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POLYHEDRA WITH SPECIFIED LINKS

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Abstract

We construct compact polyhedra with m-gonal faces whose links are generalized 3-gons. It gives examples of cocompact hyperbolic bildings of type $P(m,3)$. For $m = 3$ we get compact spaces covered by Euclidean buildings of type $\tilde{A}_2$.

1. Introduction

1.1. Preliminaries

Given a graph G we assign to each edge the length 1. The diameter of the graph is its diameter as a length metric space, its injectivity radius is half of the length of the smallest circuit.

Due to [2], [7] or [9] the following definition is equivalent to the usual one

DEFINITION 1.1. — *For a natural number m we call a connected graph G a generalized m-gon, if its diameter and injectivity radius are both equal to m.*

A graph is *bipartite* if its set of vertices can be partitioned into two disjoint subsets P and L such that no two vertices in the same subset lie on a common edge. Such a graph can be interpreted as a planar geometry, i.e. a set of points P and a set of lines L and an incidence relation $R \subset P \times L$. On the other hand each planar geometry can be considered as a bipartite graph.

Under this correspondence projective planes are the same as generalized 3-gones ([9]).

Let G be a planar geometry. For a line $y \in L$ we denote by $I(y)$ the set of all points $x \in P$ incident to y. If no confusion can arise we shall write $x \in y$ instead of $x \in I(y)$ and $y_1 \cap y_2$ instead of $I(y_1) \cap I(y_2)$. A subset S of P is called collinear if it is contained in some set $I(y)$, i.e. if all points of S are incident to a line.

Given a planar geometry G we shall denote by G' its dual geometry arising by calling lines resp. points of G points resp. lines of G'. The graphs correspomding to G and G' are isomorphic.

We will call a *polyhedron* a two-dimensional complex which is obtained from several oriented p-gons by identification of corresponding sides. Consider a point of the

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polyhedron and take a sphere of a small radius at this point. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

DEFINITION 1.2. — *Let $\mathcal{P}(p, m)$ be a tessellation of the hyperbolic plane by regular polygons with p sides, with angles π/m in each vertex where m is an integer. A* hyperbolic building *of type $\mathcal{P}(p, m)$ is a polygonal complex X, which can be expressed as the union of subcomplexes called apartments such that:*

1. Every apartment is isomorphic to $\mathcal{P}(p, m)$.

2. For any two polygons of X, there is an apartment containing both of them.

3. For any two apartments $A_1, A_2 \in X$ containing the same polygon, there exists an isomorphism $A_1 \to A_2$ fixing $A_1 \cap A_2$.

If we replace in the above definition the tessalition $\mathcal{P}(p, m)$ of the hyperbolic plane by the tessalation $\tilde{A}_2$ of the Euclidean plane by regular triangles we get the definition of the Euclidean building of type A_2.

Let C_p be a polyhedron whose faces are p-gons and whose links are generalized m-gons with $mp > 2m + p$. We equip every face of C_p with the hyperbolic metric such that all sides of the polygons are geodesics and all angles are π/m. Then the universal covering of such a polyhedron is a hyperbolic building, see [6].

In the case $p = 3$, $m = 3$, i.e. C_p is a simplicial polyhedron, we can give a Euclidean metric to every face. In this metric all sides of the triangles are geodesics of the same length. The universal coverings of these polyhedra are Euclidean buildings, see [2], [3], [7].

So, to construct hyperbolic and Euclidean buildings with compact quotients, it is sufficient to construct finite polyhedra with appropriate links.

The main result of the paper is a construction of a family of compact polyhedra with m-gonal faces (for any $m \geqslant 3$) whose links are generalized 3-gons. Fundamental groups of our polyhedra with $m \geqslant 6$ are residually finite by results of [11].

