕ, ω respectively.

In the following proof, we will use C to denote a generic constant which is independent of s, t. Taking the time derivative of the flow equation (7.7), we obtain

$$\frac{\partial^2\phi}{\partial t^2} = \Delta_\phi\left(\frac{\partial\phi(s,t)}{\partial t}\right).$$

This implies that

$$\begin{aligned}\frac{\omega^n_{\phi(s,t)}}{\omega^n} &= e^{\frac{\partial\phi(s,t)}{\partial t}} \leq \max_{x\in M} e^{\frac{\partial\phi(s,t,x)}{\partial t}}\\ &\leq \max_{x\in M} e^{\frac{\partial\phi(s,t,x)}{\partial t}}\Big|_{t=0} = \max_{x\in M}\frac{\omega^n_{\phi(s,0,x)}}{\omega^n}\\ &= \max_{x\in M}\frac{\omega^n_{\phi_0(s,x)}}{\omega^n} \leq C.\end{aligned}$$

In other words, we have a uniform upper-bound on the evolved volume form.

Following the calculation in [32], it is straightforward to show (for each fixed $s > 0$)

$$\begin{aligned}&\left(\Delta_\phi - \frac{\partial}{\partial t}\right)(exp(-\lambda\phi)(n + \Delta_0\phi))\\ &\quad\geq -exp(-\lambda\phi)\big(n^2\inf_{i\neq 1}(R_{i\bar{i}1\bar{1}})\big) - \lambda\, exp(-\lambda\phi)\left(n - \log\frac{\omega^n_\phi}{\omega^n}\right)(n + \Delta_0\phi)\\ &\qquad + (\lambda + \inf_{i\neq 1} R_{i\bar{i}1\bar{1}})\, exp(-\lambda\phi)\cdot\left(\frac{\omega^n}{\omega^n_\phi}\right)^{\frac{1}{n}}(n + \Delta\phi)^{\frac{n}{n-1}},\end{aligned}$$

where $R_{i\bar{i}1\bar{1}}$ is the bisectional curvature of the Kähler metric (corresponding to ω) and λ is a positive number such that

$$\lambda + \inf_{i\neq 1} R_{i\bar{i}1\bar{1}} > 1.$$

Multiplying $\left(\frac{\omega_\phi^n}{\omega^n}\right)^{\frac{1}{n}}$ on both sides, we get

$$\begin{aligned}
&\left(\frac{\omega_\phi^n}{\omega^n}\right)^{\frac{1}{n}}\left(\Delta_\phi - \frac{\partial}{\partial t}\right)(exp(-\lambda\phi)(n+\Delta_0\phi)) \\
&\geq -\,exp(-\lambda u)\Big(n^2 \inf_{i\neq 1}(R_{i\bar{i}1\bar{1}})\Big)\left(\frac{\omega_\phi^n}{\omega^n}\right)^{\frac{1}{n}} \\
&\quad - \lambda\, exp(-\lambda\phi)\left(n - \log\frac{\omega_\phi^n}{\omega^n}\right)\left(\frac{\omega_\phi^n}{\omega^n}\right)^{\frac{1}{n}}(n+\Delta_0\phi) \\
&\quad + (C_0 + \inf_{i\neq 1}(R_{i\bar{i}1\bar{1}}))\, exp(-C_0\phi)\cdot(n+\Delta\phi)^{\frac{n}{n-1}}.
\end{aligned}$$

If ϕ is uniformly bounded (independent of s, t), then (recall that $\left(\frac{\omega_\phi^n}{\omega^n}\right)^{\frac{1}{n}} < C$)

$$\left(\frac{\omega_\phi^n}{\omega^n}\right)^{\frac{1}{n}}\left(\Delta_\phi - \frac{\partial}{\partial t}\right)v \geq -c_1 - c_2 v + c_0 v^{\frac{n}{n-1}},$$

where c_0, c_1, c_2 are uniform positive constants and $v = exp(-\lambda\phi)(n+\Delta_0\phi)$. Therefore,

$$v(s,t) \leq v(s,0) \leq C.$$

In other words, there is a uniform constant C such that

$$0 \leq n + \Delta_0\phi(s,t) \leq C,$$

provided that $\phi(s,t)$ has a uniform C^0 bound. It is easy to see that we have uniform upper bound on $\frac{\partial\phi}{\partial t}$. On the other hand, at minimum point of ϕ, we have

$$\log\frac{\omega_\phi^n}{\omega^n} \geq 0.$$

Thus, in the barrier sense, we have

$$\frac{\partial \min_M \phi}{\partial t} \geq 0.$$

Consequently,

$$|\phi(s, t)| \leq C.$$

This concludes the proof of our first claim above.

For any sequences $s_i, t_i \to 0$, set

$$\phi_i = \phi(s_i, t_i), \quad \text{and} \quad \phi_{0i} = \phi(s_i, 0) = \phi_0(s_i, 0).$$

Passing to a subsequence if necessary, we have that ϕ_i converges to some $C^{1,1}$ Kähler potential $\tilde{\phi}_0$ (strongly in $C^{1,\alpha}(\forall \alpha < 1)$ and weakly in $W^{2,p}$ (p large enough)). Note that $\tilde{\phi}_0$ does not necessarily equal to ϕ_0 even though $t_i, s_i \to 0$!

Claim **2.** — $\omega^n_{\tilde{\phi}_0} \equiv \omega^n_{\phi_0}$ and $\frac{\omega^n_{\phi_i}}{\omega^n}$ converge strongly to $\frac{\omega^n_{\tilde{\phi}_0}}{\omega^n}$ in $L^2(M, \omega)$.

