DYNAMICS ON BLOWUPS OF THE PROJECTIVE PLANE

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CONTENTS

1. Introduction

In this paper we give a systematic construction of automorphisms of rational surfaces with positive entropy, and investigate their dynamics. Specific cases yield:

1. Surface automorphisms with the minimum possible positive entropy;
2. Attracting basins of full measure, and Julia sets of measure zero; and
3. The first examples of automorphisms of projective algebraic varieties with Siegel disks.

Surface automorphisms. — Let $F : S \to S$ be a holomorphic automorphism of a compact complex surface. By [Ca1], if the topological entropy $h(F)$ is positive, then a minimal model for S is either a K3 surface, an Enriques surface, a complex torus or a rational surface. While constructions of automorphisms with positive entropy are well-known in the first three cases, rather few are known for rational surfaces.

One can aim to construct a rational surface automorphism $F : S \to S$ with a prescribed action on the middle-dimensional cohomology. To make this precise, let

$$\pi : S \to \mathbf{P}^2$$

be a rational surface presented as the blowup of the projective plane at n distinct points $(p_1, \ldots, p_n)$. Let $\mathbf{Z}^{1,n}$ denote the lattice $\mathbf{Z}^{n+1}$ with the Minkowski inner product

$$(x \cdot x) = x^2 = x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2,$$

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let $H \subset S$ be the preimage of a generic line in the plane, and let E_i be the exceptional curve lying over p_i. Then there is a natural *marking* isomorphism

$$\phi : \mathbf{Z}^{1,n} \to H^2(S, \mathbf{Z}),$$

defined on the standard basis by $\phi(e_0) = [H]$ and $\phi(e_i) = [E_i]$, $i = 1, ..., n$. This marking sends the Minkowski inner product to the intersection pairing on $H^2(S, \mathbf{Z})$.

Any vector $\alpha \in \mathbf{Z}^{1,n}$ with $\alpha^2 = -2$ determines a reflection $\rho : \mathbf{Z}^{1,n} \to \mathbf{Z}^{1,n}$ by $x \mapsto x + (x, \alpha)\alpha$. The *Weyl group* $W_n \subset O(\mathbf{Z}^{1,n})$ is the group generated by the reflections $(s_i)_0^{n-1}$ through the vectors

$$\alpha_0 = e_0 - e_1 - e_2 - e_3 \quad \text{and}$$
$$\alpha_i = e_i - e_{i+1}, \quad i = 1, ..., n-1.$$

The *roots* $\Phi_n = \bigcup W_n(\alpha_i)$ of W_n are the orbits of the *simple roots* $(\alpha_0, ..., \alpha_{n-1})$; the latter form an integral basis for the *root lattice* $L_n = \bigoplus \mathbf{Z}\alpha_i \subset \mathbf{Z}^{1,n}$.

The Weyl groups for $3 \leq n \leq 8$ are isomorphic to the finite Coxeter groups $A_1 \times A_2$, A_4, D_5, E_6, E_7 and E_8, and are associated with classical del Pezzo surfaces. The Weyl groups for $n \geq 9$ are infinite, and for $n \geq 10$ they contain elements with spectral radius $\sigma(w) > 1$.

By a theorem of Nagata, if F is an automorphism of S, there is a unique element $w \in W_n$ making the diagram

$$\begin{array}{ccc} \mathbf{Z}^{1,n} & \xrightarrow{w} & \mathbf{Z}^{1,n} \\ \downarrow \phi & & \downarrow \phi \\ H^2(S, \mathbf{Z}) & \xrightarrow{F_*} & H^2(S, \mathbf{Z}) \end{array}$$

commute (§5). In this case we say w is *realized* by the automorphism F. By theorems of Gromov and Yomdin, the entropy of F is given by $h(F) = \log \sigma(w)$.

Coxeter elements. — The product of the generators $(s_0, ..., s_{n-1})$, taken one at a time in any order, yields a *Coxeter element* $w \in W_n$. All Coxeter elements are conjugate, so the spectral radius $\lambda_n = \sigma(w)$ is well-defined.

We can now state our first result on automorphisms of positive entropy.

Theorem **1.1.** — *For $n \geq 10$, every Coxeter element $w \in W_n$ can be realized by a rational surface automorphism with entropy $h(F_n) = \log \lambda_n > 0$.*

In fact, the automorphism $F_n : S_n \to S_n$ can be chosen to have the following additional properties.

1. The surface S_n is the blowup of n distinct points $(p_i)_1^n$ lying on a cuspidal cubic curve $X \subset \mathbf{P}^2$ (§7).

2. There is a nowhere vanishing meromorphic 2-form η on S_n with a simple pole along the proper transform Y of X.
3. The automorphism satisfies $F_n^*(\eta) = \lambda_n \cdot \eta$, and thus it expands the volume element $\eta \wedge \overline{\eta}$.
4. The Julia set $J^+(F_n)$ has measure zero, and every $z \in S_n - J^+(F_n)$ converges under iteration to the unique singular point $p \in Y$ (§9).
5. The surface S_n equipped with the **Z**-action generated by F_n is G-minimal in the sense of Manin (§12).

The first three properties determine F_n uniquely. The points $(p_i)_1^n$ admit a simple description in terms of an eigenvector satisfying $w(v) = \lambda_n^{-1} \cdot v$, namely one can take $p_i = (x_i, x_i^3) \in \mathbf{C}^2$, where $x_i = v_i + v_0/3$. This leads to concrete formulas for F_n (§11).

Lehmer's automorphism. — The smallest known Salem number is a root $\lambda_{\text{Lehmer}} \approx 1.17628081$ of Lehmer's polynomial

$$L(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1. \tag{1.1}$$

In the Appendix we will show:

Theorem **1.2.** — *If* $F : S \to S$ *is an automorphism of a compact complex surface with positive entropy, then* $h(F) \geq \log \lambda_{\text{Lehmer}}$.

It is easy to verify that $\lambda_{\text{Lehmer}} = \lambda_{10}$, and therefore:

Corollary **1.3.** — *The map* $F_{10} : S_{10} \to S_{10}$ *is a surface automorphism with the smallest possible positive entropy.*

Picture in $\mathbf{RP}^2$. — When suitably normalized, the projection $\pi : S_{10} \to \mathbf{P}^2$ transports F_{10} to a birational automorphism of the plane of the form $f_{10}(x, y) = (a, b) + (y, y/x)$. The geometry of this map is depicted in Figure 1: it blows up the vertices (p_1, p_2, p_3) of the central triangle, and blows its edges down to (p_2, p_3, p_4), where $p_4 = (a, b)$ (see §11). The remaining dots indicate the forward orbit $p_{4+i} = f^i(p_4)$, up to $p_{11} = p_1$. As shown, the points $(p_i)_1^{10}$ lie on a cuspidal cubic; blowing them up yields the surface S_{10} on which F_{10} acts. The scatter plot is an approximation to the Julia set $J^+(F_{10})$, obtained by backwards iteration of random points. As noted above, every $z \notin J^+(F_{10})$ converges under forward iteration to the cusp inside the central triangle.

Siegel disks. — A linear automorphism $R(z_1, z_2) = (\alpha z_1, \beta z_2)$ of $\mathbf{C}^2$ is an *irrational rotation* if $|\alpha| = |\beta| = 1$ and F has dense orbits on $S^1 \times S^1$. A domain $U \subset S$ is a *Siegel disk* for F if $F(U) = U$ and $F|U$ is analytically conjugate to $R|\Delta^2$ for some irrational rotation R. (Here $\Delta = \{z : |z| < 1\}$.)

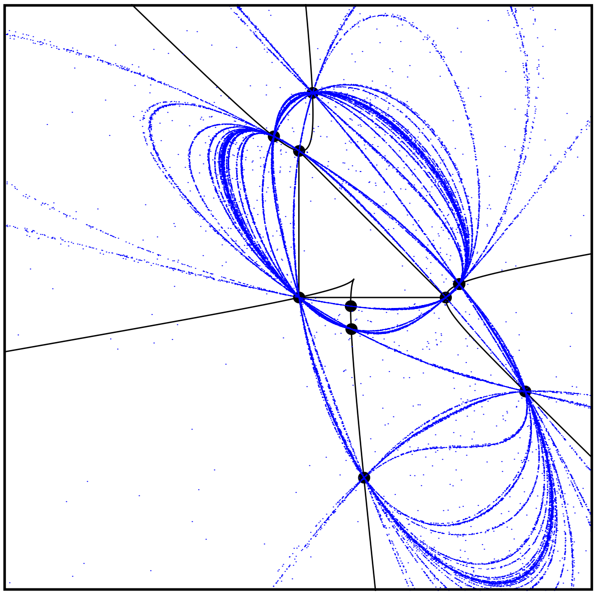

FIG. 1. — The expanding map F_{10} associated to Lehmer's number

It is easy to see an automorphism of a complex torus can never have a Siegel disk. A Siegel disk is possible on a K3 surface, but only when S is nonprojective [Mc2]. On the other hand, for rational surfaces we find (§10):

Theorem **1.4.** — *There are infinitely many n such that the standard Coxeter element $w \in W_n$ can be realized on a blowup of $\mathbf{P}^2$ by an automorphism with a Siegel disk.*

These maps arise from Galois conjugates of λ_n that lie on the unit circle. They preserve the natural volume form $\eta \wedge \overline{\eta}$, and also have positive entropy. Explicit examples, with $n = 11$ and 12, are given in §11.

Cubic curves. — The automorphisms above are all constructed using marked blowups $\pi : S \to \mathbf{P}^2$ whose basepoints $(p_i)_1^n$ lie along a *cubic curve* $X \subset \mathbf{P}^2$.

Cubics play a distinguished role because the proper transform Y of X is then an anticanonical curve on S (an element of the linear system $|-K_S|$). This facilitates the construction of useful invariants in the spirit of Hodge theory.

More precisely, by restricting line bundles from S to Y, we obtain a map

$$\rho : \mathbf{Z}^{1,n} \xrightarrow{\phi} H^2(S, \mathbf{Z}) \cong \mathrm{Pic}(S) \to \mathrm{Pic}(Y) \cong \mathrm{Pic}(X),$$

which we regard as a marking of X. The basepoints are determined by the condition $\rho(e_i) = [p_i] \in \mathrm{Pic}(X)$, so the marked pair (S, Y, ϕ) can be reconstructed from (X, ρ).

Provided X is irreducible, the marking ρ is essentially determined by its restriction to the root lattice

$$\rho_0 : L_n \to \mathrm{Pic}_0(X).$$

Here the target $\mathrm{Pic}_0(X)$ is a complex torus, $\mathbf{C}^*$ or $\mathbf{C}$ depending on whether X is a smooth, nodal or cuspidal cubic.

Fixed point formulation. — To indicate the construction of automorphisms, first suppose $w \in W_n$ is already realized by a map $F \in \mathrm{Aut}(S)$ preserving Y. Then F covers a birational map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$, stabilizing X inducing an automorphism $f_* : \mathrm{Pic}_0(X) \to \mathrm{Pic}_0(X)$ satisfying

$$\rho_0 \circ w = f_* \circ \rho_0.$$

In other words, $[\rho_0]$ is a fixed point for the natural action of w on the moduli space of markings

$$M_n(X) \subset \mathrm{Hom}(L_n, \mathrm{Pic}_0(X))/\,\mathrm{Aut}(X).$$

Conversely, to realize a given element $w \in W_n$ by a surface automorphism, we begin by locating a fixed point $[\rho_0] \in M_n(X)$. The marking determines basepoints $(p_i)_1^n$ on the cubic curve $X \subset \mathbf{P}^2$. Blowing them up, we obtain a rational surface $\pi : S \to \mathbf{P}^2$ marked by ϕ, with a distinguished anticanonical curve Y.

Now suppose $\rho_0(\alpha) \neq 0$ for all roots $\alpha \in \Phi_n$ (a generic condition). Then the basepoints $(p_i)_1^n$ satisfy no nodal relation (no two are coincident, no three are on a line, no six are on a conic, etc.). By a theorem of Nagata, this implies there is a second projection $\pi' : S \to \mathbf{P}^2$ corresponding to the marking $\phi \circ w$.

Let X′ denote the cubic curve $\pi'(Y) \subset \mathbf{P}^2$. By the assumption that $[\rho_0]$ is a fixed point for w, the marked cubic $(X', \rho \circ w)$ is isomorphic to (X, ρ). This implies the marked blowups (S, ϕ) and $(S, \phi \circ w)$ are also isomorphic. But an isomorphism between these two marked blowups is exactly an automorphism $F : S \to S$ satisfying $F_* \circ \phi = \phi \circ w$, as desired.

Cuspidal cubics. — The most flexible instance of the construction above arises when X is a cuspidal cubic. In this case we have $\mathrm{Pic}_0(X) \cong \mathbf{C}$, $[\rho_0]$ resides in the projective space

$$\mathrm{Hom}(L_n, \mathbf{C})/\mathbf{C}^* \cong \mathbf{P}^{n-1},$$

and a marking fixed by w is simply an eigenvector $v \in \mathrm{Hom}(L_n, \mathbf{C})$.

This method naturally yields a generalization of Theorem 1.1:

Theorem **1.5.** — *Suppose $w \in W_n$ has spectral radius $\sigma(w) > 1$ and no periodic roots (every orbit of $w|\Phi_n$ is infinite). Then w is realized by a surface automorphism of positive entropy.*

In fact we obtain many distinct realizations of w, depending on the choice of the eigenvector v (§7). The Siegel disk examples are constructed similarly, using reducible cubics: a conic with a tangent line, or three lines through a point.

It would be interesting to have a more complete classification of realizable elements in the Weyl group, and of the automorphisms they determine.

Notes and references. — The relationship of the Weyl group to the birational geometry of the plane is discussed in Kantor's 1895 book [Kan], and has been much developed since then [Cob], [DV], [Nag1], [Nag2], [Giz], [Lo], [Nik], [Hrw], [Ha1], [Ha4], [Ha3], [Zh], [DZ]; see also the texts [Man2] and [DO]. Similar constructions relating surface automorphisms and cubic curves appear in [Hrw] and [Ha1, §4].

