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RIGIDITY OF SYMMETRIC SPACES

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Abstract

In this note we survey the rigidity of symmetric spaces in several view points. Namely we present the marked length rigidity, the rigidity of a geometric flow, the stability of lattices in quaternionic hyperbolic space and some related results.

1. Introduction

In this note, we survey the rigidity of symmetric spaces in various view points. We want to look at the symmetric spaces as Riemannian manifolds and topological spaces, in terms of dynamics and Lie group theory tied with geometry. For Riemannian and representation theoretical point of view, we present the marked length rigidity of Zariski dense subgroups of the isometry groups, for topological point of view, we discuss the geometrically finite manifolds and the geometric flows on the manifolds modelled on the Furstenberg boundary. We also discuss that lattices in quaternionic hyperbolic space cannot be deformed continuously. Along the way, we discuss the Patterson-Sullivan measure, non-arithmeticity of non-elementary groups in rank one symmetric spaces and some properties of limit sets in the product of rank one symmetric spaces.

Some results are published and many of them are either submitted or in preparation.

2. Marked length rigidity

In this section, we announce the following results.

THEOREM 1. — *Let $\rho : G \to X$, $\phi : G \to Y$ be two Zariski dense representations where X and Y are either symmetric spaces of rank one or their product. If they have the same marked length spectrum, i.e., $l(\rho(g)) = l(\phi(g))$ for all $g \in G$, then $X = Y$ and they are conjugate.*

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This theorem is proved in [15] and [16]. The idea of the proof is using the following lemma together with the Carnot-Caratheodory metric on the geometric boundary of rank one symmetric space.

LEMMA 1. — *Let a, b be two hyperbolic isometries in CAT(-1) space X. Let a^-, b^- be the repelling fixed points of a and b, a^+, b^+ the attracting fixed points of a and b. Then*

$$\lim_{n\to\infty} e^{l(a^n)+l(b^n)-l(b^n a^n)} = [a^-, b^-, a^+, b^+]$$

where $[x, y, z, w]$ denotes the cross ratio of the four points.

This lemma implies that if two non-elementary representations have the same marked length spectrum, then there is a cross-ratio preserving homeomorphism between the two limit sets. Using the subriemannian structure, namely the Carnot-Caratheodory metric on the geometric boundary of rank one symmetric space, we can extend the map defined only on the limit set, to the whole ideal boundary, which is an isometry with respect to the Carnot metric. For the product of rank one spaces, basically we deduce the problem to each factor, which can be dealt with by the above argument.

The marked length rigidity of higher rank symmetric spaces has the following form:

THEOREM 2. — *Let Γ_1 and Γ_2 be Zariski dense subgroups in irreducible higher rank symmetric spaces X and Y. If they have the same marked length spectrum, then $X = Y$ and they are conjugate.*

This theorem is proved in [12]. The main idea of the proof is using the smallest totally geodesic space Z in $X \times Y$, left invariant under $\Gamma = \{(\gamma, \phi(\gamma)) | \gamma \in \Gamma_1, \phi$ gives the same marked length spectrum between Γ_1 and $\Gamma_2\}$. This Z has the slope 1 if they have the same marked length spectrum, and this fact gives an isometry between geometric boundaries of X and Y with respect to the Tits metrics. By the work of Tits [20], this isometry is realized by an isometry between X and Y. Along the proof we used the idea of Benoist [3].

We want to mention some corollaries deduced from these theorems and lemma.

COROLLARY 1. — *Let M be a finite volume locally symmetric manifold of rank one and N be a quotient of CAT(-1) space whose limit set is the whole ideal boundary of the CAT(-1) space. If they have the same marked length spectrum, then they are isometric.*

Proof: Using the result of [4] and lemma 1, the result follows. ∎

COROLLARY 2. — *Let M be a convex cocompact manifold with a metric g_1 which is a quotient of a symmetric space of non-compact type. Let g_2 be another symmetric metric which makes M convex cocompact. Then $I_{\mu_{BM}}(g_1, g_2) \geqslant \frac{\delta(g_1)}{\delta(g_2)}$ where $I_{\mu_{BM}}(g_1, g_2)$ is the geodesic stretch of g_2 relative to g_1 and the Bowen-Margulis measure μ_{BM} of g_1. Furthermore the followings are equivalent.*

1. $I_{\mu_{BM}}(g_1, g_2) = \frac{\delta(g_1)}{\delta(g_2)} = 1.$

2. *There is a time preserving conjugacy between* $\Omega(g_1)$ *and* $\Omega(g_2)$.