To prove this claim, choose an arbitrary smooth non-negative cut off function χ (fixed) and compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_M \chi \left(\frac{\omega^n_\phi}{\omega^n}\right)^2 \omega^n &= \int_M \chi\, \Delta_\phi \log \frac{\omega^n_\phi}{\omega^n} \cdot \left(\frac{\omega^n_\phi}{\omega^n}\right) \omega^n_\phi \\
&= \int_M \chi \left(\Delta_\phi \frac{\omega^n_\phi}{\omega^n} - \left|\nabla \log \frac{\omega^n_\phi}{\omega^n}\right|^2_\phi \frac{\omega^n}{\omega^n_\phi}\right) \omega^n_\phi \\
&\leq \int_M \Delta_\phi \chi \left(\frac{\omega^n_\phi}{\omega^n}\right) \omega^n_\phi \leq C.
\end{aligned}$$

The last inequality holds since the evolving Kähler potentials have a uniform $C^{1,1}$ upper-bound and χ is a fixed smooth function. Integrating this inequality from $t = 0$ to $t = t_i$, we have

$$\begin{aligned}
\int_M \chi \left(\frac{\omega^n_{\phi_i}}{\omega^n}\right)^2 \omega^n \Big|_{s_i, t_i} &= \int_M \chi \left(\frac{\omega^n_{\phi_{0i}}}{\omega^n}\right)^2 \omega^n \Big|_{s_i, 0} + \int_0^{t_i} \frac{d}{dt}\int_M \chi \left(\frac{\omega^n_\phi}{\omega^n}\right)^2 \omega^n dt \\
&\leq \int_M \chi \left(\frac{\omega^n_{\phi_0(s_i)}}{\omega^n}\right)^2 \omega^n + C t_i.
\end{aligned}$$

On the other hand, $\frac{\omega^n_{\phi_i}}{\omega^n}$ converges weakly to $\frac{\omega^n_{\tilde{\phi}_0}}{\omega^n}$ in $L^2(M, \omega)$. Then

$$\int_M \chi \left(\frac{\omega^n_{\tilde{\phi}_0}}{\omega^n}\right)^2 \omega^n \leq \lim_{i \to \infty} \int_M \chi \left(\frac{\omega^n_{\phi_i}}{\omega^n}\right)^2 \omega^n \tag{7.9}$$

$$\leq \lim_{i \to \infty} \left(\int_M \chi \left(\frac{\omega^n_{\phi_{0i}}}{\omega^n}\right)^2 \omega^n + C t_i\right) \tag{7.10}$$

$$= \int_M \chi \left(\frac{\omega^n_{\phi_0}}{\omega^n}\right)^2 \omega^n. \tag{7.11}$$

The last equality holds since ϕ_{0_i} converges strongly to ϕ_0 in $W^{2,p}(M, \omega)$ for p large enough (by our assumption at the beginning). This holds for any non-negative smooth cut off function in M. Consequently, we have

$$0 \leq \frac{\omega^n_{\tilde{\phi}_0}}{\omega^n} \leq \frac{\omega^n_{\phi_0}}{\omega^n}$$

a.e. in M. However,

$$\int_M \frac{\omega^n_{\tilde{\phi}_0}}{\omega^n}\omega^n = \int_M \frac{\omega^n_{\phi_0}}{\omega^n}\omega^n = vol(M)!$$

Consequently,

$$\omega^n_{\tilde{\phi}_0} \equiv \omega^n_{\phi_0} \tag{7.12}$$

in the sense of $L^2(M, \omega)$.The uniqueness of $C^{1,1}$ solution to the Monge–Ampere equation implies that $\tilde{\phi}_0 = \phi_0$. In particular, this implies the K energy $\mathbf{E}_\omega(\phi_i)$ converges to $\mathbf{E}_\omega(\phi_0)$.

On the other hand, the equality (7.12) forces equality in (7.9)–(7.11) to hold. In particular, we have

$$\int_M \chi \left(\frac{\omega^n_{\tilde{\phi}_0}}{\omega^n}\right)^2 \omega^n = \lim_{i\to\infty} \int_M \chi \left(\frac{\omega^n_{\phi_i}}{\omega^n}\right)^2 \omega^n.$$

Thus $\frac{\omega^n_{\phi_i}}{\omega^n}$ converges strongly to $\frac{\omega^n_{\tilde{\phi}_0}}{\omega^n}$ in $L^2(M, \omega)$. Our second claim is then proved.

Now we use these two claims to prove our theorem. Consider $\mathbf{E}_\omega(s, t) = \mathbf{E}_\omega(\phi(s, t))$ $(0 < s \leq 1$ and $0 \leq t < \infty)$. Set

$$A = \inf_{\phi\in\mathcal{H}_\omega} \mathbf{E}_\omega(\phi) > -\infty.$$

By our assumption on ϕ_0, we have

$$\mathbf{E}_\omega(\phi_0) = A = \lim_{s\to 0} \mathbf{E}_\omega(\phi(s, 0)) \leq \mathbf{E}_\omega(s, t), \quad \forall\, s > 0, t \geq 0.$$

For any fixed number $c_0 > 0$, it is straightforward to show that there exist sequences $s_i, t_i \to 0$ such that

$$\left.\frac{\partial \mathbf{E}_\omega(s, t)}{\partial t}\right|_{s_i, t_i} \geq -c_0.$$

In fact, we can choose s_i, t_i as follows. Suppose that (s_i, t_i) is already chosen. If

$$\max_{s\leq \frac{s_i}{2}} \max_{t\leq \frac{t_i}{2}} \frac{\partial \mathbf{E}_\omega(s, t)}{\partial t} \leq -c_0,$$

then,

$$A \le E\left(\phi\left(s, \frac{t_i}{2}\right)\right) \le E(\phi(s,0)) - c_0 \frac{t_i}{2}.$$