The Coble surfaces, obtained by blowing up the 10 double nodes of a rational plane sextic, also admit automorphisms of positive entropy [Cob, §52]. The Lehmer automorphism F_{10} first appears in the Appendix to [BK2]. Another automorphism of positive entropy, residing on $\mathbf{P}^2$ blown up at 15 points, is studied in [HV] and [Tak].

For more on dynamics on K3 surfaces, see [Sil], [Wa], [Ca2] and [Mc2].

I would like to thank I. Coskun and the referee for useful remarks, and E. Bedford for bringing [BK2] to my attention.

2. Coxeter theory

In this section we review properties of the Weyl group W_n from the perspective of Coxeter theory.

The Weyl group. — Given $n \geq 3$, let Γ_n denote the graph with vertices $S_n = \{s_0, \ldots, s_{n-1}\}$ shown in Figure 2. Define an $n \times n$ matrix by

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } s_i \text{ is joined to } s_j \text{ in } \Gamma_n, \text{ and} \\ 2 & \text{otherwise.} \end{cases}$$

The *Weyl group* associated to this diagram is the finitely-presented group

$$W_n = \langle s_0, \ldots, s_{n-1} : (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Note: when $n = 3$ the graph Γ_n has a single edge, joining s_1 to s_2.

Geometric action. — Let $V_n = \mathbf{R}^{S_n}$, equipped with the inner product

$$B_n(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$$

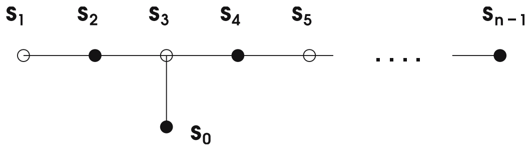

FIG. 2. — Coxeter graph Γ_n for (W_n, S_n)

on the natural basis (α_i) dual to (s_i). Any element $\alpha \in V_n$ with $B(\alpha, \alpha) = \pm 2$ determines a *reflection*

$$\rho_\alpha(x) = x - \frac{2B(x, \alpha)}{B(\alpha, \alpha)}\alpha \tag{2.1}$$

in the orthogonal group (V_n, B_n). The unique homomorphism

$$W_n \to (V_n, B_n)$$

sending s_i to ρ_{α_i} defines the *geometric action* of W_n on V_n.

Lattices. — Let $L_n \subset V_n$ denote the *root lattice* $\mathbf{Z}^{S_n} \subset \mathbf{R}^{S_n}$. Since the (α_i) form a basis for L_n, and $-2\cos(\pi/m_{ij}) = 2, 0$ or -1, the form $B_n|L_n \times L_n$ assumes integral values. Moreover L_n is invariant under the action of W_n by (2.1).

The lattice L_n is positive-definite for $n \leq 8$, semidefinite for $n = 9$ and of signature $(n-1, 1)$ for $n \geq 10$. For $n = 3, \ldots, 8$ we obtain the well-known root-lattices $A_1 \oplus A_2$, A_4, D_5, E_6, E_7 and E_8 respectively. For $n = 10$, $L_n \cong II_{9,1}$ is the unique even unimodular lattice of signature $(9, 1)$ (see e.g. [CoS, Chap. 27]).

Roots. — The basis elements $\alpha_i \in L_n$, $i = 0, 1, \ldots, n-1$ are the *simple roots* of W_n. Their orbits $\Phi_n = \bigcup_i W_n \cdot \alpha_i$ comprise the *roots* of W_n. A vector $v = \sum c_i\alpha_i \in V_n$ is *positive* if $c_i \geq 0$ for all i. The *positive roots* are denoted Φ_n^+.

Coxeter elements. — The products

$$s_{\sigma(0)}s_{\sigma(1)} \cdots s_{\sigma(n-1)}$$

of the generators of W_n, taken one at a time in any order, are the *Coxeter elements* of (W_n, S_n).

General results. — We can now state three results which follow from the general theory of Coxeter groups [Bou], [Hum, §5].

1. *The geometric representation of* W_n *is faithful.*
2. *Any root of* W_n *is positive or negative; that is,* $\Phi_n = \Phi_n^+ \cup (-\Phi_n^+)$.
3. *All Coxeter elements lie in a single conjugacy class in* W_n.

(The last statement depends on the fact that the Coxeter diagram Γ_n of W_n is a tree.)

Coxeter number. — By (3) above, all Coxeter elements $w \in W_n$ have the same order h_n. We have $h_n = 6, 5, 8, 12, 18, 30$ for $n = 3, 4, 5, 6, 7, 8$ and, as we will see below, $h_n = \infty$ for $n \geq 9$.

Spectral radius. — Let $A(\Gamma_n) = 2I - B_n$ denote the adjacency matrix of Γ_n, considered as an operator on V_n. We have $A(\Gamma_n)_{ij} = 1$ if s_i is connected to s_j by an edge in Γ_n, and $A(\Gamma_n)_{ij} = 0$ otherwise. It is straightforward to check:

Proposition **2.1.** — *The spectral radius $\sigma_n = \sigma(A(\Gamma_n))$ is a strictly increasing function of n, with $\sigma_9 = 2$.*

Bipartite theory. — A special feature of the Weyl group is that its Coxeter graph Γ_n is *bipartite*; every edge joins an even vertex to an odd vertex. This suggests splitting V_n into the direct sum $V_n^0 \oplus V_n^1$ of the spans of the roots α_i with even and odd indices i, respectively. With respect to this splitting, the adjacency matrix has the form

$$A(\Gamma_n) = \begin{pmatrix} 0 & C_n^t \\ C_n & 0 \end{pmatrix}.$$

Consider the particular Coxeter element

$$w_n = (s_0 s_2 s_4 \cdots) \cdot (s_1 s_3 s_5 \cdots) = w_n^0 \cdot w_n^1.$$

Since generators of the same parity commute, their ordering within each factor of w_n is immaterial. Using the fact that $B_n = 2I - A(\Gamma_n)$, it is easy to see that

$$w_n^0 = \begin{pmatrix} -I & C_n^t \\ 0 & I \end{pmatrix} \quad \text{and} \quad w_n^1 = \begin{pmatrix} I & 0 \\ C_n & -I \end{pmatrix}.$$

Thus the Coxeter element itself is given by

$$w_n = \begin{pmatrix} C_n^t C_n - I & -C_n^t \\ C_n & -I \end{pmatrix}$$

with respect to the splitting $V_n^0 \oplus V_n^1$.

Positivity. — By the Perron–Frobenius theorem there is a positive vector $v_n \in V_n$, unique up to scale, such that

$$A(\Gamma_n) \cdot v_n = \sigma_n v_n. \tag{2.2}$$

Let $G_n \subset V_n$ be the 2-dimensional vector space spanned by the even and odd parts of $v_n = v_n^0 + v_n^1$. By (2.2) we have $(C_n \cdot v_n^0, C_n^t \cdot v_n^1) = \sigma_n(v_n^1, v_n^0)$. Thus G_n is invariant under the Coxeter element w_n; indeed, we have

$$w_n | G_n = \begin{pmatrix} \sigma_n^2 - 1 & -\sigma_n \\ \sigma_n & -1 \end{pmatrix} \tag{2.3}$$

with respect to the basis (v_n^0, v_n^1).

Theorem **2.2.** — *The linear map* $w_n|G_n$ *is:*

- elliptic, *of order* h_n, *for* $n \leq 8$;
- parabolic, *of infinite order, for* $n = 9$; *and*
- hyperbolic, *of infinite order, for* $n \geq 10$.

Proof. — By (2.3), $w_n|G_n$ has determinant 1 and trace $\sigma_n^2 - 2$; so it is elliptic when $\sigma_n < 2$, parabolic when $\sigma_n = 2$ and hyperbolic when $\sigma_n > 2$. These alternatives correspond to $n \leq 8$, $n = 9$ and $n \geq 10$ by Proposition 2.1; and in the elliptic case $w_n|G_n$ is actually a rotation by $2\pi/h_n$ [Hum, §3.7]. □

Theorem **2.3.** — *For* $n \neq 9$, *every root* $\alpha \in \Phi_n$ *has a nonzero orthogonal projection to* $G_n \subset V_n$.

Proof. — Let β be the projection of α to G_n. We may assume $\alpha \in \Phi_n^+$. Then, since both $\alpha = \sum x_i\alpha_i$ and $v_n = \sum y_i\alpha_i$ are positive vectors, we have

$$B(v_n, \beta) = B(v_n, \alpha) = (2 - \sigma_n) \sum x_i y_i \neq 0$$

(using the fact that $\sigma_n \neq 2$ when $n \neq 9$); consequently $\beta \neq 0$. □

Corollary **2.4.** — *Let* $w \in W_n$ *be a Coxeter element.*

- *For* $n < 9$, *every orbit of* $w|\Phi_n$ *consists of* h_n *elements.*
- *For* $n > 9$, *every orbit of* $w|\Phi_n$ *is infinite; that is,* w *has no periodic roots.*

Remarks. — For $n \geq 10$, the Weyl group acts isometrically on the hyperbolic space $\mathbf{H}^{n-1} \subset \mathbf{P}V_n$ determined by the indefinite form B, and G_n corresponds to the unique hyperbolic geodesic $\gamma_n \subset \mathbf{H}^{n-1}$ stabilized by the action of w_n. In geometric terms, Theorem 2.3 states that γ_n is not contained in any of the hyperplanes $H_\alpha \subset \mathbf{H}^{n-1}$ defined by the roots $\alpha \in \Phi_n$.

Even though $h_9 = \infty$, there are roots with periods 2, 3 and 5 under the action of the Coxeter element w_9. For more on periodic roots, see [Par].

Salem and Pisot numbers. — An algebraic integer $\lambda > 1$ is a *Pisot number* if its Galois conjugates satisfy $|\lambda'| < 1$; it is a *Salem number* if its conjugates satisfy $|\lambda'| \leq 1$ and at least one conjugate lies on the unit circle. The smallest Pisot number is the root $\lambda_{\text{Pisot}} \approx 1.32471795$ of the polynomial $t^3 - t - 1$. The smallest *known* Salem number is the root $\lambda_{\text{Lehmer}} \approx 1.17628081$ of Lehmer's polynomial (Equation (1.1)).

Salem numbers arise naturally as eigenvalues of Coxeter elements. Indeed, the characteristic polynomial of a Coxeter element $w \in W_n$ is given explicitly by

$$P_n(t) = \det(tI - w) = \frac{t^{n-2}(t^3 - t - 1) + (t^3 + t^2 - 1)}{t - 1} \cdot \tag{2.4}$$

Compare [MRS, Lemma 5]. For $n \neq 9$ this polynomial has simple roots, and for $n \geq 10$ it factors as

$$P_n(t) = Q_n(t)R_n(t),$$

where $R_n(t)$ is a product of cyclotomic polynomials and $Q_n(t)$ is a Salem polynomial. The roots $t = \lambda_n^{\pm 1}$ of $Q_n(t)$ are simply the eigenvalues of $w_n|G_n$; the remaining roots lie on the unit circle.

It is easily checked that $Q_{10}(t)$ coincides with Lehmer's polynomial, and thus $\lambda_{10} = \lambda_{\text{Lehmer}}$. Inspection of (2.4) shows that as $n \to \infty$ we have

$$\lambda_n \to \lambda_{\text{Pisot}}.$$

(A similar construction shows that every Pisot number is a limit of Salem numbers [Sa, p. 30].)

Since λ_{Pisot} is not itself a Salem number, we have $\deg(Q_n) \to \infty$. By [Bi] (see also [Rum]), this implies:

Theorem **2.5.** — *As $n \to \infty$, the roots of $Q_n(t)$ other than $\lambda_n^{\pm 1}$ become equidistributed on the unit circle.*

This result will be used in Theorem 10.5 to construct Siegel disks.

Leading eigenvalues. — It is convenient to extend the factorization (2.4) to $n \leq 8$ by defining $Q_n(t)$ to be the cyclotomic polynomial for the h_n-th roots of unity, and to $n = 9$ by setting $Q_9(t) = (t-1)$. With this convention, $Q_n(t)$ is irreducible for all n.

We say $\lambda \in \mathbf{C}$ is a *leading eigenvalue* for w if $Q_n(\lambda) = 0$. The leading eigenvalues are simply the eigenvalues of $w_n|G_n$ and their Galois conjugates. Their associated eigenvectors are *leading eigenvectors*.

Theorem **2.6.** — *Let $v \in L_n \otimes \mathbf{C}$ be a leading eigenvector for a Coxeter element $w \in W_n$. Then provided $n \neq 9$, we have $v \cdot \alpha \neq 0$ for all roots $\alpha \in \Phi_n$.*

Proof. — The conclusion is formulated over $\mathbf{Q}$, so it suffices to prove the assertion when $w(v) = \lambda_n^{\pm 1} v$. In this case v belongs to $G_n \otimes \mathbf{C}$. Since $n \neq 9$, G_n is spanned by v and one of its Galois conjugates v'. If $v \cdot \alpha = 0$, then $v' \cdot \alpha = 0$ as well, so the projection of α to G_n is zero. This contradicts Theorem 2.3. □

Similar reasoning shows:

Theorem **2.7.** — *Suppose $w \in W_n$ has no periodic roots, and $w \cdot v = \lambda v$ where λ is not a root of unity. Then $0 \notin v \cdot \Phi_n$.*

Proof. — Let $S \subset V_n$ be the span of the Galois conjugates of v, and suppose $v \cdot \alpha = 0$. Then $\alpha \in S^\perp$. But $(S^\perp, B)$ is positive-definite and w-invariant, so α is periodic. □

3. The Minkowski model

Next we discuss a natural action of the Weyl group on Minkowski space.

The Minkowski lattice. — Let $\mathbf{R}^{1,n}$ denote $\mathbf{R}^{n+1}$ equipped with the Minkowski inner product

$$(x \cdot x) = x^2 = x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2.$$

The integral points $\mathbf{Z}^{1,n} \subset \mathbf{R}^{1,n}$ are a model for the unique odd unimodular lattice of signature $(1, n)$. Let $(e_0, e_1, \ldots, e_n)$ denote the standard basis in these coordinates.