3. *The two manifolds have the same marked length spectrum.*

4. g_1 *and* g_2 *are isometric.*

In [22] Thurston introduced a non-symmetric Finsler metric K on the Teichmüller space of the closed surface S, $\mathcal{T}(S)$. It is defined by

$$K(g,h) = \sup_{\alpha\in\mathcal{C}} \left\{ \log\left(\frac{l_h(\alpha)}{l_g(\alpha)}\right)\right\}$$

where $\mathcal{C}$ is the free homotopy classes of closed loops. He showed that $K(g,h)$ is equal to the minimum of global Lipschitz constants of homeomorphisms in a given homotopy class. From this it easily follows that $K(g,h) \geqslant 0$ and the equality holds iff $g = h$. To prove that K is the minimal Lipschitz constant, he uses the generalization of an earthquake on a surface, namely the cataclysm. Let $\mathcal{T}(\Gamma)$ be the space of faithful, discrete, convex cocompact representations from Γ into the isometry group of rank one symmetric space X of non-compact type. Define the distance $K(g,h)$ as above for g and h in $\mathcal{T}(\Gamma)$.

Corollary 3. — *$\delta(g) = \delta(h)$ and $K(g,h) = 0$ iff g and h are isometric for g and h in $\mathcal{T}(\Gamma)$ where $\delta(\Gamma)$ denotes the critical exponent of the Poincare series of Γ.*

We want to mention the final corollary of this section.

Corollary 4. — *Let $\Gamma \subset G$ be a non-elementary group in a semisimple Lie group G of non-compact type of rank one. Then the set of translation lengths of hyperbolic isometries in Γ is not contained in any discrete subgroup of $\mathbb{R}$.*

Proof: If the set of translation lengths is contained in $r_0\mathbb{Z}$, then by the lemma 1, the set of cross-ratios on the limit set is contained in $e^{r_0\mathbb{Z}}$. This means that 0 is the only accumulation point of cross-ratios on the limit set. Using the structure of the ideal boundary of rank one symmetric space, which is the one point compactification of a two step nilpotent group (called a generalized Heisenberg group), and using the action of isometries together with the explicit Cygan norm on the generalized Heisenberg group, one can deduce a contradiction. For the detailed proof, see [13]. ∎

3. Geometrical finiteness and Geometric flow

The (G,X) structure for the manifold M is, briefly speaking, a set of coordinate charts $\{\phi_i : U_i \to X\}$ such that transition functions are in the group G. A geometric flow on M is a flow whose one parameter group preserves this given geometric structure. In this section, we concern about the manifold modelled on the Furstenberg boundary of symmetric space. In this case G is a semisimple Lie group of non-compact type and X is the Furstenberg boundary of the symmetric space. Mainly we are concerned about the geometric boundary of a rank one symmetric space, since the extended action of isometries on the Furstenberg boundary of higher rank symmetric spaces are not satisfactory. The theorem we want to announce in this section is:

THEOREM 3. — *A non-singular geometric flow on the smooth manifold which is a rational homology sphere and is modeled on the Furstenberg boundary of a symmetric space of non-compact type, under the suitable assumptions, has a closed orbit.*

This is a special version of the Seifert conjecture saying that a smooth flow has a closed orbit or a fixed point. For the rank one case, the Furstenberg boundary and the geometric boundary coincide. Further we do not need any assumption for rank one case. For the details, see [17]. The theorem is proved in [10] for the rank one case except for the Cayley plane in a similar fashion.

We say that M is a **complete** (G, X)–manifold if dev : $\widetilde{M} \to X$ is a covering map. Note that if M is a closed (G, X)-manifold and G is a Lie group acting transitively on X with compact stabilizer G_x for some x, then M is complete. The reason is that by pulling back the G-invariant metric on X by dev, we get a geodesically complete metric on $\widetilde{M}$, so a local isometry from a complete Riemannian manifold is always a covering map. See [23]. This is the starting point of our argument in proving the main theorems.