Taking limit as $s \to 0$ in both sides, we have

$$A \le A - c_0 \frac{t_i}{2}.$$

This is contradiction! Therefore, there exists a (s_{i+1}, t_{i+1}) such that

$$\left.\frac{\partial \mathbf{E}_\omega(s,t)}{\partial t}\right|_{s_{i+1},t_{i+1}} \ge -c_0, \quad s_{i+1} \le \frac{s_i}{2}, \, t_{i+1} \le \frac{t_i}{2}.$$

We will use sequences s_i, t_i to derive an a priori estimate on the volume form.

$$\begin{aligned}
-c_0 &\le \left.\frac{\partial \mathbf{E}_\omega(s,t)}{\partial t}\right|_{s_i,t_i} \\
&= \int_M \log \frac{\omega_\phi^n}{\omega^n} \Delta_\phi \frac{\partial \phi}{\partial t} \omega_\phi^n \Big|_{s_i,t_i} - \int_M \frac{\partial \phi}{\partial t} (\mathrm{Ric}(\omega_0) - \mu \omega_\phi) \wedge \omega_\phi^{n-1} \Big|_{s_i,t_i} \\
&= -\int_M \left| \nabla \log \frac{\omega_\phi^n}{\omega^n} \right|_\phi^2 \omega_\phi^n \Big|_{s_i,t_i} \\
&\quad - \int_M \left(\log \frac{\omega_\phi^n}{\omega^n} - C \right) (\mathrm{Ric}(\omega_0) - \mu \omega_\phi) \wedge \omega_\phi^{n-1} \Big|_{s_i,t_i} \\
&\le -\int_M \left| \nabla \log \frac{\omega_\phi^n}{\omega^n} \right|_\phi^2 \omega_\phi^n \Big|_{s_i,t_i} - c \int_M \left(\log \frac{\omega_\phi^n}{\omega^n} - C \right) \omega \wedge \omega_\phi^{n-1} \Big|_{s_i,t_i}.
\end{aligned}$$

Here c, C are some uniform positive number such that

$$\log \frac{\omega_\phi^n}{\omega^n} < C + 1$$

and

$$\mathrm{Ric}(\omega_0) - \mu \omega_\phi \le c\omega.$$

Thus

$$\begin{aligned}
&\int_M \left| \nabla \log \frac{\omega_{\phi_i}^n}{\omega^n} \right|_{\phi_i}^2 \omega_{\phi_i}^n \\
&\le c_0 - c \int_M \left(\log \frac{\omega_{\phi_i}^n}{\omega^n} - C \right) (\omega + \sqrt{-1}\partial\bar\partial \phi_i - \sqrt{-1}\partial\bar\partial\phi_i) \wedge \omega_{\phi_i}^{n-1} \\
&\le c_0 + c \int_M \left(\log \frac{\omega_{\phi_i}^n}{\omega^n} - C \right) \sqrt{-1}\partial\bar\partial\phi_i \wedge \omega_{\phi_i}^{n-1} - c \int_M \left(\log \frac{\omega_{\phi_i}^n}{\omega^n} - C \right) \omega_{\phi_i}^n
\end{aligned}$$

$$\leq c_0 - c\int_M \sqrt{-1}\partial \log \frac{\omega_{\phi_i}^n}{\omega^n} \wedge \bar\partial\phi_i \wedge \omega_{\phi_i}^{n-1} - c\int_M \left(\log\frac{\omega_{\phi_i}^n}{\omega^n} - C\right)\frac{\omega_{\phi_i}^n}{\omega^n}\omega^n$$
$$\leq c_0 + c\left(\epsilon\int_M \left|\nabla \log\frac{\omega_{\phi_i}^n}{\omega^n}\right|_{\phi_i}^2 \omega_{\phi_i}^n + \frac{1}{\epsilon}\int_M \sqrt{-1}\partial\phi_i\wedge\bar\partial\phi_i\wedge\omega_{\phi_i}^{n-1}\right) + C$$
$$\leq C(\epsilon) + c\epsilon\int_M \left|\nabla\log\frac{\omega_{\phi_i}^n}{\omega^n}\right|_{\phi_i}^2 \omega_{\phi_i}^n.$$

Choose ϵ small enough so that $c\epsilon < \frac{1}{2}$. With this ϵ, we have

$$\int_M \left|\nabla\sqrt{\frac{\omega_{\phi_i}^n}{\omega^n}}\right|_\omega^2 \omega^n \leq C\int_M \left|\nabla\sqrt{\frac{\omega_{\phi_i}^n}{\omega^n}}\right|_{\phi_i}^2 \omega^n$$
$$= C\int_M \left|\nabla\log\frac{\omega_{\phi_i}^n}{\omega^n}\right|_{\phi_i}^2 \omega_{\phi_i}^n \leq C.$$

Letting $i \to \infty$, we see that

$$\int_M \left|\nabla\sqrt{\frac{\omega_{\tilde\phi_0}^n}{\omega^n}}\right|_\omega^2 \omega^n \leq C.$$

Since $\frac{\omega_{\tilde\phi_0}^n}{\omega^n} \equiv \frac{\omega_{\phi_0}^n}{\omega^n}$, we have

$$\int_M \left|\nabla\sqrt{\frac{\omega_{\phi_0}^n}{\omega^n}}\right|_\omega^2 \omega^n \leq C.$$

The theorem is then proved. □

8. The problem of uniqueness of extremal Kähler metrics

Following notations in Subsection 6.2. Suppose that ϕ_1 gives another constant scalar curvature Kähler metric. Then

$$\mathbf{E}_\omega(\phi_0) = \mathbf{E}_\omega(\phi_1) = A.$$

Since we have uniform estimates

$$|\partial\bar\partial\phi^{(l)}|_{\Sigma^{(l)}\times M} \leq C,$$

there exists a subsequence of $\varphi^{(l)}$ which converges to $\underline{\phi} \in \overline{\mathscr{H}_\omega}$ in the weak $C^{1,1}$-topology. Following Theorem 6.2.1, for any point $(s,t) \in \Sigma_1^0$, we have

$$\lim_{l\to\infty} \mathbf{E}_\omega(\phi^{(l)}(s,t)) = A = \inf_{\phi\in\mathscr{H}_\omega} \mathbf{E}_\omega(\phi).$$