Let $k_n = (-3, 1, 1, 1, \ldots, 1)$ denote the *canonical vector* in $\mathbf{Z}^{1,n}$, let

$$\mathrm{V}_n = k_n^{\perp} \subset \mathbf{R}^{1,n},$$

and let

$$\mathrm{L}_n = \mathrm{V}_n \cap \mathbf{Z}^{1,n}.$$

The stabilizer (L_n) of k_n in $\mathrm{O}(\mathbf{Z}^{1,n})$ acts faithfully on L_n.

Reflections. — Any $\alpha \in \mathrm{L}_n$ with $\alpha^2 = -2$ determines a reflection in $\mathrm{O}(\mathrm{L}_n)$ by

$$\rho_\alpha(x) = x + (x \cdot \alpha)\alpha.$$

The simplest such is the *transposition* τ_{ij}, given by reflection in the vector

$$\alpha_{ij} = e_i - e_j \tag{3.1}$$

for distinct indices $i, j \geq 1$; it simply exchanges the basis elements e_i and e_j while fixing the others.

Cremona involutions. — The *Cremona involution* $\kappa_{ijk} \in (\mathrm{L}_n)$ is given by reflection in the vector

$$\alpha_{ijk} = e_0 - e_i - e_j - e_k \tag{3.2}$$

for distinct indices $i, j, k \geq 1$. It acts by

$$\begin{aligned}
e_0 &\mapsto 2e_0 - e_i - e_j - e_k,\\
e_i &\mapsto e_0 - e_j - e_k,\\
e_j &\mapsto e_0 - e_i - e_k,\\
e_k &\mapsto e_0 - e_i - e_j, \quad \text{and}\\
e_l &\mapsto e_l \quad \text{if } l \notin \{0, i, j, k\}.
\end{aligned}$$

We will see in §5 that κ_{123} arises naturally from the standard quadratic Cremona involution on $\mathbf{P}^2$.

The Weyl group, reprise. — An integral basis for L_n is given by

$$(\alpha_0, \ldots, \alpha_{n-1}) = (\alpha_{123}, \alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1,n}).$$

In this basis, the Minkowski inner product satisfies

$$\alpha_i \cdot \alpha_j = 2\cos(\pi/m_{ij}) = -B_n(\alpha_i, \alpha_j),$$

and thus $(L_n, -x^2)$ is isometric to the root lattice $(\mathbf{Z}^n, B_n)$ for the Weyl group W_n. Identifying these two lattices, we obtain a representation

$$W_n \subset O(L_n) \subset O(\mathbf{Z}^{1,n})$$

that extends the geometric action from L_n to $\mathbf{Z}^{1,n}$. In this model, W_n is generated by the reflections

$$S_n = \{s_0, \ldots, s_{n-1}\} = \{\kappa_{123}, \tau_{12}, \tau_{23}, \ldots, \tau_{n-1,n}\}$$

through the simple roots $\alpha_0, \ldots, \alpha_{n-1} \in L_n$.

Note that the subgroup generated by $(s_1, \ldots, s_{n-1})$ is a copy of the symmetric group Σ_n, acting by permutations on the basis elements $(e_1, \ldots, e_n)$. The Weyl group also contains all the Cremona involutions, since κ_{ijk} is conjugate to κ_{123} under the action of Σ_n.

Positive roots. — The positive roots $\alpha = de_0 - \sum_1^n m_i e_i$ have a convenient form in the Minkowski model.

1. First, the conditions $k_n \cdot \alpha = 0$ and $\alpha^2 = -2$ translate into

$$\sum m_i = 3d \quad \text{and} \quad \sum m_i^2 = d^2 + 2. \tag{3.3}$$

   Positivity implies $d \geq 0$. The positive roots with $d = 0$ are given by $\alpha = \alpha_{ij}$ as in (3.1); those with $d = 1$, by $\alpha = \alpha_{ijk}$ as in (3.2).
2. Positive roots with $d > 0$ are invariant under the action of Σ_n, and satisfy $m_i \geq 0$. Thus any such root can be *normalized* so that $m_1 \geq m_2 \geq \cdots \geq m_n$.
3. A normalized positive root with $d > 1$ satisfies $0 \leq m_i$ and $m_1 + m_2 \leq d < m_1 + m_2 + m_3$. See [DO, Prop. 4, p. 74].

The normalized positive roots $\alpha = (d, -m_1, \ldots, -m_n)$ with $1 \leq d \leq 4$ are

$$(1, -1^3, 0^{n-3}), \quad (2, -1^6, 0^{n-6}), \quad (3, -2^1, -1^7, 0^{n-8}),$$
$$(4, -2^3, -1^6, 0^{n-10}), \quad (4, -3, -1^9).$$

Here k^i indicates that k is repeated i times; a given root does not occur for W_n if the exponent of 0 is negative.

4. Marked cubic curves

In this section we introduce marked cubic curves (X, ρ). These objects will form the basis for constructing rational surfaces with given automorphisms.

Cubic curves. — A *cubic curve* $X \subset \mathbf{P}^2$ is a reduced curve of degree three. We allow X to be singular or reducible, and denote its smooth points by X^*.

Picard group. — The Picard group of X is described by the exact sequence

$$0 \to \mathrm{Pic}_0(X) \to \mathrm{Pic}(X) \to H^2(X, \mathbf{Z}) \to 0,$$

where $\mathrm{Pic}_0(X)$ is isomorphic to either:

1. A compact torus $\mathbf{C}/\Lambda$ (when X is a smooth); or
2. The multiplicative group $\mathbf{C}^*$ (when X is a nodal cubic, or a conic with a transverse line, or three lines meeting in three points); or
3. The additive group $\mathbf{C}$ (when X is a cuspidal cubic, a conic with a tangent line or three lines through a single point).

(See e.g. [HM, Chap. 5B].) Every element of $\mathrm{Pic}(X)$ is represented by a divisor $D = \sum n_j p_j$ supported in X^*.

Automorphisms. — Let $\mathrm{Aut}(X)$ denote the automorphism group of X as an abstract complex variety. When X is irreducible, it is a familiar fact that $\mathrm{Aut}(X)$ acts transitively on its smooth points X^*. For general cubics, one can easily check:

Proposition **4.1.** — *Any set* E *consisting of one point from each component of* X^* *is equivalent, under* $\mathrm{Aut}(X)$, *to any other such collection.*

The derivative $D(f)$. — Let $\Omega(X)$ denote the space of sections of the dualizing sheaf ω_X. (When X is smooth, $\Omega(X)$ is just the space of holomorphic 1-forms on X.) Since X has arithmetic genus one, $\Omega(X)$ is one-dimensional. Thus we have a natural homomorphism

$$D : \mathrm{Aut}(X) \to \mathbf{C}^*$$

characterized by $f^*\omega = D(f)\omega$ for all $\omega \in \Omega(X)$. Equivalently, $D(f)$ is the derivative of f_* on the tangent space to the origin in $\mathrm{Pic}_0(X)$. If X is irreducible, then f acts on the universal cover $\widetilde{X^*} \cong \mathbf{C}$ by $f(z) = D(f)z + c$.

Proposition **4.2.** — *If* $\mathrm{Pic}_0(X) \not\cong \mathbf{C}$ *then* $D(f)$ *is a kth root of unity, where* $k = 1, 2, 3, 4$ *or* 6.

Proof. — For any cubic curve we have $\mathrm{Pic}_0(X) \cong \mathbf{C}/\Lambda$ for some discrete group $\Lambda \subset \mathbf{C}$, and $D(f)\Lambda = \Lambda$ for all $f \in \mathrm{Aut}(f)$. Thus $D(f)$ must be a root of unity as above, unless Λ is trivial. □

Particular cubics. — The cubic curves with $\mathrm{Pic}_0(X) \cong \mathbf{C}$ – namely the cuspidal cubic, a conic with a tangent line and three lines through a point – will play a leading role in the sequel. In these cases $f \in \mathrm{Aut}(X)$ acts on $\mathrm{Pic}_0(X)$ by $E \mapsto D(f)E$, where $D(f) \in D(\mathrm{Aut}(X)) = \mathbf{C}^*$. For example, the automorphisms of a cuspidal cubic act on $X^* \cong \mathbf{C}$ by $f(t) = at+b$, with $D(f) = a$. The fact that we can have $|D(f)| > 1$ makes these cubics suitable for the construction of automorphisms with positive entropy.

Marked cubics. — A *marked cubic curve* (X, ρ) is an abstract curve X equipped with a homomorphism $\rho : \mathbf{Z}^{1,n} \to \mathrm{Pic}(X)$, such that

1. The sections of the line bundle $\rho(e_0)$ provide an embedding $X \hookrightarrow \mathbf{P}^2$, making X into a cubic curve; and
2. There are distinct *basepoints* $p_i \in X^*$ such that $\rho(e_i) = [p_i]$ for $i = 1, 2, \ldots, n$.

The basepoints p_i are uniquely determined by ρ, since X^* embeds into $\mathrm{Pic}(X)$. Conversely, a cubic embedding $X \hookrightarrow \mathbf{P}^2$ together with a choice of distinct points $p_i \in X^*$ determines a marking of X.

We emphasize that different markings of X can yield different projective embeddings $X \hookrightarrow \mathbf{P}^2$ (e.g. different locations for its flexes); but these embeddings are all equivalent under the action of $\mathrm{Aut}(X)$.

Isomorphism. — An *isomorphism* $(X, \rho) \cong (X', \rho')$ is a biholomorphic map $f : X \to X'$ such that $\rho' = f_* \circ \rho$. We define

$$\begin{aligned} W(X, \rho) &= \{w \in W_n : (X, \rho \circ w) \text{ is a marked cubic}\}, \quad \text{and} \\ \mathrm{Aut}(X, \rho) &= \{w \in W(X, \rho) : (X, \rho) \cong (X, \rho \circ w)\}. \end{aligned}$$

Action of Aut(X). — It is convenient to break up the marking ρ of X into two pieces: the map

$$\rho_0 : \mathrm{Ker}(\deg \circ \rho) \to \mathrm{Pic}_0(X),$$

and

$$\deg \circ \rho : \mathbf{Z}^{1,n} \to H^2(X, \mathbf{Z}).$$

Proposition 4.1 readily implies:

Theorem **4.3.** — *The maps* ρ_0 *and* $\deg \circ \rho$ *determine* (X, ρ) *up to isomorphism.*

When X is irreducible, we have $\deg(\rho(u)) = -u \cdot k_n$, and thus:

Corollary **4.4.** — *An irreducible marked cubic* (X, ρ) *is determined up to isomorphism by* $\rho_0 : L_n \to \mathrm{Pic}_0(X)$.

Moduli spaces. — The moduli space of markings of X is given by:

$$\mathrm{M}_n(\mathrm{X}) = \{\rho : \mathbf{Z}^{1,n} \to \mathrm{Pic}(\mathrm{X}) : (\mathrm{X}, \rho) \text{ is a marked cubic}\}/\operatorname{Aut}(\mathrm{X}).$$

A second model for $\mathrm{M}_n(\mathrm{X})$ is obtained by fixing an embedding $\mathrm{X} \subset \mathbf{P}^2$; then a marking is determined by a choice of basepoints, and we have

$$\mathrm{M}_n(\mathrm{X}) \cong ((\mathrm{X}^*)^n - \Delta)/\operatorname{Aut}(\mathbf{P}^2, \mathrm{X}).$$

Here $\Delta = \{(p_i) : p_j = p_k \text{ for some } j \neq k\}$.

In the irreducible case, another rather explicit model is given by

$$\mathrm{M}_n(\mathrm{X}) \cong \{\rho_0 : \mathrm{L}_n \to \mathrm{Pic}_0(\mathrm{X}) : \rho_0(e_i - e_j) \neq 0 \ \forall i > j \geq 1\}/\operatorname{Aut}(\mathrm{X}).$$

When $\mathrm{Pic}_0(\mathrm{X}) \cong \mathbf{C}$, this model exhibits $\mathrm{M}_n(\mathrm{X})$ as the complement of finitely many hyperplanes in the projective space

$$\mathrm{Hom}(\mathrm{L}_n, \mathbf{C})/\mathbf{C}^* \cong \mathbf{C}^n/\mathbf{C}^* = \mathbf{P}^{n-1}.$$

5. Marked blowups

In this section we summarize the connection between the Weyl group W_n and the blowups of the projective plane at n points. See also [Man2], [Ha1] and [DO].

Marked blowups. — A *marked blowup* (S, ϕ) is a smooth projective surface S equipped with an isomorphism

$$\phi : \mathbf{Z}^{1,n} \to \mathrm{H}^2(\mathrm{S}, \mathbf{Z})$$

such that:

1. The marking ϕ sends the Minkowski inner product to the intersection pairing;
2. There exists a birational morphism $\pi : \mathrm{S} \to \mathbf{P}^2$, presenting S as the blowup of the projective plane at n distinct *basepoints* $p_1, \ldots, p_n$; and
3. The marking satisfies $\phi(e_0) = [\mathrm{H}]$ and $\phi(e_i) = [\mathrm{E}_i]$, $i = 1, \ldots, n$, where $\mathrm{H} = \pi^{-1}(\mathrm{L})$ is the preimage of a generic line in $\mathbf{P}^2$ and $\mathrm{E}_i \subset \mathrm{S}$ is the exceptional curve $\pi^{-1}(p_i)$.

The marking determines $\pi : \mathrm{S} \to \mathbf{P}^2$ up to post-composition with an automorphism of $\mathbf{P}^2$. Note that the canonical class of S is given by

$$\mathrm{K}_\mathrm{S} = \left[-3\mathrm{H} + \sum \mathrm{E}_i\right] = \phi(k_n).$$

As in the preceding sections, we assume $n \geq 3$.

Moduli. — An *isomorphism* $(S, \phi) \cong (S', \phi')$ is given by a biholomorphic map $F : S \to S'$ such that the diagram

(**5.1**)
$$\begin{array}{ccc} \mathbf{Z}^{1,n} & = & \mathbf{Z}^{1,n} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ H^2(S, \mathbf{Z}) & \xrightarrow{F_*} & H^2(S', \mathbf{Z}) \end{array}$$

commutes. In this case $p_i' = g(p_i)$ for some $g \in PGL(3, \mathbf{C}) \cong Aut(\mathbf{P}^2)$. Thus the moduli space of marked blowups is given by the configuration space

$$\mathscr{P}_n = ((\mathbf{P}^2)^n - \Delta)/\,PGL_3(\mathbf{C}),$$

where $\Delta = \{(p_i) : p_j = p_k \text{ for some } j \neq k\}$.