The main ingredient of the proof is the following lemma:

LEMMA 2. — *Let $\{\phi_t\}$ be a closed non-compact one-parameter group of $\mathscr{C}_{\mathrm{Iso}(H_{\mathbb{F}}^n)}(\rho(\pi_1(M)))$ where $\mathscr{C}$ denotes the centralizer. Then either one of the following is true:*

(1) M is (G,X)-equivalent to X.

(2) ϕ_t fixes a unique fixed point $\{\infty\}$ with $\mathrm{dev}^{-1}(\infty) = \varnothing$.

(3) ϕ_t fixes exactly two points $\{[0,0], \infty\}$ with $\mathrm{dev}^{-1}(\{[0,0], \infty\}) = \varnothing$.
Furthermore in cases (2) and (3), $\mathrm{Aut}(M)$ is compact.

For the proof, see [17] and [10].

This theorem somehow reflects the rigidity of symmetric spaces at infinity. The other theorem we want to present here is about the topological picture of non-compact locally symmetric manifolds.

In hyperbolic 3-manifold theory, a geometrically finite manifold plays a very important role in many different aspects. For example it is homeomorphic to a 3-manifold with boundary and with finitely many ends. This property is the reminiscence of the Scott's core in 3-manifold topology. Apart from this topological nature, a geometrically finite manifold has a geometric simpleness such as the thick part of the convex core has finite volume and is compact. From the dynamical point of view, a geometrically finite group has a limit set consisting of parabolic fixed points and conical limit points. Thurston [24] also showed that a geometrically finite manifold has a finite sided fundamental domain in the universal cover.

In [5] Bowditch proved four equivalent definitions of geometrical finiteness for negatively curved Hadamard manifolds. In [18], he basically showed that the characterization of geometrical finiteness in negatively curved Hadamard manifolds persists in general non-positively curved symmetric spaces except for the nature of the parabolic fixed points. But while writing this he realized that such a characterization seems meaningful only for the product of rank one symmetric spaces. In this section we give examples of geometrically finite manifolds by analizing the limit set.

In [18], he showed that the following definitions are equivalent for locally symmetric manifolds.

DEFINITION 1. — $\Gamma \subset \mathrm{Iso}(X)$ *is geometrically finite if* X/Γ *is homeomorphic to the interior of a topological manifold* $M(\Gamma)$ *such that* $M(\Gamma)$ *has a finite number of ends and* $M(\Gamma)$ *with the ends truncated off is compact. Furthermore all the corresponding ends in* X/Γ *are union of parabolic cusps, i.e., the end component of the thin part* $(X/\Gamma)_{(0,\epsilon]}$ *is the union of* $X_\epsilon/\Gamma_\epsilon$*'s such that*

1. $\Gamma_\epsilon(X_\epsilon) = X_\epsilon$.
2. Γ_ϵ *contains a parabolic isometry and* $d(x, \gamma(x)) \leqslant \epsilon$ *for all* $x \in X_\epsilon$ *and for some parabolic isometry* $\gamma \in \Gamma_\epsilon$. *Furthermore* Γ_ϵ *is a maximal parabolic subgroup of* Γ.

DEFINITION 2. — $\Gamma \subset \mathrm{Iso}(X)$ *is geometrically finite if for* $\epsilon \in (0, \epsilon(n))$, *where* $\epsilon(n)$ *is the Margulis constant,* $C(\Gamma) \cap (M_{[\epsilon,\infty)} - \{\textit{cusp ends}\})$ *is compact where* $M = X/\Gamma$ *and* $C(\Gamma)$ *is the convex core of* Γ.

DEFINITION 3. — $\Gamma \subset \mathrm{Iso}(X)$ *is geometrically finite if* $N_\eta C(\Gamma)$ *has finite volume for some* $\eta > 0$ *where* $N_\eta C(\Gamma)$ *is a* η*-neighborhood of* $C(\Gamma)$.