In the discussion below, we fix an arbitrary interior point $(s, t) \in {\Sigma_1}^0$. Theorem 7.1.1 implies that ${\omega_{\phi^{(l)}}}^n(s, t, \cdot)$ converges strongly to $\omega^n_{\underline{\phi}}(s, t, \cdot)$. By Theorem 7.2.4, we have $\frac{\omega^n_{\underline{\phi}}(s,t,\cdot)}{\omega^n} > \epsilon_0$ as long as it is positive. The set of points where the volume ratio $\frac{\omega^n_{\underline{\phi}}(s,t,\cdot)}{\omega^n}$ vanishes must have measure 0. Otherwise, it contradicts to the fact that $\sqrt{\frac{\omega^n_{\underline{\phi}}(s,t,\cdot)}{\omega^n}}$ is in $W^{1,2}(M, \omega)$ (cf. Theorem 7.3.1). Thus,

$$\frac{\omega^n_{\underline{\phi}}(s, t, \cdot)}{\omega^n} > \epsilon_0$$

for all points in M except at most a set of measure 0. Normalizing this volume ratio in the L^2 sense, we obtain

$$\frac{\omega^n_{\underline{\phi}}(s, t, x)}{\omega^n} > \epsilon_0, \qquad \forall\, x \in M.$$

Since $\underline{\phi}$ has a uniform $C^{1,1}$ bound, this implies that

$$\omega_{\underline{\phi}}(s, t, \cdot) \geq c_0 \omega$$

for some positive constant c_0. In other words, the metric $g_{\underline{\phi}}$ is equivalent to g_0. For any locally supported test function ξ, we have[26]

$$\int_M \log \frac{\omega^n_{\underline{\phi}}}{\omega^n} \sqrt{-1}\partial\bar{\partial}\xi \wedge \omega^{n-1}_{\underline{\phi}} = \int_M \xi(\mathrm{Ric}(\omega_0) - \omega_{\underline{\phi}}) \wedge \omega^{n-1}_{\underline{\phi}}.$$

Write

$$\omega_{\underline{\phi}} = \left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \underline{\phi}}{\partial w_\alpha \partial w_{\bar{\beta}}} \right) dw^\alpha dw^{\bar{\beta}} = g_{\underline{\phi},\alpha\bar{\beta}} dw^\alpha dw^{\bar{\beta}},$$

and

$$f = \frac{\omega^n_{\underline{\phi}}}{\omega^n}.$$

Since $\log f$ is in $W^{1,2}$, we then have

$$-\int_M g^{\alpha\bar{\beta}}_{\underline{\phi}} \frac{\partial \log f}{\partial w_\alpha} \frac{\partial \xi}{\partial w^{\bar{\beta}}} f = -\int_M \xi(\mathrm{Ric}(\omega_0) - \omega_{\underline{\phi}}) \wedge \omega^{n-1}_{\underline{\phi}}$$

[26] In any fixed open set, $\omega_{\underline{\phi}}$ can be approximated by a sequence of smooth Kähler metrics which have a uniform positive lower bound. Thus one can do small deformations in arbitrary directions. Consequently, one can establish the Euler–Lagrange equation in the weak sense.

for any locally supported test function ξ. Hence $\log f$ satisfies the following 2nd order non-linear equation in the weak sense:

$$\frac{1}{f}\frac{\partial}{\partial w_{\bar\beta}}\left(g^{\underline{\phi},\alpha\bar\beta} f \frac{\partial}{\partial w_\alpha}\log f\right) = g^{\underline{\phi},\alpha\bar\beta}\,\mathrm{Ric}(\omega_0)_{\alpha\bar\beta} - n.$$

Note that this is a uniformly elliptic second order non-linear partial differential equation with uniformly bounded coefficients, while the right hand side is in L^∞. According to the Hölder estimate (due to de Giorgi), there exists a small constant $\alpha \in (0,1)$ such that $\log f \in C^\alpha$ for any interior points. Since $\partial M = \emptyset$, this implies that f is $C^\alpha(M)$. Using the Monge–Ampere equation

$$\det\left(g_{\alpha\bar\beta} + \frac{\partial^2 \underline{\phi}}{\partial w_\alpha \partial w_{\bar\beta}}\right) = f,$$

one can deduce $\underline{\phi} \in C^{2,\alpha}$. Returning to the original equation of divergence form, we have

$$g_{\underline{\phi}}^{\alpha\bar\beta}\frac{\partial^2}{\partial w_\alpha \partial \bar w_\beta}(\log f) = \left(g_{\underline{\phi}}^{\alpha\bar\beta}\,\mathrm{Ric}_{\alpha\bar\beta}(\omega) - n\right) f.$$

Here the left hand side is a uniformly elliptic operator with C^α coefficients, the right hand side is also C^α continuous. The standard elliptic regularity theory implies that $\log f = \log \frac{\omega_{\underline{\phi}}^n}{\omega^n} \in C^{2,\alpha}$. This in turn implies that $\phi \in C^{4,\alpha}$ or the right hand side is in $C^{2,\alpha}$. By repeating this boot-strapping between these two equations, one shows that $\underline{\phi}$ is smooth. Consequently, it must be of constant scalar curvature. It is easy to see $\frac{\partial \underline{\phi}}{\partial s} = 0$, and $\underline{\phi}(s,t)$ $(0 \le t \le 1)$ satisfies the geodesic equation:

$$\frac{\partial^2 \underline{\phi}}{\partial t^2} - g_{\underline{\phi},\alpha\bar\beta}\eta^\alpha \eta^{\bar\beta} = \Delta_z \underline{\phi} - g_{\underline{\phi},\alpha\bar\beta}\eta^\alpha\eta^{\bar\beta} = 0,$$

where

$$\eta^\alpha = -g_{\underline{\phi}}^{\alpha\bar\beta}\frac{\partial^2 \underline{\phi}}{\partial w_\alpha \partial \bar w_\beta}.$$