Role of the Weyl group. — Now suppose there are two birational morphisms $\pi, \pi' : S \to \mathbf{P}^2$, exhibiting S as the blowup of $\mathbf{P}^2$ at (p_i) and (p_i') respectively. Then there is a birational map f making the diagram

(**5.2**)
$$\begin{array}{ccc} S & = & S \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathbf{P}^2 & \overset{f}{\dashrightarrow} & \mathbf{P}^2 \end{array}$$

commute; and the corresponding markings are related by $\phi' = \phi \circ w$ for a unique $w \in (\mathbf{Z}^{1,n})$.

The following signal result shows these different blowups are related by the action of the Weyl group.

Theorem **5.1** *(Nagata).* — *Let* (S, ϕ) *be a marked blowup, and* $w \in O(\mathbf{Z}^{1,n})$. *If* $(S, \phi \circ w)$ *is also a marked blowup, then* $w \in W_n$.

See [Nag2, p. 283], [DO, p. 90, Thm. 2].

Cremona involutions. — Let

$$W(S, \phi) = \{w \in W_n : (S, \phi \circ w) \text{ is a marked blowup}\}.$$

The right action of the symmetric group simply reorders the basepoints of a blowup, so we have

(**5.3**) $$\Sigma_n \subset W(S, \phi).$$

To obtain more elements in $W(S, \phi)$, consider the standard quadratic *Cremona involution* $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$, given by

(**5.4**) $$f[x, y, z] = [yz, xz, xy].$$

This map blows up the three points $(q_1, q_2, q_3) = ([1, 0, 0], [0, 1, 0], [0, 0, 1])$ and blows down the lines between them.

Theorem **5.2.** — *If* $p_k \notin \overline{p_i p_j}$ *for all distinct indices with* $i, j \in \{1, 2, 3\}$, *then* $(S, \phi \circ \kappa_{123})$ *is a marked blowup.*

Proof. — Choose coordinates so that $p_i = q_i$ for $i = 1, 2, 3$; then $\pi' = f \circ \pi : S \to \mathbf{P}^2$ is a birational morphism, presenting $(S, \phi \circ \kappa_{123})$ as a marked blowup with basepoints p_1, p_2, p_3 and $f(p_i)$, $i \geq 4$. These points are distinct so long as $(p_4, \ldots, p_n)$ lie outside the lines blown down by f. □

Nodal roots. — We say $\alpha \in \Phi_n$ is a *nodal root* for (S, ϕ) if $\phi(\alpha) \in H^2(S, \mathbf{Z})$ is represented by an effective divisor D. In this case D projects to a curve of degree $d > 0$ on $\mathbf{P}^2$, and hence $\alpha = de_0 - \sum m_i e_i$ is a positive root with $d > 0$.

A nodal root is *geometric* if we can take D to be a sum of smooth rational curves.

Example. — To see the distinction, let E be a smooth elliptic curve through points $(p_i)_1^9$ in general position, subject only to the condition that (p_1, p_2, p_3) lie on a line L. Identifying these curves with their strict transforms on the blowup with basepoints $(p_i)_1^9$, we obtain a nodal root

$$\alpha = \left(e_0 - \sum_1^3 e_i\right) + \left(3e_0 - \sum_1^9 e_i\right) = (4, -2^3, -1^6)$$

represented uniquely by the effective divisor L + E. Since E is irrational, α is not geometric.

Theorem **5.3.** — *If* (p_i) *has three collinear points, then* (S, ϕ) *has a geometric nodal root.*

Proof. — Let L be a line passing through three or more of the basepoints (p_i). After reordering, we can assume the points on L are $(p_1, \ldots, p_k)$. Then the strict transform C of L gives a smooth rational curve on S with $[C] = [H - \sum_1^k E_i]$, and thus $[C + \sum_4^k E_i] = \phi(\alpha_{123})$. □

Theorem **5.4.** — *If* (S, ϕ) *has no geometric nodal roots, then* $W(S, \phi) = W_n$.

Proof. — If (S, ϕ) has no geometric nodal roots and $w \in W(S, \phi)$, then $(S, \phi \circ w)$ also has no geometric nodal roots. Thus it suffices to show the generators of W_n belong to $W(S, \phi)$. This is immediate by Equation (5.3) for the transpositions $\tau_{12}, \tau_{23}, \ldots$, and for κ_{123} it follows from the preceding two results. □

Corollary **5.5.** — *A marked surface has a nodal root iff it has a geometric nodal root.*

Proof. — Suppose (S, ϕ) has no geometric nodal roots, then $(S, \phi \circ w)$ is a marked blowup for all $w \in W_n$. If $[D] = \phi(\alpha)$ is a nodal root, then D maps to a plane curve of positive degree under each corresponding projection $S \to \mathbf{P}^2$, and thus

$$0 < w(e_0) \cdot \alpha = e_0 \cdot w^{-1}(\alpha)$$

for all $w \in W_n$. Taking w to be reflection through α yields a contradiction. □

Realization. — Let

$$\mathrm{Aut}(S, \phi) = \{w \in W(S, \phi) : (S, \phi) \cong (S, \phi \circ w)\}.$$

There is a natural surjection $\mathrm{Aut}(S) \to \mathrm{Aut}(S, \phi)$. Shifting focus, we say $w \in W_n$ is *realized* on (S, ϕ) if $w \in \mathrm{Aut}(S, \phi)$. In this case there is an $F \in \mathrm{Aut}(S)$ making the diagram

$$\begin{array}{ccc} \mathbf{Z}^{1,n} & \xrightarrow{w} & \mathbf{Z}^{1,n} \\ \downarrow \phi & & \downarrow \phi \\ H^2(S, \mathbf{Z}) & \xrightarrow{F_*} & H^2(S, \mathbf{Z}) \end{array}$$

commute. The map F is unique so long there are 4 points in general position among the $(p_i)_1^n$.

Note that $F : S \to S$ covers a unique birational map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$, yielding a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{F} & S & = & S \\ \downarrow \pi & & \pi' \downarrow & & \downarrow \pi \\ \mathbf{P}^2 & = & \mathbf{P}^2 & \overset{f}{\dashrightarrow} & \mathbf{P}^2. \end{array} \tag{5.5}$$

Here π and π' correspond to the markings ϕ and $\phi \circ w$. This diagram is a combination of (5.1) and (5.2); the right square indicates that ϕ and $\phi \circ w$ are both marked blowups, and the left square indicates they are isomorphic.

Examples.

1. The whole Weyl group W_4 is realized by automorphisms when the basepoints $(p_i)_1^4$ are in general position (because $\mathrm{Aut}\,\mathbf{P}^2$ acts transitively on such configurations of points).
2. On the other hand, there is no realization of $\kappa_{123} \in W_8$. Indeed, any realization would give a quadratic Cremona involution $f : \mathbf{P}^2 \to \mathbf{P}^2$ based at (p_1, p_2, p_3) and fixing $(p_4, \ldots, p_8)$; but a quadratic involution has only 4 fixed points.

3. Let $K_9 \subset W_9$ be the kernel of the natural map to $W_8/(\pm I)$, and let S be the elliptic surface obtained by blowing up the 9 basepoints of a generic pencil of cubics. Then every element $w \in K_9$ is realized on S; indeed, we have $K_9 \cong \mathrm{Aut}(S) \cong \mathbf{Z}^8 \ltimes (\mathbf{Z}/2)$. See [Cob, §52], [Giz], [DO, p. 106].
4. Let $W_{10}(2)$ denote the congruence subgroup of W_{10} that acts trivially on $L_{10}/2L_{10}$, and let S be a surface obtained by blowing up the 10 nodes of a generic rational sextic in the plane. Then every element of $W_{10}(2)$ is realized on S [Cob, §52].

Fixed point formulation. — Let $\mathscr{P}_n^* \subset \mathscr{P}_n$ denote the moduli space of marked blowups without nodal roots. Then W_n acts on $\mathscr{P}_n^*$ by Theorem 5.4, and:

The fixed points of $w \in W_n$ in $\mathscr{P}_n^*$ correspond to the surfaces on which it is realized.

In particular the question of whether or not w can be realized by a surface automorphism depends only on its conjugacy class, provided we restrict attention to surfaces without nodal roots.

6. Synthesis

In this section we observe that a marked cubic curve determines a marked blowup S with a distinguished anticanonical curve Y. Their automorphism groups are related by the following two results.

Theorem **6.1.** — *Let* (S, Y, ϕ) *be the marked pair obtained by blowing up* (X, ρ). *Then we have* $\mathrm{Aut}(S, Y, \phi) = \mathrm{Aut}(X, \rho) \cap W(S, \phi)$.

Theorem **6.2.** — *If* X *is irreducible and* $0 \notin \rho(\Phi_n)$, *then* (S, ϕ) *has no nodal roots and*

$$\mathrm{Aut}(S, \phi) \supset \mathrm{Aut}(X, \rho).$$

If $n \geq 10$ *then equality holds.*

These results *linearize* the problem of constructing surface automorphisms, by reducing it to the study of maps $\rho : \mathbf{Z}^{1,n} \to \mathrm{Pic}(X)$.

Marked pairs. — Let (S, ϕ) be a marked blowup. An *anticanonical curve* is a reduced curve $Y \subset S$ whose cohomology class in $H^2(S, \mathbf{Z})$ satisfies

$$[Y] = \left[3H - \sum E_i\right] = -K_S. \qquad (\mathbf{6.1})$$

A *marked pair* (S, Y, ϕ) is a marked blowup with a distinguished anticanonical curve. An *isomorphism* between marked pairs (S, Y, ϕ) and (S', Y', ϕ') is a biholomorphic map $F : S \to S'$, compatible with markings, that sends Y to Y′.

Proposition **6.3.** — *If* $n \geq 10$, *then* S *carries at most one irreducible anticanonical curve* Y.

Proof. — In this case we have $Y^2 = 9 - n < 0$. $\square$

From surfaces to cubics. — Let $\pi : S \to \mathbf{P}^2$ be a projection compatible with ϕ, presenting S are a blowup with basepoints $(p_i)_1^n$. Then (6.1) implies that

$$X = \pi(Y) \subset \mathbf{P}^2$$

is a cubic curve, passing through each basepoints p_i with multiplicity one. Moreover, the fact that $E_i \cdot Y = 1$ implies $\pi : Y \to X$ is an isomorphism. Combining the identification $H^2(S, \mathbf{Z}) = \mathrm{Pic}(S)$ with the restriction map $r : \mathrm{Pic}(S) \to \mathrm{Pic}(Y)$, we obtain a natural marking

$$\rho : \mathbf{Z}^{1,n} \xrightarrow{\phi} H^2(S, \mathbf{Z}) = \mathrm{Pic}(S) \xrightarrow{r} \mathrm{Pic}(Y) \xrightarrow{\pi_*} \mathrm{Pic}(X).$$

Thus a marked pair (S, Y, ϕ) canonically determines a marked cubic curve (X, ρ).

From cubics to surfaces. — Conversely, let (X, ρ) be a marked cubic curve. Then we have basepoints $p_i \in X$ determined by $\rho(e_i)_1^n$, and an embedding $X \subset \mathbf{P}^2$ determined by $\rho(e_0)$. Letting (S, ϕ) be the marked blowup with basepoints $p_i \in \mathbf{P}^2$, and $Y \subset S$ the strict transform of X, we obtain a marked pair

$$(S, Y, \phi) = \mathrm{Bl}(X, \rho),$$

which we call the *blowup* of (X, ρ). It is easy to see that this functorial construction inverts the preceding one; summing up, we have:

Theorem **6.4.** — *The functor* $(X, \rho) \mapsto (S, Y, \phi) = \mathrm{Bl}(X, \rho)$ *establishes an equivalence between the category of marked cubic curves and the category of marked pairs.*

Volume. — Since Y is an anticanonical curve, there is a meromorphic $(2, 0)$ form η on S, unique up to scale, with a simple pole along Y and no other poles or zeros.

The form η determines a natural volume measure

$$\mathrm{vol}(U) = \int_U \eta \wedge \overline{\eta},$$

locally finite on $S - Y$ but of infinite total mass.

Determinant. — Let $\mathrm{Aut}(S, Y)$ denote the group of automorphisms of S stabilizing Y. The forms η and $F^*\eta$ are proportional for any $F \in \mathrm{Aut}(S, Y)$, and thus we have a natural homomorphism

$$\delta : \mathrm{Aut}(S, Y) \to \mathbf{C}^*$$

characterized by

$$F^*\eta = \delta(F) \cdot \eta.$$

We call $\delta(F)$ the *determinant* of F, since $\delta(F) = \det DF_p$ for all $p \in S - Y$ fixed by F.

Note that every $F \in \mathrm{Aut}(S, Y)$ covers a birational map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ stabilizing X.

Theorem **6.5.** — *For any* $F \in \mathrm{Aut}(S, Y)$*, we have* $D(f|X) = \delta(F)$.

Proof. — The Poincaré residue map [GH, p. 500] sends η to a nonzero 1-form $\omega \in \Omega(Y) \cong \Omega(X)$, satisfying $F^*\eta/\eta = f^*\omega/\omega$. □

Realizations. — An element $w \in W_n$ is realized by $F \in \mathrm{Aut}(S, Y)$ if $\phi \circ w = F_* \circ \phi$ on $\mathbf{Z}^{1,n}$. Let

$$\mathrm{Aut}(S, Y, \phi) \subset \mathrm{Aut}(S, \phi) \subset W_n$$

be the group of elements so realized. We can now show that $\mathrm{Aut}(S, Y, \phi) = \mathrm{Aut}(X, \rho) \cap W(S, \phi)$.