Now we want to give examples of geometrically finite manifolds of higher rank which are not lattices. Let Z be a product of rank one symmetric spaces X and Y and $\Gamma \subset \mathrm{Iso}(X) \times \mathrm{Iso}(Y)$ be a non-elementary group. The geometric boundary of Z can be identified with $\partial X \times \partial Y \times [0, \infty]$ where $[0, \infty]$ denotes the direction associated with the Riemannian product.

The following theorem is proved in [14].

THEOREM 4. — *Let* $\Gamma \subset \mathrm{Iso}(X) \times \mathrm{Iso}(Y)$ *be a non-elementary group. Then the limit set* Λ *of* Γ *is* $\Lambda = F \times P$ *where* F *is a projection of* Λ *onto* $\partial X \times \partial Y$, *and* P *a projection on* $\mathbb{R}$. *Furthermore* F *is a minimal closed set under* Γ *and* $P = I$ *is an interval where* I *is the closure of the set of ratios* $\{\frac{l(\beta)}{l(\alpha)} \mid (\alpha, \beta) \in \Gamma\}$.

Let $\Gamma_1 \subset \mathrm{Iso}(X)$ and $\Gamma_2 \subset \mathrm{Iso}(Y)$ be geometrically finite discrete groups in X and Y. Then $\Gamma = \Gamma_1 \times \Gamma_2$ has the limit set $\Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2} \times P$. Since Γ is a product, obviously P contains 0 and ∞, which implies that $P = [0, \infty]$. Let $\widetilde{C}(\Gamma)$ be the minimal convex set in Z containing the limit set $\Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2} \times [0, \infty]$. We want to show that $\widetilde{C}(\Gamma)$ is equal to

$\widetilde{C}(\Gamma_1) \times \widetilde{C}(\Gamma_2)$. Obviously the ideal boundary of $\widetilde{C}(\Gamma_1) \times \widetilde{C}(\Gamma_2)$ contains $\Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2} \times [0, \infty]$. Conversely, let $l_1 \subset \widetilde{C}(\Gamma_1)$, $l_2 \subset \widetilde{C}(\Gamma_2)$ be biinfinite geodesics whose end points are limit points. Since the ideal boundary of $\widetilde{C}(\Gamma)$ contains $\{l_1(\pm\infty)\} \times \{l_2(\pm\infty)\} \times [0, \infty]$, $\widetilde{C}(\Gamma)$ should contain $l_1 \times l_2$. Since this is true for any pairs of such biinfinite geodesics, $\widetilde{C}(\Gamma)$ should contain $\widetilde{C}(\Gamma_1) \times \widetilde{C}(\Gamma_2)$.

Then $\widetilde{C}(\Gamma)/\Gamma = \widetilde{C}(\Gamma_1)/\Gamma_1 \times \widetilde{C}(\Gamma_2)/\Gamma_2$, which is of finite volume since each factor is. By definition 3 of geometrical finiteness, Γ is geometrically finite.

4. Rigidity of quaternionic hyperbolic space and geometry of quaternionic Kähler manifolds

In this section we present some results related to quaternionic Kähler manifolds. First we want to present the rigidity of lattices in quaternionic hyperbolic space.

In [21] Toledo proved that if the integral, over the closed Riemann surface S, of the pull back of the Kähler form on complex hyperbolic manifold M by the homotopy equivalence map f between S and M is equal to $\pm 2\pi\chi(S)$, then the image of S under f lies on the quotient of complex geodesic. This is the special case of the conjecture of Goldman and Milson in [8]. They conjectured that if Γ is a cocompact group in $SU(n, 1)$ and if $\rho \in \mathrm{Hom}(\Gamma, SU(n+k, 1))$ satisfies $\int_{H^n_{\mathbb{C}}/\Gamma} \rho^* \omega^n = \mathrm{Vol}(H^n_{\mathbb{C}}/\Gamma)$, where ω is a Kähler form on the complex hyperbolic space $H^{n+k}_{\mathbb{C}}$, then ρ is Fuchsian i.e., ρ is a faithful discrete torsion free representation stabilizing a totally geodesic $H^n_{\mathbb{C}}$ in $H^{n+k}_{\mathbb{C}}$. The conjecture is settled by Corlette in [7].