The second variation of the K energy must be identically 0 in the direction of t, which implies

$$\int_0^1 dt \int_M \left|\frac{\partial \eta^\alpha}{\partial w_{\bar\beta}}\right|_{\underline{\phi}}^2 \omega_{\underline{\phi}}^{[n]} = 0$$

or

$$\frac{\partial \eta^\alpha}{\partial w_{\bar{\beta}}} \equiv 0, \quad \forall\, \alpha, \beta = 0, \forall\, 1, 2, ..., n.$$

Thus, this path represents a path of holomorphic isometric transformations. The uniqueness is then proved for cscK metric.

For general extremal Kähler metrics, we can apply the above arguments to the modified K energy $\tilde{\mathbf{E}}_\omega$ defined in (6.6). According to [15], there is a unique extremal holomorphic vector field Y in a given Kähler class up to holomorphic conjugation. By Proposition 6.1.3, we only need to prove the uniqueness of extremal Kähler metrics which are invariant under action of Im(Y). For invariant metrics, the above arguments go through for $\tilde{\mathbf{E}}_\omega$ in (6.6) on the space of invariant Kähler metrics. Thus we have

Theorem **8.0.1.** — *In any Kähler class, the extremal Kähler metric is unique up to holomorphic transformations.*

This concludes our proof of Theorem 1.1.1.

9. Appendix

The purpose of this appendix is to give a proof for Lemma 3.3.1. The lemma is more or less known to experts in the field, although it is difficult to find exact statement in literatures. The proof presented here is shown to us by Professor E. Lupercio. It uses some standard theory of loop groups. We will be very explicit in our presentation here for the sake of completeness.

To simplify the following explanation let us suppose that $G = GL_n(\mathbf{C})$. The *loop group* $\mathscr{L}G$ of a Lie group G is the space of maps from the unit circle S^1 in $\mathbf{C}$ to the corresponding group G. In this note, the space $\mathscr{L}G$ is endowed with the structure of an infinite dimensional *polarized* manifold. By a *polarization* of a vector space H we mean a class of decompositions $H_+ \oplus H_-$ that differ only "by a finite amount." We will be more precise below. A manifold is polarized if its tangent bundle is polarized at every fiber.

There are several important subgroups of the loop group that deserve consideration. The first of them is the subgroup $\mathscr{L}^+GL_n(\mathbf{C})$ of loops $\gamma \in \mathscr{L}GL_n(\mathbf{C})$ that extend to holomorphic maps of the closed unit disc D^2 on the complex plane $\tilde{\gamma} : D^2 \to GL_n(\mathbf{C})$.

The loop group has very important homogeneous spaces that posses very nice geometrical interpretations. The most important of them is the *restricted Grassmannian of a Hilbert space* H. The fundamental idea is that the loop group acts transitively on the restricted Grassmannian, and the stabilizer of a point is the subgroup $\mathscr{L}^+GL_n(\mathbf{C})$.

This action thus realizes the restricted Grassmannian as a homogeneous space for the loop group of the form $\mathscr{L}GL_n(\mathbf{C})/\mathscr{L}^+GL_n(\mathbf{C})$. To define this Grassmannian we need the concept of *polarization* for the Hilbert space. Let just say that if we realize our Hilbert space as the space of functions on the circle $H^{(n)} = L^2(S^1, \mathbf{C}^n) \cong H \otimes \mathbf{C}^n$, then the natural polarization for $H^{(n)}$ is given by

$$H^{(n)} = H_+^{(n)} \oplus H_-^{(n)} \tag{9.1}$$

where $H_+^{(n)}$ consists of those elements of $H^{(n)}$ that are boundary values for a holomorphic map on the unit disc D^2, and $H_-^{(n)}$ is the orthogonal complement of $H_+^{(n)}$ in $H^{(n)}$. In other words $H_+^{(n)}$ is the space of functions $f(z)$ so that in its Fourier expansion no negative powers of z appear.

We define the *restricted Grassmannian* $Gr(H^{(n)})$ of $H^{(n)}$ to be the space of all closed subspaces W of $H^{(n)}$ so that the projections $W \to H_+^{(n)}$ and $W \to H_-^{(n)}$ are respectively a Fredholm and a Hilbert–Schmidt operator. This definition is crafted in such a way that W is then 'comparable' in a suitable sense with $H_+^{(n)}$, that is to say, the decomposition $W \oplus W^\perp$ is also a polarization. With this definition the restricted Grassmannian is an infinite dimensional complex manifold with charts modeled on the Hilbert space $\mathscr{I}_2(W; W^\perp)$ of Hilbert–Schmidt operators $W \to W^\perp$. This shows that the restricted Grassmannian group is a polarized manifold.

More relevant to this discussion is the Grassmannian $Gr^{(n)} \subseteq Gr(H^{(n)})$ consisting of those $W \in Gr(H^{(n)})$ so that $zW \subseteq W$. The index of the projection $W \to H_+^{(n)}$ is called the virtual dimension of W. The *virtual dimension* index the connected components of the Grassmannian, that is to say that it can be thought as an isomorphism $\pi_0 Gr(H^{(n)}) \cong \mathbf{Z}$.