Proof of Theorem 6.1. — Let $w \in W_n$ lie in the intersection of the groups on the right. Since $w \in W(S, \phi)$, there is a projection $\pi' : S \to \mathbf{P}^2$ corresponding to the marking $\phi' = \phi \circ \rho$. The corresponding basepoints p_i' lie on the cubic curve

$$X' = \pi' \circ \iota(X) \subset \mathbf{P}^2,$$

furnishing it with a marking ρ'. By construction, (X', ρ') is isomorphic to $(X, \rho \circ w)$. Since $w \in \mathrm{Aut}(X, \rho)$, there is an automorphism $g : \mathbf{P}^2 \to \mathbf{P}^2$ sending X to X′ and p_i to p_i'. This map is covered by a unique $F \in \mathrm{Aut}(S)$ realizing w. Since $g \circ \pi = \pi' \circ F$, we also have $F(Y) = Y$, and thus $w \in \mathrm{Aut}(S, Y, \phi)$.

For the reverse inclusion, observe that any $F \in \mathrm{Aut}(S, Y)$ covers a birational map with $f|X \in \mathrm{Aut}(X)$. □

A similar argument appears in [Ha1, Cor. 4.4].

The irreducible case. — Let us specialize the preceding discussion to the case where X is irreducible. We first remark:

Theorem **6.6.** — *If* $0 \notin \rho_0(\Phi_n)$*, then* (S, ϕ) *has no nodal roots.*

Proof. — By Corollary 5.5, if (S, ϕ) has a nodal root then it also has a root α represented by an effective sum D of smooth rational curves. By irreducibility, the strict transform Y of X is either a singular rational curve or a smooth elliptic curve (recall we only blowup smooth points of X), so it is not contained in D. Thus

$$\rho(\alpha) = [D \cap Y] \in \mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$$

is a line bundle of degree zero represented by an effective divisor, so it is trivial. □

Proof of Theorem 6.2. — Once there are no nodal roots, we have $W(S, \phi) = W_n$, and thus $\mathrm{Aut}(S, Y, \phi) = \mathrm{Aut}(X, \rho)$ by Theorem 6.1; and Proposition 6.3 implies $\mathrm{Aut}(S, Y, \phi) = \mathrm{Aut}(S, \phi)$ when $n \geq 10$. □

Comparison to K3 surfaces. — The surface $S - Y$ behaves in many ways like a K3 surface, with its canonical bundle trivialized by η. One can similarly regard Theorem 6.4 as an elementary Torelli theorem, stating that (S, Y) is determined by its 'periods' $H^2(S, \mathbf{Z}) \to \mathrm{Pic}(Y)$. Compare [Lo], [Ha2].

7. Cuspidal cubics

In this section we will establish:

Theorem **7.1.** — *Suppose λ is an eigenvalue of $w \in W_n$, λ is not a root of unity and w has no periodic roots. Then there is a unique marked pair* (S, Y, ϕ) *and* $F \in \mathrm{Aut}(S, Y)$ *such that* Y *is irreducible,* $\delta(F) = \lambda$ *and* F *realizes* w.
Moreover $\langle F \rangle$ *has finite index in the full group* $\mathrm{Aut}(S)$.

The surface S is constructed explicitly by blowing up the points $(x_i, x_i^3)_1^n$ on the cuspidal cubic $y^3 = x$ in $\mathbf{C}^2$, where $x_i = -v_i - v_0/3$ and $w(v) = \lambda^{-1} v$.

More generally, let $\mathbf{C}^{1,n} = \mathbf{Z}^{1,n} \otimes \mathbf{C}$ with the complex bilinear Minkowski form, and let

$$(\mathbf{C}^{1,n})^* = \left\{ v \in \mathbf{C}^{1,n} : 0 \notin v \cdot \Phi_n \right\}.$$

For each $v \in (\mathbf{C}^{1,n})^*$, we define

$$W_n^v = \left\{ w \in W_n : [v] \in \mathbf{C}^{1,n}/\mathbf{C} k_n \text{ is an eigenvector for } w \right\}.$$

(This group includes all w for which v itself is an eigenvector; the formulation above allows for a uniform treatment of the case $n = 9$.) We will construct a marked blowup along a cuspidal cubic for every $v \in (\mathbf{C}^{1,n})^*$, and establish:

Theorem **7.2.** — *For every* $v \in (\mathbf{C}^{1,n})^*$, *we have* $\mathrm{Aut}(S^v, Y^v, \phi^v) = W_n^v$.

Corollary **7.3.** — *If* $n \geq 10$, *we have* $\mathrm{Aut}(S^v) \cong W_n^v$.

Corollary **7.4.** — *A Coxeter element* $w \in W_n$ *is realized on* (S^v, ϕ^v) *whenever* v *is a leading eigenvector for* w *and* $n \neq 9$.

These corollaries are immediate from Theorems 6.2 and 2.6.

Structure of the automorphism group. — It is easy to see, as in the proof of Theorem 7.1, that W_n^v is virtually abelian whenever v is timelike or null ($v \cdot v \geq 0$). This is also true for $n = 10$ by [Bor, Thm. 3.9.1]. (I am grateful to D. Allcock for this reference.)

Question. — Is the group W_n^v, $v \in (\mathbf{C}^{1,n})^*$, always virtually abelian?

Blowups along a cuspidal cubic. — Let $X^* \subset \mathbf{C}^2 \subset \mathbf{P}^2$ be the smooth locus of the cuspidal cubic defined by $y = x^3$, parameterized by $p(t) = (t, t^3)$. Given $v \in (\mathbf{C}^{1,n})^*$, define $(t_i)_0^n$ and $(p_i)_0^n$ by

$$\begin{aligned} 3t_0 &= v \cdot e_0, \\ t_i &= v \cdot e_i, \quad i > 0, \quad \text{and} \\ p_i &= p(t_i - t_0). \end{aligned}$$

Then $p_i = (x_i, x_i^3)$ with $x_i = -v_i - v_0/3$ as above. The condition $0 \notin v \cdot \Phi_n$ implies

$$v \cdot \alpha_{ij} = v \cdot (e_i - e_j) = t_i - t_j \neq 0,$$

so the points $(p_i)_1^n$ are distinct. Thus the given embedding $X \subset \mathbf{P}^2$, together with the basepoints $(p_i)_1^n$, determines a marking $\rho^v : \mathbf{Z}^{1,n} \to \mathrm{Pic}(X)$.

We let (S^v, ϕ^v) denote the marked blowup of $\mathbf{P}^2$ at the basepoints of (X, ρ^v), and $Y^v \subset S^v$ the strict transform of X. Note that (S^v, ϕ^v) only depends on the location of v in the quotient $\mathbf{C}^{1,n}/\mathbf{C}k_n$.

Theorem **7.5.** — *The marking homomorphism* $\rho_0^v : L_n \to \mathrm{Pic}_0(X)$ *is given by*

$$\rho_0^v(u) = (u \cdot v)D,$$

where $D = [p(1) - p(0)]$.

Proof. — Identify $\mathrm{Pic}_0(X)$ with $\mathbf{C}$ by $\sum n_j p(s_j) \mapsto \sum n_j s_j$; then $D = 1$. Let $u = de_0 + \sum_1^n m_i e_i$ be an element of L_n; then $3d + \sum m_i = 0$. Since p_0 is the unique flex of X, we have $\rho^v(e_0) = [3p_0]$, and hence:

$$\begin{aligned} \rho^v(u) &= \left[3dp_0 + \sum m_i p_i\right] = \left[3dp(0) + \sum m_i p(t_i - t_0)\right] \\ &= \left(\sum m_i t_i\right) - t_0 \left(\sum m_i\right) = 3dt_0 + \sum m_i t_i = u \cdot v. \end{aligned}$$

□

Corollary **7.6.** — *We have* $(X, \rho^v \circ w^{-1}) \cong (X, \rho^{w(v)})$.

Proof. — By Theorem 4.3, ρ_0 determines (X, ρ) up to isomorphism, and we have $\rho_0^v \circ w^{-1} = \rho_0^{w(v)}$ because $w^{-1}(u) \cdot v = u \cdot w(v)$. □

Corollary **7.7.** — *We have* $(X, \rho^a) \cong (X, \rho^b)$ *iff* $a = \lambda b + \mu k_n$ *for some* $\lambda, \mu \in \mathbf{C}$.

Proof. — An isomorphism is given by an $f \in \mathrm{Aut}(X)$ satisfying $f_* \circ \rho_0^a = \rho_0^b$. Since $\mathrm{Aut}(X)$ acts on $\mathrm{Pic}_0(X) \cong \mathbf{C}$ by scalar multiplication, the result follows. □

Proof of Theorem 7.2. — The assumption $v \in (\mathbf{C}^{1,n})^*$ implies $0 \notin (v \cdot \Phi_n)D = \rho^v(\Phi_n)$, and thus $\mathrm{Aut}(S^v, Y^v, \phi^v) = \mathrm{Aut}(X, \rho^v)$ by Theorem 6.2. By the preceding observations, $w \in \mathrm{Aut}(X, \rho^v)$ if and only $[v]$ is an eigenvector for w on $\mathbf{C}^{1,n}/\mathbf{C}k_n$. □

Proof of Theorem 7.1. — Since λ is an eigenvalue of w, so is λ^{-1}. Let $v \in \mathbf{C}^{1,n}$ be a corresponding eigenvector. By Theorem 2.7, we have $v \in (\mathbf{C}^{1,n})^*$ and thus w is realized by an automorphism $F \in \mathrm{Aut}(S^v, Y^v)$, covering a birational map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$. This realization satisfies

$$D(f)\rho_0^v = f_*\rho_0^v = \rho_0^v \circ w = \rho_0^{w^{-1}(v)} = \rho_0^{\lambda v} = \lambda \rho_0^v,$$

and thus $\delta(F) = D(f) = \lambda$ as required.

For uniqueness, observe that we have $\mathrm{Pic}_0(Y) \cong \mathbf{C}$ by Proposition 4.2, and thus $X = \pi(Y)$ is a cuspidal cubic, by irreducibility of Y. Hence (S, Y, ϕ) has the form (S^u, Y^u, ϕ^u) for some u satisfying $w(u) = \lambda^{-1}u$. Since w preserves a form of signature $(1, n)$, any eigenvalue which is not a root of unity has multiplicity one; thus u is proportional to v, and hence $(S, Y, \phi) \cong (S^v, Y^v, \phi^v)$.

To analyze the full group $\mathrm{Aut}(S) \cong W_n^v$, we note that $\sigma(w) > 1$ and thus w acts by translation along a geodesic γ in the hyperbolic space $\mathbf{H}^n \subset \mathbf{PR}^{1,n}$. The Galois conjugates of $[v]$ include one of the endpoints $[u]$ of $\gamma \subset \overline{\mathbf{H}^n}$. It then follows readily from discreteness of W_n that $W_n^v = W_n^u$ stabilizes γ, so W_n^v is a finite extension of $\langle w \rangle$. Consequently $\mathrm{Aut}(S)$ is a finite extension of $\langle F \rangle$. □

Modular perspective. — The discussion above can be summarized on the level of moduli spaces as follows: there is a W_n-equivariant map

$$\mathbf{P}(\mathbf{C}^{1,n})^* \cong M_n(X)^* \to \mathscr{P}_n^* \subset \mathscr{P}_n,$$

given by $[v] \mapsto (S^v, \phi^v)$, and hence fixed points (eigenvectors) for w in $\mathbf{P}(\mathbf{C}^{1,n})^*$ furnish fixed points (realizations) for w in $\mathscr{P}_n^*$.

Nodal cubics. — Parallel results can be formulated for other irreducible cubics. For example, let X be the nodal cubic $y^2 = 4x^2(x - 1)$, parameterized by

$p(t) = (q(t), q'(t))$ where $q(t) = 1/\sin^2(t)$. Let (S^v, ϕ^v) be the marked blowup with basepoints $p_i = p(v \cdot (e_i - e_0/3))$. This corresponds to the marking $\rho_0^v : \mathbf{Z}^{1,n} \to \mathrm{Pic}_0(X) \cong \mathbf{C}^*$ given by

$$\rho_0^v(u) = \exp(2iu \cdot v).$$

Using the fact that Aut(X) acts on $\mathrm{Pic}_0(X) \cong \mathbf{C}^*$ by $z \mapsto z^{\pm 1}$, Theorem 6.2 readily implies:

Theorem **7.8.** — *If* $\rho_0^v \circ w(u) = (\rho_0^v(u))^{\pm 1}$, *and* $1 \notin \exp(2iv \cdot \Phi_n)$, *then* w *is realized on* (S^v, ϕ^v).

The case of an elliptic curve is similar.

8. Reducible cubics

When $n \geq 8$, any n points on a reducible cubic curve satisfy a nodal relation. Nevertheless, some interesting automorphisms can be realized in this configuration. In this section we give two such constructions: one for three lines through a point, and the other for a conic with a tangent line. These are the reducible cubics with $\mathrm{Pic}_0(X) \cong \mathbf{C}$.

Standard Coxeter element. — Let $\pi_n = s_1 s_2 \cdots s_{n-1} \in \Sigma_n \subset W_n$ be the cyclic permutation $(123...n)$. We will study realizations of the *standard* Coxeter element, defined by

$$w = \pi_n \circ \kappa_{123} \in W_n.$$

This element satisfies

$$\begin{aligned} w(e_0) &= 2e_0 - e_2 - e_3 - e_4, \\ w(e_1) &= e_0 - e_3 - e_4, \\ w(e_2) &= e_0 - e_2 - e_4, \\ w(e_3) &= e_0 - e_2 - e_3, \end{aligned} \tag{8.1}$$

$w(e_i) = e_{i+1}$ for $4 \leq i < n$, and $w(e_n) = e_1$.

I. *Three lines through a point.* — Let $X_j \subset \mathbf{P}^2$ be the line defined in affine coordinates (x, y) by $y = j$, and let $X = X_1 \cup X_2 \cup X_3$. Then X is a reducible cubic, consisting of three lines meeting in a single point at infinity. Their fundamental classes determine a natural basis $([X_j])_1^3$ for $H^2(X, \mathbf{Z}) \cong \mathbf{Z}^3$.

Assume $n \equiv 0 \mod 3$. We will show the standard Coxeter element can be realized by an automorphism that cyclically permutes the irreducible components of X.