In this section we use the Toledo type invariant in quaternionic hyperbolic space in a similar fashion as in complex hyperbolic space and prove that Goldman and Milson conjecture holds for n=1, i.e., the representation of real hyperbolic four manifold into a quaternionic hyperbolic space.

We use the canonical quaternionic Kähler 4-form to compute the Toledo type invariant. The theorem we want to present is:

THEOREM 5. — *For any uniform lattice $\Gamma \subset PSp(n, 1)$, and any dimension $m \geqslant n \geqslant 1$, there is no quasifuchsian deformation of its inclusion $\Gamma \subset PSp(n, 1) \subset PSp(m, 1)$.*

This theorem is proved in [2]. When $n = 2$, it is a straighforward corollary of Corlette's superrigidity. The most difficult case is when $n = 1$.

Let M be a real hyperbolic 4-manifold, and $\rho : \pi_1(M) \to PSp(n, 1)$ be a representation. Let X_ρ be the associated flat $H^n_{\mathbb{H}}$-bundle and s a section from M to X_ρ. This section s defines a ρ-equivariant map $\tilde{f} : H^4_{\mathbb{R}} \to H^n_{\mathbb{H}}$. Since its pull back form $f^*\omega$ is $\pi_1(M)$-invariant, it descends to a four form $f^*\omega$ (with the same notation) on the 4-manifold M. We define the character of the representation ρ as:

$$c(\rho) = \int_M f^* \omega .$$

This is independent of the choice of a section s since $H^n_{\mathbb{H}}$ is contractible. Furthermore the flat bundle X_ρ is completely determined by the representation ρ. By thinking of $Sp(n,1)$ as a subset in $O(4n,4)$, ρ is a point in an algebraic variety. Then it is a standard fact that two flat bundles determined by representations in the same component of the representation variety are bundle isomorphic. Therefore the character $c(\rho)$ is a constant function on each component of the representation variety.

We can define $f^*\omega$ as the following straight 4-cochain. Let $x = (x_1, \cdots, x_5)$ be ordered distinct five points in $H^n_{\mathbb{H}} \cup \partial H^n_{\mathbb{H}}$. Let $\Sigma_{12} \subset H^n_{\mathbb{H}}$ be a unique $\mathbb{H}$-line through the points x_1 and x_2, and $\Pi : H^n_{\mathbb{H}} \to \Sigma_{12}$ its orthogonal projection. We denote by $\triangle(\Pi(x))$ the geodesic simplex in $\Sigma_{12} \cong H^4_{\mathbb{R}}$ with vertices $\Pi(x_i)$, and consider a straight geodesic simplex $\triangle(x) \subset H^n_{\mathbb{H}}$ with vertices x_i inductively defined as follows (see [9] section 1.2). For a given $l \geqslant 1$ we define an l-dimensional simplex with ordered vertices $x_1, \cdots, x_{l+1}$ as the geodesic cone from x_l over $(l-1)$-dimensional straight simplex spanned by the first l vertices $x_1, \cdots, x_l$. Then

$$\int_{\triangle(x)} \omega = \int_{\triangle(\Pi(x))} \omega.$$

A crucial lemma to prove above theorem is:

LEMMA 3. — *Let $M = H^4_{\mathbb{R}}/\Gamma$ be a closed oriented real hyperbolic 4-manifold, $\pi_1(M) = \Gamma \subset \mathrm{Iso}_+ H^4_{\mathbb{R}}$, and $\rho : \Gamma \to PSp(n,1)$ be a representation of its fundamental group with character $c(\rho)$, $|c(\rho)| = |f^*\omega([M])| = \mathrm{Vol}(M)$. Then $\rho(\pi_1(M))$ leaves invariant an $\mathbb{H}$-line in $H^n_{\mathbb{H}}$.*

Using this lemma and the fact that the character is constant on each component of the representation variety, the theorem follows.

Quaternionic hyperbolic space is quaternionic Kähler since its isotropy group is $Sp(n)Sp(1) = Sp(n) \times_{\mathbb{Z}_2} Sp(1)$. We just want to mention the following theorem which seems to be irrelevant to the symmetric space, none the less it seems that it might prove to be useful to study quaternionic hyperbolic space.