One of the main tools is a bijection T of a special type between points and lines of a finite projective plane G. If such a bijection exists, we can construct a family of compact polyhedra with m-gonal faces, with any $m \geqslant 3$ whose links are generalized 3-gons. The existence of T in known for the projective planes over finite fields of characteristique $\neq 3$ (chapter 3). But for the projective plane of order 3 such a bijection exists as well.

So, if one can prove the existence of T for a finite projective plane G (even non-desarguesian), then chapters 2.2 and 2.3 immediately give the existence of buildings with G as the link.

We note, that some hyperbolic buildings with links, which are finite projective planes were constructed also in [8].

1.2. Polygonal presentation.

We recall the definition of the polygonal presentation, given in [10].

Definition. Suppose we have n disjoint connected bipartite graphs $G_1, G_2, \dots G_n$. Let P_i and L_i be the sets of black and white vertices respectively in G_i, $i = 1, \dots, n$; let $P = \cup P_i$, $L = \cup L_i$, $P_i \cap P_j = \emptyset$ $L_i \cap L_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to L$.

A set $\mathcal{K}$ of k-tuples $(x_1, x_2, \dots, x_k)$, $x_i \in P$, will be called a *polygonal presentation* over P compatible with λ if

(1) $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \dots, x_k, x_1) \in \mathcal{K}$;

(2) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ for some $x_3, \dots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i;

(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

If there exists such $\mathcal{K}$, we will call λ a *basic bijection.*

Polygonal presentations for $n = 1$, $k = 3$ were listed in [5] with the incidence graph of the finite projective plane of order two or three as the graph G_1. Some polygonal presentations for $n > 1$ can be found in [10].

1.3. Construction of polyhedra.

One can associate a polyhedron X on n vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \dots, x_k)$ from the definition we take an oriented k-gon on the boundary of which the word $x_1 x_2 x_3 \dots x_k$ is written. To obtain the polyhedron we identify the sides with the same label of our polygons, respecting orientation. We will say that the polyhedron X *corresponds* to the polygonal presentation $\mathcal{K}$.

The following lemma was proved in [10]:

LEMMA 1.3. — *A polyhedron X which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \dots, G_n$ as the links.*

Remark. Consider a polygonal presentation $\mathcal{K}$. Let s_i be the number of vertices of the graph G_i and t_i be the number of edges of G_i, $i = 1, \dots, n$. If the polyhedron X corresponds to the polygonal presentation $\mathcal{K}$, then X has n vertices (the number of vertices of X is equal to the number of graphs), $k \sum_{i=1}^{n} s_i$ edges and $\sum_{i=1}^{n} t_i$ faces, all faces are polygons with k sides.

2. Main Construction.

2.1. Crucial lemma

Let G be a finite projective plane and let P resp. L denote the set of its points resp. lines.

Assume that a bijection $T : P \to L$ is given and satisfies the following properties

1. For each $x \in P$ the point x and the line $T(x)$ are not incident.
2. For each pair x_1, x_2 of different points in P the points x_1, x_2 and $T(x_1) \cap T(x_2)$ are not collinear.

LEMMA 2.1. — *Let $T : P \to L$ be as above, $y \in L$ a line. Then the map $T^* : I(y) \to I(y)$ given by $T^*(x) = T(x) \cap I(y)$ is a bijection.*

Proof. — By the first property of T the map T^* is well defined, by the second property it must be injective. Since $I(y)$ is finite, the statment follows. □

Let $G, P, L, T : P \to L$ be as above. Let $P = \{x_1, ..x_p\}$ be a labelling of points in P and set $y_i = T(x_i)$. Consider the following set $O \subset P \times P \times P$, consisting of all triples (x_i, x_j, x_k) satisfying $x_i \in y_k$, $x_j \in y_i$ and and $x_j \in y_k$.

Remark. — The conditions on $(x_i, x_j, x_k) \in K$ are not cyclic. We require $x_j \in y_k$ and not $x_k \in y_j$!! For this reason in the polygonal presentations defined below dual graphs of G appear.