Let $[i] \in \{1, 2, 3\}$ denote the residue class of $i \mod 3$. Define $R : \mathbf{Z}^{1,n} \to H^2(X, \mathbf{Z})$ by

$$R(e_i) = \begin{cases} [X_1] + [X_2] + [X_3] & \text{if } i = 0, \text{ and} \\ [X_{[i]}] & \text{if } i \geq 1. \end{cases}$$

Then R transports the action of the Coxeter element $w|\mathbf{Z}^{1,n}$ to the order 3 automorphism of $H^2(X, \mathbf{Z})$ that sends $[X_i]$ to $[X_{[i+1]}]$.

Given $v \in \mathbf{C}^{1,n}$, let $t_i = v \cdot e_i$ and define $(p_i)_1^n$ in (x, y)-coordinates by

$$p_i = \begin{cases} (t_i - t_0, 1) & \text{if } [i] = 1, \\ (-t_i/2, 2) & \text{if } [i] = 2, \text{ and} \\ (t_i, 3) & \text{if } [i] = 3. \end{cases}$$

Assuming the points p_i are distinct, we then obtain a marked cubic (X, ρ^v).

Theorem **8.1.** — *The marked cubic* (X, ρ^v) *satisfies* $\deg(\rho^v(u)) = R(u)$ *and* $\rho_0^v(u) = u \cdot v$, *for a suitable choice of isomorphism* $\text{Pic}_0(X) \cong \mathbf{C}$.

Proof. — Observing that a transverse line meets X_j in a single point, we have $\deg(\rho^v(e_0)) = [X_1]+[X_2]+[X_3] = R(e_0)$; and similarly, $\deg(\rho^v(e_i)) = \deg([p_i]) = R(e_i)$ for $i \geq 1$. This shows $\deg(\rho^v(u)) = R(u)$.

For the second part, note that if $(x_1, 1)$, $(x_2, 2)$ and $(x_3, 3)$ lie on a line, then $x_1 - 2x_2 + x_3 = 0$. Thus we have an isomorphism $\text{Pic}_0(X) \cong \mathbf{C}$ given by

$$\sum n_k(x_k, y_k) \mapsto \Big(\sum_{y_k=1} n_k x_k\Big) - 2\Big(\sum_{y_k=2} n_k x_k\Big) + \Big(\sum_{y_k=3} n_k x_k\Big).$$

Note also that if $u = de_0 + \sum m_i e_i$ and $\deg \rho(u) = 0$, then $d + \sum_{[i]=1} m_i = 0$. Hence

$$\begin{aligned} \rho_0^v(u) &= \sum_{[i]=1} m_i(t_i - t_0) - 2\sum_{[i]=2} m_i(-t_i/2) + \sum_{[i]=3} m_i t_i \\ &= dt_0 + \sum_1^n m_i t_i = u \cdot v, \end{aligned}$$

as desired. □

Realizations. — Let (S^v, ϕ^v) be the marked blowup determined by (X, ρ^v), and let Y^v be the strict transform of X.

Theorem **8.2.** — *Let v be a leading eigenvector for $w = \pi_n \circ \kappa_{123}$, where $n \equiv 0 \mod 3$ and $n \neq 9$. Then w is realized by an automorphism on the marked pair* (S^v, Y^v, ϕ^v).

Proof. — It suffices, by Theorem 6.1, to show that (i) (X, ρ^v) is a marked cubic, (ii) $w \in \mathrm{Aut}(X, \rho^v)$, and (iii) $w \in W(S^v, \phi^v)$.

Let v be a leading eigenvector for w; then $0 \notin v \cdot \Phi_n$, by Theorem 2.6. Hence $\rho_0^v(e_i - e_j) \neq 0$ for $i > j \geq 1$; thus the points $(p_i)_1^n$ are distinct, and we have (i).

Next, consider the automorphism of X given by $f(x, j) = (\lambda x, [j+1])$, where $w(v) = \lambda^{-1} v$. Since f cyclically permutes the lines (X_1, X_2, X_3), we have

$$\deg \circ \rho^v \circ w = R \circ w = f_* \circ \rho^v;$$

and since f acts on $\mathrm{Pic}_0(X) \cong \mathbf{C}$ by multiplication by λ, we have

$$\rho_0^v \circ w = \rho_0^{w(v)} = \lambda \rho_0^v = f_* \circ \rho_0^v.$$

Hence Theorem 4.3 implies $(X, \rho^v \circ w) \cong (X, f_* \rho^v) \cong (X, \rho^v)$; thus $w \in \mathrm{Aut}(X, \rho^v)$, and we have (ii).

For (iii), suppose $p_k \in \overline{p_i p_j}$ for three distinct indices (i, j, k), with $i, j \in \{1, 2, 3\}$. Then there is exactly one point on each of the three lines X_1, X_2, X_3. For these three points to be collinear, the divisor $[p_i + p_j + p_k]$ must be linearly equivalent to the hyperplane section, which implies $\rho_0^v(\kappa_{ijk}) = 0$. But we have seen that $0 \notin v \cdot \Phi_n$, so no such triple (i, j, k) exists; consequently $\kappa_{123} \in W(S^v, \phi^v)$ by Theorem 5.2. The permutation π_n simply reorders the basepoints (cf. Equation (5.3)), so we also have $w = \pi_n \circ \kappa_{123} \in W(S^v, \phi^v)$, as desired. □

Theorem **8.3.** — *The standard Coxeter element $w = \pi_9 \circ \kappa_{123}$ is realized on (S^v, ϕ^v) whenever $w(v) = \lambda v$, $v \neq 0$, and λ is a primitive 5th root of unity.*

Proof. — One can check that the eigenvector has the form $v = (v_0, v_1, \ldots, v_9)$ with $v_1 = v_5$ and $v_4 = v_9$, but no other coincident entries. Since the $1 \neq 5 \mod 3$ and $4 \neq 9 \mod 3$, the corresponding points $(p_i)_1^9$ on X are distinct. Similarly, a direct computation shows the only lines through three basepoints are X_1, X_2, X_3 and $\overline{p_5 p_7 p_9}$. Thus Theorem 5.2 applies again, to show $\kappa_{123} \in W(S^v, \phi^v)$ and hence $w \in \mathrm{Aut}(S^v, \phi^v)$. □

II. *Conic and a tangent line.* — Now let $X = X_1 \cup X_2 \subset \mathbf{P}^2$, where X_2 is the conic $xy = 1$, and X_1 is its tangent line $y = 0$. Their fundamental classes $[X_1]$, $[X_2]$ determine a basis for $H^2(X, \mathbf{Z}) \cong \mathbf{Z}^2$.

Assume n is odd. By a similar argument, we will show the standard Coxeter element can be realized by an automorphism that exchanges X_1 and X_2.

Let $[i] = 2$ if $i \in \{1, 2, 3\}$ or i is even, and let $[i] = 1$ otherwise. Define $R : \mathbf{Z}^{1,n} \to H^2(X, \mathbf{Z})$ by

$$R(e_i) = \begin{cases} [X_1] + 2[X_2] & \text{if } i = 0, \text{ and} \\ [X_{[i]}] & \text{if } i \geq 1. \end{cases}$$

Then R transports the action of the Coxeter element $w|\mathbf{Z}^{1,n}$ to the involution of $H^2(X, \mathbf{Z})$ that exchanges $[X_1]$ and $[X_2]$.

Given $v \in \mathbf{C}^{1,n}$, let $t_i = v \cdot e_i$ and define $(p_i)_1^n$ by

$$p_i = \begin{cases} (t_0 - t_i, 0) & \text{if } [i] = 1, \text{ and} \\ (t_i, 1/t_i) & \text{if } [i] = 2. \end{cases}$$

Assuming the points p_i are distinct, we again obtain a marked cubic (X, ρ^v).

Theorem **8.4.** — *The marked cubic* (X, ρ^v) *satisfies* $\deg(\rho^v(u)) = R(u)$ *and* $\rho_0^v(u) = u \cdot v$, *for a suitable choice of isomorphism* $\mathrm{Pic}_0(X) \cong \mathbf{C}$.

Proof. — Observing that a transverse line meets X_j in j points, we have $\deg(\rho^v(e_0)) = [X_1] + 2[X_2] = R(e_0)$; and similarly, $\deg(\rho^v(e_i)) = \deg([p_i]) = R(e_i)$ for $i \geq 1$. This shows $\deg(\rho^v(u)) = R(u)$.

For the second part, note that if $(a, 0)$, $(b, 1/b)$ and $(c, 1/c)$ lie on a line, then $a = b + c$. Thus we have an isomorphism $\mathrm{Pic}_0(X) \cong \mathbf{C}$ given by

$$\sum n_k(x_k, y_k) \mapsto \left(\sum_{y_k \neq 0} n_k x_k\right) - \left(\sum_{y_k = 0} n_k x_k\right).$$

Note also that if $u = de_0 + \sum m_i e_i$ and $\deg \rho(u) = 0$, then $d + \sum_{[i]=1} m_i = 0$. Hence

$$\rho_0^v(u) = \sum_{[i]=2} m_i t_i - \sum_{[i]=1} m_i(t_0 - t_i) = dt_0 + \sum_1^n m_i t_i = u \cdot v,$$

as desired. □

Realizations. — Letting (S^v, Y^v, ϕ^v) be the marked blowup determined by (X, ρ^v) as before, we can now state:

Theorem **8.5.** — *Assume n is odd and $n \neq 9$. Then the standard Coxeter element $w = \pi_n \circ \kappa_{123}$ belongs to* $\mathrm{Aut}(S^v, Y^v, \phi^v)$ *whenever v is a leading eigenvector for w.*

Theorem **8.6.** — *The standard Coxeter element $w = \pi_9 \circ \kappa_{123}$ is realized on (S^v, ϕ^v) whenever $w(v) = \lambda v$, $v \neq 0$, and λ is a primitive 5th root of unity.*

The proofs follow the same lines as the proofs of Theorems 8.2 and 8.3.

9. Expanding dynamics

In this section we describe the global dynamics of $F \in \mathrm{Aut}(S, Y)$ when the natural volume form $\eta \wedge \overline{\eta}$ is expanded by F.

Fixed points. — Let $F : S \to S$ be an automorphism of a blowup of $\mathbf{P}^2$, preserving an anticanonical curve Y. Let $X = \pi(Y) \subset \mathbf{P}^2$ as usual, and assume $\mathrm{Pic}_0(X) \cong \mathbf{C}$; then X is either a cuspidal cubic, three lines through a point or a conic with a tangent line.

The unique singular point $p \in Y \cong X$ is necessarily fixed by F, and it is straightforward to compute the eigenvalues (λ_1, λ_2) of DF_p in terms of $\delta = \delta(F)$. Indeed, since Y has multiplicity two at p, $F|Y$ determines DF_p, and we find:

$$(\mathbf{9.1}) \qquad (\lambda_1, \lambda_2) = \begin{cases} (\delta^{-2}, \delta^{-3}) & \text{for a cuspidal cubic;} \\ (\epsilon\delta^{-1}, \epsilon^{-1}\delta^{-1}) & \text{for three lines through a point; and} \\ (\epsilon\delta^{-2}, \epsilon^{-1}\delta^{-1}) & \text{for a conic with a tangent line.} \end{cases}$$

Here ϵ is a primitive kth root of unity, with $k = 1, 2$ or 3 depending on the period of $F^*|H^2(X, \mathbf{Z})$.

Expanding maps. — We say $F \in \mathrm{Aut}(S, Y)$ is *expanding* if $|\delta(F)| > 1$. This guarantees $\mathrm{Pic}_0(X) \cong \mathbf{C}$ as assumed above, by Proposition 4.2. By Equation (9.1), F has an attracting fixed point at p.

The *Julia set* $J^+(F) \subset S$ is the smallest closed set such that the forward iterates $\langle F^n, n > 0\rangle$ form a normal family when restricted to $S - J^+(F)$.

Theorem **9.1.** — *If* F *is expanding, then* $J^+(F)$ *has measure zero, and* $F^n(z) \to p$ *for all* $z \in S - J^+(F)$.

Proof. — For concreteness we treat the case where $X \cong Y$ is a cuspidal cubic. The other two cases are similar.

Let U be the open set of $z \in S$ such that $F^n(z) \to p$. We will first show that

$$\mathrm{vol}(S - U) = \int_{S-U} \eta \wedge \overline{\eta} < \infty.$$

Let $x : Y \to \mathbf{P}^1$ be the inverse of the normalization of the singular curve Y, sending Y^* to $\mathbf{C}$ and p to infinity. Composing with a translation, we can assume $F|Y$ has the form $x \mapsto \delta x$; then $x = 0$ gives the unique fixed point $q \in Y^*$. Since $|\delta| > 1$, all other points of Y converge to p under iteration, and thus $Y - \{q\} \subset U$.

We can extend x to a local coordinate system (x, y) on S, in which $q = (0, 0)$, Y is the x-axis, and $\eta = dx\, dy/y$. We assume (x, y) maps a neighborhood of q to the polydisk $\Delta^2 = \{(x, y) : |x|, |y| < 1\}$. Since $F(Y) = Y$, in this coordinate system we have

$$(x', y') = F(x, y) = (\delta x, 0) + O(|y|).$$

Equivalently, there is a $C > 0$ such that $|y'| \leq C|y|$ and $|x' - \delta x| \leq C|y|$.

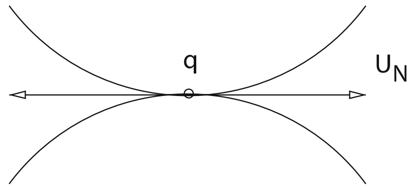

FIG. 3. — The region U_N lies in the basin of p

Consider the region $U_N = \{(x, y) \in \Delta^2 : |y| < |x|^N/N\}$ shown in Figure 3. If $(x, y) \in U_N$ and $(x', y') = F(x, y) \in \Delta^2$, then for N sufficiently large we have

$$|x'| \geq |\delta x| - C|y| \geq |\delta x|(1 - |x|^{N-1}/N) \geq |\delta|^{1/2}|x|. \tag{9.2}$$

Choose N larger still, we can assume $|\delta|^{N/2} > C$; then we have

$$|y'| \leq C|y| \leq C|x|^N/N \leq |x'|^N/N. \tag{9.3}$$

In other words, (x', y') is also in U_N.