Let M be a quaternionic Kähler manifold, i.e., the linear holonomy group is in $Sp(n)Sp(1)$. Let P denote the principal $Sp(n)Sp(1)$ bundle of M. Let E, H denote the standard complex representations of $Sp(n), Sp(1)$ on $\mathbb{C}^{2n}, \mathbb{C}^2$ respectively. Note here that $Sp(n)$ acts on the left and $Sp(1)$ on the right. Also there are invariant skew forms $\omega_H \in \wedge^2 H^*, \omega_E \in \wedge^2 E^*$, for example $\omega_H((a,b),(c,d)) = ad - bc$ and similar for ω_E. Then H and H^* can be identified by $h \to \omega_H(\cdot, h)$. A **standard** basis of H is a unitary basis of the form $\{h, \tilde{h}\}$ such that $\omega_H(h, \tilde{h}) = 1$. Let H be a locally defined vector bundle over M defined by

$$H = \tilde{P} \times_{Sp(n)\times Sp(1)} H$$

where $\tilde{P}$ is a lifting of P to a principal $Sp(n) \times Sp(1)$ bundle which is locally defined over M.

DEFINITION 4. —

$$\mathrm{Spin}^{\mathbb{H}}(n) = (\mathrm{Spin}(n) \times Sp(1))/\mathbb{Z}_2.$$

It is said that X has a $\mathrm{Spin}^{\mathbb{H}}$ structure if there is a principal $\mathrm{Spin}^{\mathbb{H}}(n)$ bundle over $P_{SO(n)}$. From the exact sequence of the Sheaf cohomology

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}^{\mathbb{H}}(n) \to SO(n) \times SO(3) \to 1$$
$$(\alpha, \beta) \in \mathrm{Spin}^{\mathbb{H}}(n) \to (\bar{\alpha}, \bar{\beta}) \in SO(n) \times SO(3)$$
$$0 \to H^1(X, \mathbb{Z}_2) \to H^1(X, \mathrm{Spin}^{\mathbb{H}}(n)) \to H^1(X, SO(n)) \oplus H^1(X, SO(3))$$
$$\to H^2(X, \mathbb{Z}_2) \cdots,$$

the existence of $\mathrm{Spin}^{\mathbb{H}}$ structure is equivalent to the existence of a $SO(3)$ bundle L such that

$$\omega_2(X) = \omega_2(L)(\mathrm{mod}\, 2).$$

But since X is a quaternionic Kähler manifold, S^2H is a globally defined $SO(3)$ bundle whose $\mathbb{Z}_2$ reduction $\omega_2(S^2H)$ of its first chern class is equal to $\omega_2(X)$ for n odd [19]. So every quaternionic Kähler manifold has a canonical $\mathrm{Spin}^{\mathbb{H}}$ structure for n odd, specially every real 4-dimensional Riemannian manifold has a canonical $\mathrm{Spin}^{\mathbb{H}}$ structure since $Sp(1) \cdot Sp(1) = SO(4)$. From

$$0 \to \mathbb{Z}_2 \to Sp(1) \to SO(3) \to 0$$

the lift of a $SO(3)$ bundle is a $Sp(1)$ bundle. The canonical lift of S^2H is H. The following theorem is proved in [11].

THEOREM 6. — *Let X be a symplectic 4-manifold with $b^{2^+} \geqslant 2$. Then the Seiberg-Witten invariant for the canonical characteristic $Sp(1)$ bundle H is 1.*

5. Patterson-Sullivan theory on the product of Hadamard manifolds

In this section we want to present theorems related to the Patterson-Sullivan theory. All the theorems mentioned here hold not just for the product of rank one symmetric spaces but also for the product of Hadamard manifolds with curvature $\leqslant -a^2 < 0$.

This subject is intensively studied by [1], [6]. The following theorem is proved in [14].