If we let the loop group $\mathscr{L}GL_n(\mathbf{C})$ act on $H^{(n)}$ by matrix multiplication (every element $\gamma(z)$ of the loop group is a matrix valued function on S^1) then the action induces a corresponding action on $Gr^{(n)}$ – this is the purpose of the condition $zW \subseteq W$ in the definition of this Grassmannian. The action is transitive and the isotopy group of $H_+^{(n)}$ is precisely $\mathscr{L}^+GL_n(\mathbf{C})$. This produces the identification $\mathscr{L}GL_n(\mathbf{C})/\mathscr{L}^+GL_n(\mathbf{C}) \cong Gr^{(n)}$. A version of the maximum modulus principle furthermore implies that

$$\Omega U_n \cong \mathscr{L}U_n/U_n \cong \mathscr{L}GL_n(\mathbf{C})/\mathscr{L}^+GL_n(\mathbf{C}) \cong Gr^{(n)}. \tag{9.2}$$

There is natural stratification of $Gr^{(n)}$ whose strata are indexed by homomorphisms $S^1 \to GL_n(\mathbf{C})$. Every such homomorphism can be written in the form

$$z^{\mathbf{k}} = \begin{pmatrix} z^{k_1} & & & & \\ & z^{k_2} & & & \\ & & z^{k_3} & & \\ & & & \ddots & \\ & & & & z^{k_n} \end{pmatrix}$$

where $\mathbf{k} = (k_1, k_2, k_3, ..., k_n)$ is a integer partition of the non-negative integer number k, namely $k_1 + k_2 + \cdots + k_n = k$.

The *Birkhoff factorization theorem*[27] establishes that any loop $\gamma(z) \in \mathscr{L}\mathrm{GL}_n(\mathbf{C})$ can be factored in the form

$$\gamma(z) = \gamma_-(z) z^{\mathbf{k}} \gamma_+(z) \tag{9.3}$$

where $\gamma_+(z), \gamma_-(1/z) \in \mathscr{L}^+\mathrm{GL}_n(\mathbf{C})$ and $\mathbf{k}$ is well defined up to the ordering of the k_i's.

We will say that $\mathbf{k}$ is the *multi-index* (or Grothendieck index) of $\gamma(z)$ whenever (9.3) holds. We will tolerate the ordering ambiguity in this definition.

Let us return to the description of the stratification of $\mathrm{Gr}^{(n)}$. Notice that given a loop $\gamma(z) \in \mathscr{L}\mathrm{GL}_n(\mathbf{C})$ of index $\mathbf{k}$, then multiplying on the right by any element $\phi_+(z) \in \mathscr{L}^+\mathrm{GL}_n(\mathbf{C})$ will not affect the multi-index, that is, $\gamma(z)\phi_+(z)$ still has multi-index $\mathbf{k}$. From this we conclude that the multi-index is constant along orbits of the right-action of $\mathscr{L}^+\mathrm{GL}_n(\mathbf{C})$ in $\mathscr{L}\mathrm{GL}_n(\mathbf{C})$. In other words every point of $\mathrm{Gr}^{(n)}$ has a well defined multi-index.

Let us define $\mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$ by declaring that $\phi_-(z) \in \mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$ if and only if $\phi_-(1/z) \in \mathscr{L}^+\mathrm{GL}_n(\mathbf{C})$. Again the action of $\mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$ doesn't affect te multi-index of an element. Then the orbits of the action of $\mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$ in $\mathrm{Gr}^{(n)}$ are precisely the same as the sets of elements in with the same multi-index and all its permutations. This is once more a consequence of the Birkhoff factorization. To avoid the problem of the permutations we will have to consider a smaller subgroup N^- of $\mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$. The group N^- consists of those elements in $\gamma_- \in \mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$ so that,

$$\gamma_-(\infty) = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For every partition $\mathbf{k}$ of k (and here the order is important) we define the subspaces $\mathrm{H}_{\mathbf{k}} \in \mathrm{Gr}^{(n)}$ as

$$\mathrm{H}_{\mathbf{k}} = z^{\mathbf{k}} \mathrm{H}_+^{(n)} = \left\{ f(z) = (f_1(z), ..., f_n(z)) : f_i(z) = \sum_{j=k_i}^{\infty} a_j^i z^j, \ \ a_j^i \in \mathbf{C} \right\} \tag{9.4}$$

and we define $\Sigma_{\mathbf{k}}^{\sigma}$ to be the orbit of $\mathrm{H}_{\mathbf{k}}$ under the action of $\mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$ in $\mathrm{Gr}^{(n)}$.

Define now $\Sigma_{\mathbf{k}}$ to be $\mathrm{N}^- \cdot \mathrm{H}_{\mathbf{k}} \subset \mathrm{Gr}^{(n)}$ (again, the order in $\mathbf{k}$ in important.) We have

[27] The Birkhoff factorization theorem is equivalent to the theorem of Grothendieck that states that every holomorphic bundle of rank n over the Riemman sphere can be uniquely written in the form of powers of the Hopf bundle $\mathrm{O}(k_1) \oplus \cdots \oplus \mathrm{O}(k_n)$.

Proposition **9.0.1.** — *The set of all elements in* $\mathrm{Gr}^{(n)}$ *of multi-index* $\mathbf{k}$ *if precisely* $\Sigma^{\sigma}_{\mathbf{k}}$.

Proof. — Take any element $\gamma(z) \in \mathscr{L}\mathrm{GL}_n(\mathbf{C})$ of index $\mathbf{k}$, and write $\mathrm{W} = \gamma(z)\mathrm{H}^{(n)}_+ \in \mathrm{Gr}^{(n)}$. Using the Birkhoff factorization $\gamma(z) = \gamma_-(z)z^{\mathbf{k}}\gamma_+(z)$ we have $\mathrm{W} = \gamma_-(z)z^{\mathbf{k}}\gamma_+(z)\mathrm{H}^{(n)}_+ = \gamma(z) = \gamma_-(z)z^{\mathbf{k}}\mathrm{H}^{(n)}_+ = \gamma_-(z)\mathrm{H}_{\mathbf{k}}$, showing thus that every W is in some $\Sigma^{\sigma}_{\mathbf{k}}$. □

In fact a more refined statement is true.