The following lemma is crucial for the later construction:

LEMMA 2.2. — *A pair (x_i, x_k) resp. (x_i, x_j) resp. (x_j, x_k) is a part of at most one triple $(x_i, x_j, x_k) \in K$ and such a triple exists iff $x_i \in y_k$ resp. $x_j \in y_i$ resp. $x_j \in y_k$ holds.*

Proof. — The conditions stated at the end are certainly necessairy.

1) Let $x_i \in y_k$ be given. Then y_i and y_k are different and the point $x_j = y_i \cap y_k$ is uniquely defined.

2) Let $x_j \in y_i$ be given. Then x_j and x_i are different, so there is exactly one line y_k containing x_j and x_l.

3) Let $x_j \in y_k$ be given. Then (x_i, x_j, x_k) is in K iff for the map $T^* : I(y_k) \to I(y_k)$ of Lemma 2.1 the equality $T^*(x_i) = x_j$ holds. By Lemma 2.1 the point x_i is uniquely defined. □

2.2. Euclidean polyhedra

Now we are ready for the polygonal presentations. Let the notations be as above, G_1 and G_2 two projective planes with isomorphisms $J^t : G \to G_t$ and G_3 a projective plane with an isomorphism $J^3 : G' \to G_3$ of the dual projective plane G' of G. For $t = 1, 2$ we set $x_i^t = J^t(x_i)$, $y_i^t = J^t(y_i)$ and for $t = 3$ we set $x_i^3 = J^3(y_i)$ and $y_i^3 = J^3(x_i)$.

Let P_t resp. L_t be the set of lines of G_t. For $P = \cup P_t$ and $L = \cup L_t$ we consider the bijection $\lambda : P \to L$ given by $\lambda(x_i^t) = y_i^{t+1}$ ($t + 1$ is taken modulo 3).

Now consider the subset $\mathcal{T}$ of $P \times P \times P$ consisting of all triples (x_i^1, x_j^2, x_k^3) with $(x_i, x_j, x_k) \in K$ and all cyclic permutation of such triples.

The stament of Lemma 2.2 can be now reformulated as:

PROPOSITION 2.3. — *The subset $\mathcal{T}$ of $P \times P \times P$ defines a polygonal presentation compatible with λ.*

The polyhedron X which corresponds to $\mathcal{T}$ by the construction of Lemma 1.3 has triangular faces and exactly three vertices with two links naturally isomorphic to G and one link naturally isomorphic to the dual G' of G. By [2] or [7] the universal covering of X is a Euclidean building.

2.3. Hyperbolic polyhedra

We continue to use the same notation. We have a projective plane G, with points $P = \{x_1, ..., x_p\}$ and lines $L = \{y_1, ..., y_p\}$ and a subset $K \subset P \times P \times P$.

Let $w = z_1 \dots z_n$ be a word of length n in three letters a, b, c with $z_1 = a, z_2 = b, z_3 = c$ that does not contain proper powers of the letters a, b, c. (I.e. $z_z \neq z_{t+1}$ and $z_n \neq a$). For example $w = abcbcab$ is a possible choice.

Set $\mathrm{Sign}(ab) = \mathrm{Sign}(ba) = \mathrm{Sign}(ac) = 1$ and $\mathrm{Sign}(cb) = \mathrm{Sign}(ca) = \mathrm{Sign}(ba) = -1$. For $t = 1, ..., n$ let G_t be isomorphic to G resp. to G' if $\mathrm{Sign}(z_t z_{t+1}) = 1$ resp. $\mathrm{Sign}(z_t z_{t+1}) = -1$.

Fixed isomorphisms induce as above a natural labelling of the points and lines of G_t: $P_t = (x_1^t, ..., x_q^t)$ and $L_t = (y_1^t, ..., y_q^t)$.

For $P = \cup P_t$ and $L = \cup L_t$ we define a basic bijection $\lambda : P \to L$ by $\lambda(x_i^t) = y_i^{t+1}$.