THEOREM 7. — *Let Γ_1, Γ_2 be convex cocompact groups in Hadamard manifolds X and Y with curvatures bounded from above by $-a^2$. Then the Patterson-Sullivan measure is supported on $F \times I_\mu(\Gamma_1, \Gamma_2)$ where I_μ is the geodesic stretch with respect to the Bowen-Margulis measure μ.*

We also mention that the shadow lemma and the counting lemma hold as in symmetric spaces.

THEOREM 8. — *Let Γ be a irreducible discrete subgroup of* $\mathrm{Iso}(X) \times \mathrm{Iso}(Y)$ *and σ a (β,θ)-density on ∂X. Then there exists $C > 0$ such that for all $K > C$,*

$$\sigma_{x_0}(S(x_0; B(\gamma(x_0), K)) \sim e^{-\beta d(x_0, P^{\theta}(\gamma x_0))}.$$

COROLLARY 5. — *Let Γ be a discrete irreducible group in* $\mathrm{Iso}(X) \times \mathrm{Iso}(Y)$ *where X and Y are pinched Hadamard manifolds. If σ is a (β,θ)-density for $\theta \neq 0, \infty$, then $|\Gamma x_0 \cap B^{\theta}(x_0, l)| \leqslant Ce^{\beta l} l$ for some constant $C > 0$.*

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References

[1] P. ALBUQUERQUE, *Patterson-Sullivan theory in higher rank symmetric spaces*, GAFA, vol. 9 (1999) 1–28.

[2] B. APANASOV and I. KIM, *Numerical invariants and deformations in quaternionic and octonionic hyperbolic spaces*, submitted.

[3] Y. BENOIST, *Propriété asymptotiques des groupes linéaires*, GAFA. vol 7 (1997), 1–47.

[4] M. BOURDON, *Sur le birapport au bord des CAT(−1)-espaces*, Publications Mathématiques IHES, n 83 (1996), 95–104.

[5] B. BOWDITCH, *Geometrical finiteness with variable negative curvature*, Duke J. Math. 77 (1995), 229–274.

[6] M. BURGER, *Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2*, Internation Mathematics Research Notices, No. 7 (1993), 217–225.

[7] K. CORLETTE, *Flat G-bundles with canonical metrics. J. Differential Geometry*, 28 (1988) 361–382.

[8] W. GOLDMAN, J. MILSON, *Local rigidity of discrete groups acting on complex hyperbolic space*, Inv. Math., 88, (1987), 495–520.

[9] M. GROMOV, *Volume and bounded cohomology*, Publ. Math. IHES vol. 56 (1982), 5–99.

[10] Y. KAMISHIMA, *Geometric Flows on Compact Manifolds and Global Rigidity*, Topology, v. 35, No. 2 (1996), 439–450.

[11] Y. KAMISHIMA and I. KIM, *The Seiberg-Witten theory on quaternionic Kähler manifolds*, preprint.

[12] F. DAL'BO and I. KIM, *Marked length rigidity of higher rank symmetric spaces*, in preparation.

[13] F. DAL'BO and I. KIM, *Length spectrum of rank one symmetric spaces are not arithmetic*, in preparation.

[14] F. DAL'BO and I. KIM, *Geometry on the product of Hadamard manifolds*, in preparation.

[15] I. KIM, *Marked length rigidity of symmetric spaces of rank one and their product*, submitted.

[16] I. KIM, *Ergodic theory and Rigidity on the symmetric space of non-compact type*, submitted.

[17] I. Kim, *Geometric Flow and Rigidity on Symmetric Spaces of Non-compact Type*, to appear in the Trans. of the AMS.

[18] I. Kim, *Geometrical finiteness of symmetric spaces of non-compact type*, to appear in Forum Math.

[19] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. 67 (1982), 143–171.

[20] J. Tits, *Buildings of spherical type and finite BN-pairs*, Springer LNM, vol. 386 (1974).

[21] D. Toledo, *Representations of surface groups in complex hyperbolic space*, J. Differential Geometry, 29 (1989), 125–133.

[22] W. Thurston, *Minimal stretch maps between hyperbolic surfaces*, preprint.

[23] W. Thurston, *Three Dimensional Geometry and Topology*, The Geometry Center, unpublished manuscript.

[24] W. Thurston, *Three Dimensional Geometry and Topology*, Princeton lecture Notes, 1983.

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