Proof. — We use the Pressley–Segal identification of $\mathrm{H}^{(n)} = \mathrm{L}^2(\mathrm{S}^1, \mathbf{C}^n) \to \mathrm{L}^2(\mathrm{S}^1, \mathbf{C}) = \mathrm{H}$ given by $(f_1, \ldots, f_n) \mapsto \tilde{f}(\zeta) = \sum_{i=1}^{n} \zeta^{i-1} f_i(\zeta^n)$. We define $\check{\mathrm{W}} = \bigcup_m \mathrm{W} \cap \zeta^m \mathrm{H}_-$. Choose an algebraic vector space basis of $\check{\mathrm{W}}$ by considering the subset of $\check{\mathrm{W}}$ consisting of elements of the form $\zeta^s + \sum_{k=-\infty}^{s-1} a_k \zeta^k$, and choosing one such element for each possible value of s. Call S the set of all the values of s appearing in this construction. Denote by $w_s = \zeta^s + \sum_{k=-\infty}^{s-1} a_k \zeta^k$ the chosen element so that our basis is $\mathscr{B} = \{w_s \colon s \in \mathrm{S}\}$. Let $\mathrm{H}_{\mathrm{S}} = \{\sum_{s \in \mathrm{S}} a_s \zeta^s\}$ be the Hilbert space generated by the ζ^s. We may suppose that the orthogonal projection $\mathrm{W} \to \mathrm{H}_{\mathrm{S}}$ sends $w_s \mapsto \zeta^s$ (by using reduction of the basis $\mathscr{B}$ to its reduced echelon form.) This induces an isomorphism $\mathrm{W} \cong \mathrm{H}_{\mathrm{S}}$. Since $z\mathrm{W} \subset \mathrm{W}$ then $s \in \mathrm{S}$ implies $s + n \in \mathrm{S}$ (cf. p. 98 in [9]). There are n elements r_i in S so that $r_i - n$ is not in S, and $\mathrm{S} = \bigcup_{q \in \mathbf{N}} \{r_1 + qn, \ldots, r_n + qn\}$. Writing $r_i = nk_i + i - 1$ we conclude immediately that $\mathrm{H}_{\mathrm{S}} = \mathrm{H}_{\mathbf{k}}$.

We can find the element $\gamma_-(z)$ we are seeking for using the isomorphism $\mathrm{W} \to \mathrm{H}_{\mathbf{k}}$ by the following procedure. We define the smooth function $v_i(z) \in \mathrm{W}$ to be the element in W that projects to $(0, \ldots, z^{k_i}, \ldots, 0) = \zeta^{i-1+nk_i} = \zeta^{r_i}$, so that we can write a basis of W as $\mathscr{B}' = \{z^k v_i(z) \colon 1 \le n, k \ge 0\}$. The matrix of smooth functions $v(z) = (v_1(z), \ldots, v_n(z))$ defines an element in $\mathscr{L}\mathrm{GL}_n(\mathbf{C})$. Clearly $v(z) \cdot \mathrm{H}_+ = \mathrm{W}$. Define $\gamma_-(z) = v(z) \cdot z^{-\mathbf{k}}$. Then $\gamma_-(z) \cdot \mathrm{H}_{\mathbf{k}} = \mathrm{W}$. Since $v(z)$ projects to $(0, \ldots, z^{k_i}, \ldots, 0) \in \mathrm{H}_{\mathbf{k}}$ then $\gamma_i(z) = v_i(z) z^{-k_i}$ projects to $(0, \ldots, 1, \ldots, 0) \in \mathrm{H}_+$, therefore no positive powers of z appear in the expansion of $\gamma_-(z)$ and moreover the constant term in the Laurent expansion $\gamma_-(\infty)$ is upper triangular. We also have that $\deg\det \gamma_-(z) + v.dim\,\mathrm{W} = v.dim\,\mathrm{H}_{\mathbf{k}}$, and therefore $\deg\det \gamma_-(z) = 0$. From this we conclude that $\gamma_-(z) \in \mathrm{N}^-$. □

Proposition **9.0.2.** — *The Grassmannian* $\mathrm{Gr}^{(n)}$ *admits a partition*

$$\mathrm{Gr}^{(n)} = \coprod_{\mathbf{k}} \Sigma^{\sigma}_{\mathbf{k}}.$$

Moreover, each $\Sigma^{\sigma}_{\mathbf{k}}$ *is the union of the* $\Sigma_{\mathbf{k}}$*'s for all permutations in the order of* $\mathbf{k}$, *namely*

$$\Sigma^{\sigma}_{\mathbf{k}} = \coprod_{\epsilon \in \mathrm{S}_n} \Sigma_{\epsilon(\mathbf{k})}.$$

Proof. — It is easy to see that for each permutation ϵ we have that $H_{\epsilon(\mathbf{k})} \in \Sigma^{\sigma}_{\mathbf{k}}$ and hence $\Sigma^{\sigma}_{\mathbf{k}} \supseteq \bigcup_{\epsilon \in S_n} \Sigma_{\epsilon(\mathbf{k})}$. Since the whole $\mathrm{Gr}^{(n)}$ is the disjoint union of $\Sigma_{\mathbf{j}}$ for all possible $\mathbf{j}$ and since $N^- \subset \mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$, it is enough to show that $\Sigma^{\sigma}_{\mathbf{k}}$ does not contain any $H_{\mathbf{j}}$ for a $\mathbf{j}$ that is not a permutation of $\mathbf{k}$. To see this we associate a sequence $\omega(W)$ to each W by $\omega_i(W) = \mathbf{dim}(W \cap \zeta^i H_-)$. It is immediate to see that the sequence ω is $\mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$-invariant, and it nevertheless distinguishes $H_{\mathbf{k}}$ from $H_{\mathbf{j}}$ (because of the proof of the previous proposition we can recover the ordered multi-index from ω.) □

Theorem **9.0.3.** — *The set* $\Sigma_{\mathbf{k}}$ *is a contractible submanifold of* $\mathrm{Gr}^{(n)}$ *of codimension*

$$cd(\mathbf{k}) = \sum_{i<j} |k_i - k_j| - \varrho(\mathbf{k}),$$

where $\varrho(\mathbf{k})$ *is the number of inversions of* $\mathbf{k}$.