For each triple $(x_i, x_j, x_l) \in K$ we consider the unique n-tuple in P^n such that at the t-th place stands x_i^t resp. x_j^t resp. x_k^t if z_t is equal to a resp. b resp. c. Consider the subset $T_n \in P^n$ of all such tuples together with all their cyclic permutations.

>From Lemma 2.2 we immediatly see:

PROPOSITION 2.4. — *The subset $T_n \in P^n$ is a polygonal presentation over λ. By Lemma 1.3 it defines a polyhedron X whose faces are n-gones and whose n-vertices have as links G resp. G'.*

3. An algebraic construction

Let $F = F_q$ be a finite field of charakteristik $p \neq 3$ with q elements. Consider the field $K = F_{q^3}$ as an extension of F of degree 3. In the sequel we shall denote by g elements of K and by a, b, c elements of F and call them scalars. We denote by Gr_1 resp. Gr_2 the set of 1- resp. 2-dimensional F vector spaces of K.

The multplicative group K^* operates on the sets Gr_1 and Gr_2 by multiplication. The kernel of this operation is precisely F^* and K^*/F^* operates on both sets simply transitively. Especially we can write each element of Gr_1 as gF for some $g \in K^*$.

Let Tr be the trace map $Tr : K \to F$ of the extension $F \subset K$.

Denote by $E \in Gr_2$ the 2-dimensional kernel of $Tr : K \to F$. We define a map $T : Gr_1 \to Gr_2$ by $T(gF) = gE$. The map T is well defined bijective and K^* invariant.

PROPOSITION 3.1 (A.Lytchak, private communication). — *For the map $T : Gr_1 \to Gr_2$ and arbitrary $l \neq l_1 \in Gr_1$ holds:*

1. *The image $T(l)$ does not contain l.*
2. *The l, l_1 and $T(l) \cap T(l_1)$ generate the vector space K.*

Proof. — Since T is K^* invariant, we may assume $l = F$. Since $Tr(1) = 1$, F does not lie in $T(F) = E$. Now assume that $l_1 = gF$. If the statment is wrong, some non zero element of the form $bg - a$ must be in $T(F) \cap T(gF) = E \cap gE$. Since 1 is not in E and G is not in gF, we may assume (replacing g by a scalar multiple) that this non zero element is $g - 1$. So $g - 1 \in E$ and $g - 1 \in gE$.

The first inclusion is equivalent to $Tr(g) = 1$ and the second one to $Tr(\frac{1}{g}) = 1$. Let's prove, that if for an element $g \in K^*$ the equalities $Tr(g) = Tr(\frac{1}{g}) = 1$ hold, then g is equal to 1. Assume $g \neq 1$. Then g is not in F. Let $m(x) = x^3 + ax^2 + bx + c$ be the minimal polynom of g. Then $c \neq 0$ and $\tilde{m}(x) = x^3 + \frac{b}{c}x^2 + \frac{a}{c}x + \frac{1}{c}$ is the minimal polynom of $\frac{1}{g}$. The condition $Tr(g) = Tr(\frac{1}{g}) = 1$ means $a = \frac{b}{c} = -1$. I.e. $m(x) = x^3 - x^2 + bx - b = (x^2 + 1)(x - b)$ is reducible. Contradiction. So, $g = 1$.

Now we get a contradiction to $l \neq l_1$. □

COROLLARY 3.2. — *For the projective plane $\mathcal{P}^2(\mathbb{F}_q)$ over finite field $\mathbb{F}_q$ of charakteristique $\neq 3$ there is a bijection T between the set P of points and the set L of lines, $T : P \to L$, that satisfies the following properties*

1. *For each $x \in P$ the point x and the line $T(x)$ are not incident.*
2. *For each pair x_1, x_2 of different points in P the points x_1, x_2 and $T(x_1) \cap T(x_2)$ are not collinear.*

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