Because of (9.2), every point $z \in U_N$ eventually escapes from Δ^2 under iteration, but by (9.3) it remains in U_N until it does so. For N large, the escaping points are very close to Y yet a definite distance from q; thus they lie in U, and consequently $U_N \subset U$.

Since the measure V is locally finite on $S - Y$, and U contains all of Y except q, to show $\mathrm{vol}(S - U)$ is finite it suffices to show the volume V_N of $\Delta^2 - U_N$ with respect to the form $\eta \wedge \overline{\eta}$ is finite. But this is straightforward, since $\int_{\Delta^2} \eta \wedge \overline{\eta}$ is only borderline divergent. In detail, we have:

$$\begin{aligned} V_N = 4 \int_{\Delta^2 - U_N} |dx|^2\, |dy|^2/|y|^2 &= 4\pi \int_{\Delta} (N|y|)^{2/N} |dy|^2/|y|^2 \\ &= 8\pi^2 \int_0^1 (Nr)^{2/N} dr/r < \infty. \end{aligned}$$

Thus $\mathrm{vol}(S - U)$ is finite. The closed set $S - U$ is also invariant under F, so we have

$$\mathrm{vol}(S - U) = \mathrm{vol}(F(S - U)) = |\delta|^2 \,\mathrm{vol}(S - U).$$

Since $|\delta| > 1$, this implies $S - U$ has measure zero. Clearly $J^+(F) \subset S - U$, so the Julia set also has measure zero.

Finally suppose $z \notin J^+(F)$; then $\langle F^n|B\rangle$ forms a normal family on some ball B containing z. Since U has full measure, it must meet B, and hence $F^n|B$ converges to the constant function p; in particular $F^n(z) \to p$, as desired. □

Corollary **9.2.** — *Any* F*-invariant probability measure is supported on* $J^+(F)$*, and hence it is singular with respect to Lebesgue measure.*

In particular the unique measure of maximal entropy for F is singular. For more on the maximal measure, see [BD], [Duj] and references therein.

10. Siegel disks

In this section we show:

Theorem **10.1.** — *For all n sufficiently large with $n \neq 2, 4 \mod 6$, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a Siegel disk.*

The proof rests on a more general statement that also yields concrete examples.

Irrational rotations. — A pair of numbers $\alpha, \beta \in \mathbf{C}^*$ are *multiplicatively independent* if they satisfy no relation of the form $\alpha^i \beta^j = 1$ with $(i, j) \neq (0, 0)$. (In particular, neither α nor β is a root of unity.)

A linear map $R : \mathbf{C}^2 \to \mathbf{C}^2$ is an *irrational rotation* if its eigenvalues α, β lie on the unit circle and are multiplicatively independent. (We exclude rotations that preserve a pencil of curves of the form $x^i = ty^j$.)

Let $F : S \to S$ be a surface automorphism with a fixed point p. A domain $U \subset S$ is a *Siegel disk* for F if $F(U) = U$ and $F|U$ is analytically conjugate to an irrational rotation $R|\Delta^2$. Using results from transcendence theory, one can show [Mc2, Thm. 5.1]:

Theorem **10.2.** — *If* $F(q) = q$ *and* DF_q *is an irrational rotation with algebraic eigenvalues, then* F *has a Siegel disk centered at* q.

Eigenvalues. — Now let $F : S \to S$ be an automorphism of a blowup $\pi : S \to \mathbf{P}^2$, preserving an anticanonical curve Y with $\pi(Y) = X$. We will assume F has the following properties.

1. The determinant $\delta = \delta(F)$ lies on the unit circle, but is not a root of unity.
2. The Lefschetz number $L(F) = 2$; equivalently, $\operatorname{Tr} F^*|H^2(X, \mathbf{C}) = 0$.
3. F has only isolated fixed points.
4. No irreducible component of Y is invariant under F.

These assumptions imply that X is either three lines through a point or a conic with a tangent line; that F fixes the unique singular point $p \in Y$; that F has exactly one other fixed point q; and that $q \notin Y$. We aim to produce a Siegel disk centered at q.

Let (α, β) be the eigenvalues of DF_q. Since $q \notin Y$, we have $\alpha\beta = \det DF_q = \delta$. We also have

$$2 + (\alpha/\beta) + (\beta/\alpha) = T_m(\delta),$$

where $T_m(\delta)$ is a rational function whose form depends on the number of irreducible components m of Y.

Theorem **10.3.** — *The function* $T_m(\delta)$ *is given by*

$$T_m(\delta) = \begin{cases} \delta(1+\delta)^2/(1+\delta+\delta^2)^2 & \textit{when } m = 2, \textit{ and} \\ \delta/(1+\delta)^2 & \textit{when } m = 3. \end{cases}$$

Proof. — Let $(d_1, d_2) = (\det(DF_p), \det(DF_q))$, and let $(t_1, t_2) = (\operatorname{Tr} DF_p, \operatorname{Tr} DF_q)$. By assumption (4) above, the m irreducible components of Y are cyclically permuted by F. Thus ϵ is a primitive mth root of unity in Equation (9.1), and hence

$$(d_1, t_1) = \begin{cases} (\delta^{-3}, -\delta^{-1} - \delta^{-2}) & \text{when } m = 2, \text{ and} \\ (\delta^{-2}, -\delta^{-1}) & \text{when } m = 3. \end{cases}$$

We also have $d_2 = \delta$ as already noted. By the Atiyah–Bott formula [AB, (4.9–4.10)], we also have:

$$L^r(F) = \sum_{s=0}^{4} (-1)^s \operatorname{Tr} F^* | H^{r,s}(S) = \sum_{F(z)=z} \frac{\operatorname{Tr} \wedge^r DF_z}{\det(I - DF_z)};$$

applying this formula with $r = 0$ and $r = 1$, and using the fact that $\det(I - A) = 1 - \operatorname{tr}(A) + \det(A)$, we obtain

$$w_1 + w_2 = 1 \quad \text{and} \quad t_1 w_1 + t_2 w_2 = 0,$$

where $1/w_i = 1 - t_i + d_i$. These relations imply

$$T_m(\delta) = \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{t_2^2}{d_2} = \frac{(t_1 w_1)^2}{d_2(1-w_1)^2},$$

and the formulas stated above then follow. □

Note that $T_m(\delta) \in \mathbf{R}$ when $\delta \in S^1$.

Theorem **10.4.** — *Suppose* $T_m(\delta) \in [0, 4]$ *but* $T_m(\delta') \notin [0, 4]$, *where* $\delta' \in S^1$ *is a Galois conjugate of* δ. *Then* F *has a Siegel disk centered at* q.

Proof. — The eigenvalues of DF_q satisfy $\alpha\beta = \delta$ and $(\alpha/\beta) + (\beta/\alpha) + 2 = T_m(\delta)$; since δ is algebraic, so are α and β. The condition $T_m(\delta) \in [0, 4]$ implies $|\alpha/\beta| = 1$; since $|\alpha\beta| = |\delta| = 1$ as well, the eigenvalues lie on the unit circle.

To check multiplicative independence, let (α', β') be Galois conjugates of (α, β) corresponding to δ'; then $|\alpha'\beta'| = |\delta'| = 1$ but $|\alpha'/\beta'| \neq 1$, because $F(\delta') \notin [0, 4]$. Thus if $\alpha^i \beta^j = 1$, we have $(\alpha')^i(\beta')^j = 1$ and hence $i = j$; but then $(\alpha\beta)^i = \delta^i$ and hence $i = 0$, since δ is not a root of unity.

By Theorem 10.2, F has a Siegel disk centered by q. □

Theorem **10.5.** — *For all n sufficiently large, any Coxeter element* $w \in W_n$ *has a pair of conjugate leading eigenvalues* (δ, δ') *with* $T_m(\delta) \in [0, 4]$ *and* $T_m(\delta') \notin [0, 4]$.

Proof. — Note that $T_2(S^1) = [4/9, \infty]$ and $T_3(S^1) = [1/4, \infty]$. In either case we can find open intervals $I, I' \subset S^1$ such that $T_m(I) \subset [0, 4]$ and $T_m(I') \cap [0, 4] = \emptyset$. Then by the equidistribution Theorem 2.5, for all n sufficiently large the Salem factor $Q_n(t)$ of $\det(tI - w)$ has at least one root $\delta \in I$ and another $\delta' \in I'$. □

Proof of Theorem 10.1. — For any δ as above, the constructions of §8 apply (when n is odd or divisible by three) to yield a realization of the standard Coxeter element with $\delta(F) = \delta$. We have $\mathrm{Tr}(F^*|H^2(S, \mathbf{Z})) = \mathrm{Tr}(w|\mathbf{Z}^{1,n}) = 0$, F cyclically permutes the irreducible components of Y, and F has no curve of fixed points (e.g. by Theorem 11.1 below); thus F has a Siegel disk by Theorem 10.4. □

Remark. — A similar discussion in the setting of K3 surfaces appears in [Mc2].

11. Examples

This section presents some specific examples of automorphisms of rational surfaces, including expanding maps and maps with Siegel disks.

The birational model. — Consider the birational map $f : \mathbf{P}^2 \to \mathbf{P}^2$ given in affine coordinates by

$$f(x, y) = (a, b) + (y, y/x) \qquad (\mathbf{11.1})$$

for some $(a, b) \in \mathbf{C}^2$. This map blows up the vertices of the triangle $\Delta(p_1, p_2, p_3)$, and blows down its sides to yield the triangle $\Delta(p_2, p_3, p_4)$, where $p_1 = (0, 0)$, $p_2 = (\infty, 0)$, $p_3 = (0, \infty)$ and $p_4 = (a, b)$. (The points p_2 and p_3 are on the line at infinity.) See Figure 4.

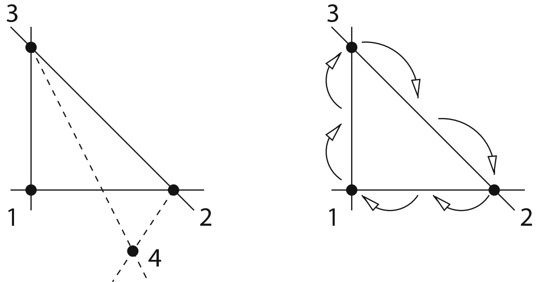

Fig. 4. — Birational dynamics $f : \mathbf{P}^2 \to \mathbf{P}^2$, with p_1, p_2, p_3 and p_4 labeled. The case $p_4 = p_1$, shown at the right, has order 6

When the parameters (a, b) are chosen so that $p_4 = p_1$, the triangle $\Delta(p_1, p_2, p_3)$ is invariant under f. Upon blowing up these three points we obtain a realization of the standard Coxeter element in W_3, which has order six. More generally, defining $p_{i+4} = f^i(p_4)$, we have:

Theorem **11.1.** — *Realizations of the standard Coxeter element in* W_n *correspond to values of* $(a, b) \in \mathbf{C}^2$ *such that*

$$p_i \notin \overline{p_1p_2} \cup \overline{p_2p_3} \cup \overline{p_3p_1}, \quad 4 \le i \le n,$$

and $p_{n+1} = p_1$.

Proof. — Assume the orbit of p_4 cycles as above, and let $\pi : S \to \mathbf{P}^2$ be the marked blowup with basepoints $(p_i)_1^n$. Then certainly f lifts to a morphism $F_0 : S \to \mathbf{P}^2$, since the points of indeterminacy $\{p_1, p_2, p_3\}$ have been replaced by lines. But now every $p_i \in \mathbf{P}^2$ is the image $F_0(L_i)$ of a line in S, so F_0 lifts to an automorphism $F : S \to S$ covering f.

To compute the element $w \in W_n$ realized by F, note that f sends a generic line to a conic through (p_2, p_3, p_4), and thus $w(e_0) = 2e_0 - e_2 - e_3 - e_4$. The point p_1 blows up to the line $\overline{p_3p_4}$, so we have $w(e_1) = e_0 - e_3 - e_4$. By similar reasoning we can compute $w(e_i)$ for the remaining basis elements, and observe that the result agrees with the answer for standard Coxeter transformation (given by Equation (8.1)).

Conversely, if an automorphism of a marked blowup $F : S \to S$ realizes the standard Coxeter transformation $w = \pi_n \circ \kappa_{123}$, we can normalize the basepoints so that $(p_1, p_2, p_3) = ((0, 0), (\infty, 0), (0, \infty))$; then the birational map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ covered by F is a composition of the standard Cremona involution (5.4) with an automorphism sending (p_1, p_2) to (p_2, p_3). Such a map has the form $f(x, y) = (a', b') + (Ay, By/x)$; conjugating by $(x, y) \mapsto (Bx, By/A)$ puts it into the form $f(x, y) = (a, b) + (y, y/x)$. ☐

Corollary **11.2.** — *If* $3 \le n \le 8$ *then* f *is periodic, and its period agrees with the Coxeter number* $h_n = 6, 5, 8, 12, 18$ *or* 30.

The expanding series; $n \ge 10$. — Recall that for each $n \ge 10$, a Coxeter element $w \in W_n$ has a unique eigenvalue $\lambda_n > 1$ outside the unit circle; that λ_n is a Salem number; and that the other eigenvalues of w are conjugates of λ_n or roots of unity (§2).