Proof. — Let us define an open neighborhood of $H_{\mathbf{k}}$. Let $L_0^- \subset \mathscr{L}^-\mathrm{GL}_n(\mathbf{C})$ be is the subgroup of elements $\gamma(z)$ so that $\gamma(\infty)$ is the identity matrix. Consider the map $\mathscr{L}\mathrm{GL}_n(\mathbf{C}) \to \mathrm{Gr}^{(n)}: \gamma(z) \mapsto \gamma(z) \cdot H_{\mathbf{k}}$. Let $U_{\mathbf{k}}$ be the image under this map of $z^{\mathbf{k}} L_0^- z^{-\mathbf{k}}$. Clearly $H_{\mathbf{k}} \in U_{\mathbf{k}}$.

Now $U_{\mathbf{k}}$ we prove that is an open set. Here we return to the proof of Lemma 1. When we proved that $\gamma_-(z) \in N^-$ we should point out that the same argument actually proves a little bit more. Indeed we have that $z^{-\mathbf{k}} v_i(z) z^{\mathbf{k}} \in H$ is of the form $(0, ..., 1, ..., 0) + h_-(z) \in H$ where $h_-(z) \in H_-$. Therefore $\gamma_-(z) \in z^{\mathbf{k}} L_0^- z^{-\mathbf{k}}$. Without loss of generality we assume $\mathbf{k} = (0, ..., 0)$, otherwise we shift by the appropiate $z^{\mathbf{k}}$. In this case the Birkhoff factorization can be refined to state that every loop $\gamma(z)$ factorizes uniquely as $\gamma(z)\gamma_+(z)$ where $\gamma_-(z) \in L_0^-$ and $\gamma_+(z) \in \mathscr{L}^+\mathrm{GL}_n(\mathbf{C})$. This implies that $L_0^- \cong U_0$ is an open chart of $\mathrm{Gr}^{(n)} = \mathscr{L}\mathrm{GL}_n(\mathbf{C})/\mathscr{L}^+\mathrm{GL}_n(\mathbf{C})$, and therefore $U_{\mathbf{k}}$ is open. Also notice that this argument also implies that $\Sigma_{\mathbf{k}} \subseteq U_{\mathbf{k}}$.

We want to show then that the codimension of the inclusion $\Sigma_{\mathbf{k}} \subseteq U_{\mathbf{k}}$ is given by the formula of the statement of the theorem. Since we know that in the proof of Lemma 1 we actually have $\gamma_-(z) \in (z^{\mathbf{k}} L_0^- z^{-\mathbf{k}}) \cap N^-$. In fact $\mathscr{L}\mathrm{GL}_n(\mathbf{C}) \to \mathrm{Gr}^{(n)}: \gamma(z) \mapsto \gamma(z) \cdot H_{\mathbf{k}}$ induces an identification $(N^- \cap z^{\mathbf{k}} L_0^- z^{\mathbf{k}}) \to \Sigma_{\mathbf{k}}$. We claim that the multiplication in $\mathscr{L}\mathrm{GL}_n(\mathbf{C})$ indices an identification $(N^- \cap z^{\mathbf{k}} L_0^- z^{-\mathbf{k}}) \times (N^+ \cap z^{\mathbf{k}} L_0^- z^{-\mathbf{k}}) \to z^{\mathbf{k}} L_0^- z^{-\mathbf{k}} = U_{\mathbf{k}}$, where N^+ is just as N^- except that we talk of lower triangular matrices.

All that remains then is to compute the dimension of $N^+ \cap z^{\mathbf{k}} L_0^- z^{-\mathbf{k}}$. This is done as follows. By taking Laurent expansions of the entries of en element $\gamma(z) \in (N^+ \cap z^{\mathbf{k}} L_0^- z^{-\mathbf{k}})$ we conclude that for $i < j$ then $\gamma_{ii}(z) = 1$, $\gamma_{ij}(z) = \sum_{l=1}^{k_i - k_j - 1} a_l z^l$ and $\gamma_{ji} = \sum_{l=0}^{k_j - k_i - 1} a_l z^l$. By counting coefficients we obtain the desired formula. □

Remark **9.0.4.** — Notice that the proof of the previous theorem actually shows more. It shows that the codimension of elements $\gamma(z)$ of multi-index $\mathbf{k}$ inside

$\mathscr{L}\mathrm{GL}_n(\mathbf{C})$ is given by the same formula. This is done by considering $(\mathrm{N}^- \cap z^{\mathbf{k}} \mathrm{L}_0^- z^{-\mathbf{k}}) \times (\mathrm{N}^+ \cap z^{\mathbf{k}} \mathrm{L}_0^- z^{-\mathbf{k}}) \times \mathscr{L}^+\mathrm{GL}_n(\mathbf{C})$ instead of $(\mathrm{N}^- \cap z^{\mathbf{k}} \mathrm{L}_0^- z^{-\mathbf{k}}) \times (\mathrm{N}^+ \cap z^{\mathbf{k}} \mathrm{L}_0^- z^{-\mathbf{k}})$.

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