By §7, for each $n \ge 10$ there is also a unique realization of $w = \pi_n \circ \kappa_{123}$ by an automorphism $F_n : (S_n, Y_n) \to (S_n, Y_n)$ such that Y_n is irreducible and $\delta(F_n) = \lambda_n$. The entropy of this map satisfies

$$h(F_n) = \log \lambda_n,$$

TABLE 5. — The expanding series

n	(a_n, b_n)	$\delta(F) = \lambda_n$	Salem polynomial $Q_n(t)$
10	$(0.49949650, -0.08373582)$	1.17628081	$t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$
11	$(0.58778739, -0.04883156)$	1.23039143	$t^{10} - t^7 - t^5 - t^3 + 1$
12	$(0.64057213, -0.03135002)$	1.26123096	$t^{10} - t^8 - t^5 - t^2 + 1$
13	$(0.67470764, -0.02114197)$	1.28063815	$t^8 - t^5 - t^4 - t^3 + 1$
14	$(0.69769622, -0.01469740)$	1.29348595	$t^{10} - t^8 - t^7 + t^5 - t^3 - t^2 + 1$
15	$(0.71359046, -0.01042890)$	1.30226880	$t^{12} - t^{11} - t^7 + t^6 - t^5 - t - 1$
16	$(0.72478872, -0.00750881)$	1.30840900	$t^{16} + t^{15} - t^{13} - t^{12} - \cdots - t^3 + t + 1$
50	$(0.75487582, -0.00000044)$	1.32471696	$t^{50} + t^{49} - t^{47} - t^{46} - \cdots - t^3 + t + 1$
∞	$(0.75487766, 0)$	1.32471795	$t^3 - t - 1$

since λ_n is the largest eigenvalue of $F_n^*|H^*(S_n)$ (see the Appendix). The basepoints $(p_i)_1^n$ for S_n lie on a cuspidal cubic and are immediately computable from the corresponding eigenvector for w. Indeed, if $w(v) = \lambda_n^{-1} v$, then one can take $p_i = (x_i, x_i^3) \in \mathbf{C}^2$ where $x_i = v_i + v_0/3$, as in §7.

Changing coordinates, one easily obtains the coefficients for the birational map $f_n(x, y) = (a_n, b_n) + (y, y/x)$ covered by F_n. The results for several values of n are summarized in Table 5; the last column gives the Salem polynomial satisfied by λ_n.

As noted in §2 we have $\lambda_n \to \lambda_{\text{Pisot}}$ as $n \to \infty$, and one can similarly check that $(a_n, b_n) \to (a_\infty, b_\infty) = (1/\lambda_{\text{Pisot}}, 0)$. We have included these values in Table 5 under $n = \infty$; they can be regarded as the parameters for an automorphism of a blowup with infinitely many basepoints.

Lehmer's automorphism. — As previously remarked, we have $h(F_{10}) = \log(\lambda_{\text{Lehmer}})$, and thus F_{10} is a surface automorphism with the smallest possible positive entropy. Its behavior over the real projective plane is depicted in Figure 1 of the Introduction.

The reducible series. — The standard Coxeter element can also be realized by an expanding map on a blowup over a reducible cubic when n is odd (using a conic with a tangent line) or $n \equiv 0 \mod 3$ (using three lines through a single point). These realizations, constructed in §8, also satisfy $\delta(F) = \lambda_n$. Examples for $n = 11$ and $n = 12$ are shown in Figure 6; the corresponding parameter values are $(a, b) = (-0.92607569, 0.61173015)$ and $(-2.26123096, 1.79287619)$ respectively.

As in Figure 1, the basepoints $(p_i)_1^n$ are shown as round dots lying on the cubic X (some are outside of the frame of the figure). The points (p_1, p_2, p_3) form the vertices of the central right triangle. The scatter plot gives an approximation to the Julia set $J^+(F)$, obtained by backward iteration of random points in $\mathbf{RP}^2$. By Theorem 9.1 all other points converge, under forward iteration, to the unique singular point $p \in Y$ (the point of tangency when $n = 11$, and the triple point when $n = 12$).

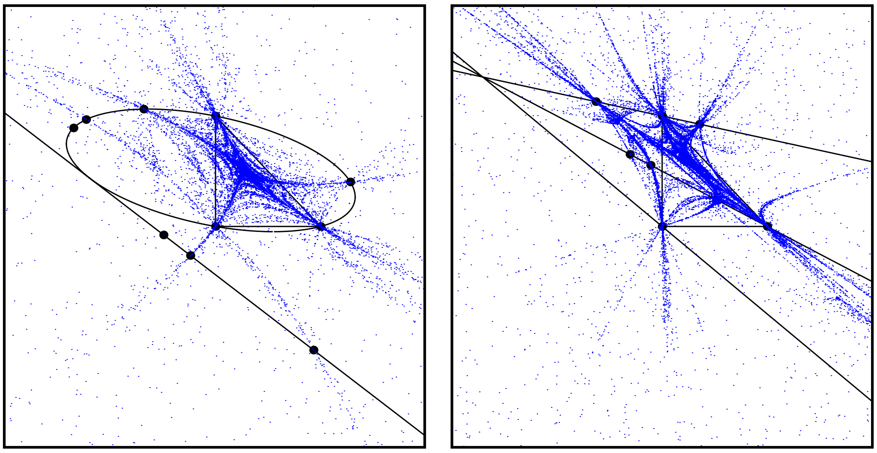

FIG. 6. — Automorphisms preserving a conic with a tangent line ($n = 11$) and three lines through a point ($n = 12$)

$n = 9$: *Pencils of cubics.* — By Theorems 8.3 and 8.6, the standard Coxeter element in W_9 can also be realized by blowing up points along reducible cubics. We obtain four such realizations, one for each choice of a 5th root of unity. The corresponding parameter values are $(a, b) = (a, -\overline{a})$, where a is a root of the polynomial $t^4 + 3t^3 + 4t^2 + 2t + 1$.

It can be verified that the realizations obtained using a conic with a tangent line are the same as those obtained using three lines through a point. Thus there is a pencil of conics through the basepoints $(p_i)_1^9$, generated by these two reducible curves.

We remark that the maps of the form $f(x, y) = (a, -a) + (y, y/x)$ also leave invariant a pencil of conics, and were studied as early as 1945 by Lyness (see e.g. [BR], [PR]). These maps (for $a \neq 0, 1$) realize the standard Coxeter element in W_9 in a generalized sense (it is necessary to blow up infinitely near points on $\mathbf{P}^2$).

Siegel disks. — We conclude with two explicit examples of surface automorphisms with Siegel disks.

The first is obtained using the two roots

$$\delta \approx -0.09996672 + 0.99499078i,$$
$$\delta' \approx -0.63841984 + 0.76968831i$$

of the Salem polynomial $Q_{11}(t)$. Applying the conic and tangent line construction of §8, we obtain a realization of the standard Coxeter in W_{11} by an automorphism with $\delta(F) = \delta$. Since $T_2(\delta) \approx 2.81$ lies in $[0, 4]$ while $T_2(\delta') \approx 9.43$ does not, Theorem 10.4 implies that F has a Siegel disk centered at the unique fixed point $q \notin Y$.

The corresponding birational map is given by $f(x, y) = (a, -\overline{a}) + (y, y/x)$ with $a \approx 0.04443170 - 0.44223856i$. The orbit closure of a typical point near q is a totally real torus; the projection of such an orbit to the real plane is shown in Figure 7.

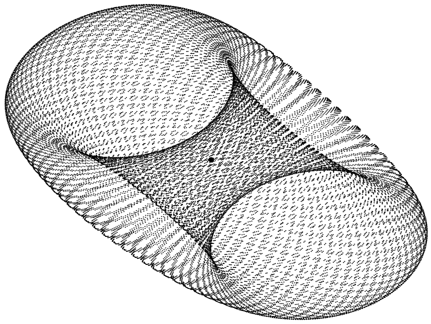

FIG. 7. — An orbit in a Siegel disk

The second example is constructed using the two roots

$$\delta \approx -0.77526329 + 0.63163820i,$$
$$\delta' \approx -0.96079798 + 0.27724941i$$

of $Q_{12}(t)$ to obtain an automorphism F of a marked blowup along three lines through a single point, satisfying $\delta(F) = \delta$. Since $T_3(\delta) \approx 2.22$ while $T_3(\delta') \approx 12.75$, we again obtain a Siegel disk. The corresponding birational map is given by $f(x, y) = (a, -\overline{a}) + (y, y/x)$ with $a \approx -0.22473670 - 0.63163820i$.

Notes and references. — Bedford and Kim determine the possible values of the entropy (or degree growth) for birational maps in the family $f(x, y) = (a, b) + (y, y/x)$ in [BK2]; the values $\log \lambda_n = h(F_n)$ appear as special cases, as do the parameter values for $n = 9$ and $n = 10$ obtained above. Theorem 11.1 and its Corollary explain the occurrence of Coxeter numbers in [BK2, Theorem 2].

12. Minimality

A surface automorphism $F : S \to S$ is *minimal* if for any birational morphism $\pi : S \to S'$ and $F' \in \mathrm{Aut}(S')$ that makes the diagram

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow{\pi} & & \downarrow{\pi} \\ S' & \xrightarrow{F'} & S' \end{array}$$

commute, the map π is an isomorphism. (This agrees with Manin's notion of a G-minimal surface [Man1], where the action of $G \cong \mathbf{Z}$ is generated by F.) In this section we will show:

Theorem **12.1.** — *Suppose $w \in W_n$ has infinite order, no periodic roots, and there is no $v \in W_n(e_1)$ fixed by w. Then any realization of w is minimal.*

Corollary **12.2.** — *Any realization of a Coxeter element $w \in W_n$, $n \geq 10$ is minimal.*

Proof. — We have already seen in §2 that for $n \geq 10$, w has infinite order and no periodic roots. It also follows from Equation (2.4) for $\det(tI - w)$ that the kernel of $(I - w)|\mathbf{Z}^{1,n}$ is rank one, generated by k_n. Since $e_1 \notin \mathbf{Z}k_n$ and W_n fixes k_n, w has no fixed vector in $W_n(e_1)$. □

Iterated blowups. — An *iterated blowup* is a surface S equipped with a sequence of birational morphisms

$$S = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots S_1 \xrightarrow{\pi_1} S_0 = \mathbf{P}^2, \qquad (\mathbf{12.1})$$

such that S_i is the blowup S_{i-1} at a single point $p_i \in S_{i-1}$. Any birational morphism $\pi : S \to \mathbf{P}^2$ can be factored as above.

Let E_i denote the effective divisor $(\pi_i \circ \cdots \circ \pi_n)^{-1}(p_i) \subset S$, and let $H \subset S$ be the preimage of a generic line in $\mathbf{P}^2$. Then the data above determines a natural marking

$$\phi : \mathbf{Z}^{1,n} \to H^2(S, \mathbf{Z})$$

sending e_0 to [H] and e_i to $[E_i]$.

The following two results from [Nag1] and [Nag2] are stated explicitly in [Ha3, Cor 1.2] and [Ha1, Thm. 0.1]; the second is a more general form of Theorem 5.1.

Theorem **12.3.** — *If a rational surface* S *admits an automorphism* F *such that* $F^*|H^2(S, \mathbf{Z})$ *has infinite order, then* S *is an iterated blowup of* $\mathbf{P}^2$.

Theorem **12.4.** — *If both* (S, ϕ) *and* $(S, \phi \circ w)$ *are marked iterated blowups, then* $w \in W_n$.

Proof of Theorem 12.1. — Let F be a realization of w on a marked surface (S, ϕ), and let $\pi : S \to S'$ present $F' \in \mathrm{Aut}(S')$ as a quotient of F. Then by Theorem 12.3 there is a birational morphism $\pi' : S' \to \mathbf{P}^2$. By the structure of birational morphisms [Ha, V.5], each term in the composition $\pi' \circ \pi : S \to \mathbf{P}^2$ can be factored to yield a presentation of S as an iterated blowup (12.1), with $S' = S_k$ for some k. By Theorem 12.4, the corresponding marking of S satisfies $\psi = \phi \circ g$ for some $g \in W_n$.

If $k = n$ then $\pi : S \to S'$ is an isomorphism as desired. If $k = n - 1$, then $\pi : S \to S'$ has a unique exceptional fiber E_n, which must be preserved by F (and mapped by π to a fixed point of F′). Since $[E_n] = \phi \circ g(e_n)$, the vector $g(e_n) \in W_n(e_n) = W_n(e_1)$ is fixed by w, contrary to assumption.

Similarly, if $k < n-1$, then F stabilizes the curve

$$C = E_{k+1} \cup E_{k+2} \cup \cdots \cup E_n$$

consisting of the critical points of π. An iterate F^j, $j > 0$ then stabilizes each irreducible component of C, and hence fixes the class $[E_n]-[E_{n-1}]$. Consequently w^j fixes the root $\alpha = g(e_n - e_{n-1})$, contrary to our assumption that w has no periodic roots. □

We remark that the blowups along cubics considered in this paper, and the blowups along sextics considered by Coble, both produce surfaces with effective pluri-anticanonical divisors. It is natural to ask the following:

Question. — Let $F : S \to S$ be a minimal realization of an element of infinite order in W_n. Then does some negative power of K_S admit a holomorphic section?

A similar question is formulated in [Ha3], which shows the answer is *no* if one drops the hypothesis of minimality.[1]

A. Appendix: Entropy of surface automorphisms

This appendix presents a general lower bound for the entropy of a surface automorphism.

Entropy. — Let $F : S \to S$ be an automorphism of a compact complex manifold, not necessarily projective. When S is a Kähler, results of Gromov and Yomdin show that the topological entropy of F is given simply by the spectral radius of its action on the cohomology [Gr, p. 233]; that is:

$$h(F) = \log \sigma(F^*|H^*(S, \mathbf{C})). \tag{A.1}$$

In the case of dimension two we find:

Theorem **A.1.** — *Let* $F : S \to S$ *be an automorphism of a compact complex surface. Then either* $h(F) = 0$ *or* $h(F) \geq \log \lambda_{\text{Lehmer}} \approx 0.16235761$.

Proof. — Suppose $h(F) > 0$. By [Ca2], a minimal model for S is either a K3 surface, an Enriques surface, an Abelian surface, or a rational surface. In particular, S is Kähler. In the first three cases, F descends to the minimal model and $h(F) = \log \sigma(F^*|H^2(S, \mathbf{C}))$ is the logarithm of a Salem number of degree at most 22 over $\mathbf{Q}$. It is known that λ_{Lehmer} is the smallest such Salem number [FGR], so the desired bound holds.

Now assume S is rational. Then by the results of Nagata from §12, S is an iterated blowup of $\mathbf{P}^2$ and $h(F) = \log \sigma(w)$ for some $w \in W_n$. Since W_n is a Coxeter group, we have $\sigma(w) \geq \lambda_{\text{Lehmer}}$ by [Mc1], completing the proof. □

[1] A negative answer in the minimal case is announced in [BK1].

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