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## BENDING INVARIANTS FOR HYPERSURFACES

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A very long-standing problem in classical differential geometry is the bendability problem for compact hypersurfaces in euclidean spaces. By a hypersurface we mean a connected Riemannian  $n$ -manifold ( $n \geq 2$ ) which is  $C^2$ -isometrically immersed in  $\mathbb{R}^{n+1}$ . A hypersurface  $M$  is said to be bendable if it can be isometrically and non-trivially deformed in  $\mathbb{R}^{n+1}$ . It is said rigid if every hypersurface which is  $C^2$  isometric to  $M$  is congruent to  $M$  (i.e is obtained by applying an ambient rigid motion). The bendability problem is to decide whether there exist bendable compact hypersurfaces or not. Clearly rigidity implies unbendability. Only few results are known in particular cases. Cohn-Vossen has proved that smooth compact convex surfaces in  $\mathbb{R}^3$  are rigid. In higher dimensions  $n \geq 3$ , Beez has proved rigidity under the hypothesis that the second fundamental form has everywhere rank at least 3, and Sacksteder [Sac] proved that complete convex hypersurfaces are rigid provided the second fundamental form has rank at least 3 *at some point*. We refer to the book of M. Spivak [Spi] for an excellent account of the subject. The bendability problem has also been studied infinitesimally, locally and for complete hypersurfaces. More results are known in these latter cases, we refer again to M. Spivak's book and to the more recent survey of I. Sabitov [Sab]. In the piecewise-linear category, R. Connelly has given an example of an embedded polyhedra in  $\mathbb{R}^3$  admitting non-trivial isometric deformations (cf. [Co]).

In this paper, we are concerned with a closely related problem. We are interested in geometric quantities defined on the hypersurface which are extrinsic -that is, depend on the way the surface is immersed in  $\mathbb{R}^{n+1}$  and not only on the metric on  $M$ - but are invariant under isometric deformations of  $M$ . Results in this direction may be viewed as unbendability results in a weak sense.

In order to state our main theorem we introduce some notations. Let  $M$  be a compact oriented boundaryless manifold of dimension  $n$  and consider an immersion  $\phi : M \rightarrow \mathbb{R}^{n+1}$ . We endow  $M$  with the metric induced by the euclidean one on  $\mathbb{R}^{n+1}$ . Associated to the second fundamental form of  $\phi$  there are  $n$  invariants, namely, the ele-

mentary symmetric functions  $S_r$  of the principal curvatures  $k_1, k_2, \dots, k_n$  :

$$S_r = \sum_{i_1 < \dots < i_r} k_{i_1} \dots k_{i_r} \quad (1 \leq r \leq n)$$

The  $r$ -mean curvature  $H_r$  of  $\phi$  is defined by  $\binom{n}{r} H_r = S_r$ . So, for example,  $H_1$  is the mean curvature,  $H_2$  the scalar curvature and  $H_n$  the Gauss-Kronecker curvature of  $\phi$ .

An isometric deformation of the immersion  $\phi$  is a  $C^2$  mapping:  $\Phi : (-\epsilon, \epsilon) \times M \rightarrow \mathbb{R}^{n+1}$  such that  $\phi_t : M \rightarrow \mathbb{R}^{n+1}$ ,  $t \in (-\epsilon, \epsilon)$ , defined by  $\phi_t(p) = \Phi(t, p)$ ,  $p \in M$ , is an isometric immersion and  $\phi_0 = \phi$ . Denote by  $dA$  the volume element on  $M$ .

Our main result is the following:

**THEOREM.** — For each  $r, 1 \leq r \leq n$ , the total  $r$ -mean curvature  $\int_M H_r dA$ , is invariant under isometric deformations of the immersion  $\phi$ .

This result is due, for  $n = 2$ , to F. Almgren and I. Rivin [Al-Ri] who first derived a similar result for polyhedra and then extended it to smooth surfaces using geometric measure theory methods. Independently from our work, I. Rivin and J-M. Schlenker [Ri-Sc] gave a more direct proof and proved the stronger result that the  $H_r$  are pointwise invariant for  $r \geq 2$ .

*Proof.* — Denote, for each  $t$ , by  $H_r(t)$ , the  $r$ -mean curvature of  $\phi_t$ . Since the deformation is isometric the volume element of  $M$  is invariant under the deformation and because the  $\phi_t$  satisfy the same hypothesis as  $\phi$  it is sufficient to show that:

$$\frac{d}{dt} \Big|_{t=0} \int_M H_r(t) dA = 0$$

which amounts to the same to show that:

$$\frac{d}{dt} \Big|_{t=0} \int_M S_r(t) dA = 0$$

For this we need to know the derivative:  $S'_r(0)$ . Consider, for each  $t$ , a  $C^1$  unit field  $N_t$ , normal to the immersion  $\phi_t$ , and depending  $C^1$  on  $t$ . Denote by  $B_t$  the shape operator, with respect to  $N_t$ , associated to the immersion  $\phi_t$  and call  $D$  be the Levi-Civita connection on  $\mathbb{R}^{n+1}$ . We have:  $B_t u = -D_u N_t$ , for  $u \in TM$ . Also, denote by  $\xi$  the variation vector field of  $\Phi$ , that is,  $\xi(p) = \frac{\partial \Phi}{\partial t}(p) \Big|_{t=0}$ , and let  $\nabla$  be the Levi-Civita connection on  $M$ . We have the following formula (see e.g [Ro], where the formula is derived for "normal" deformations but the proof extends easily to general ones):

$$\langle B'(o)u, v \rangle = -\langle \nabla_u N'(o), v \rangle - \langle D_{Bu} \xi, v \rangle, \quad u, v \in TM \tag{0}$$

In the above formula,  $N'(0)$  denotes the vector field  $\frac{d}{dt} N(0)$  and it is tangent to  $M$  since  $N_t$  is unitary for each  $t$ .

In order to go further, consider (after R. Reilly [Rei]) the Newton transformations (or  $(0, 2)$  tensors),  $T_r, 0 \leq r \leq n$ , defined by  $T_r = S_r \text{Id} - S_{r-1} B + \dots (-1)^r B^r$ , or, inductively, by  $T_0 = \text{Id}, T_r = S_r \text{Id} - B T_{r-1}$ . These Newton transformations enjoy the following properties (cf. [Rei] or [Ro]):

- (1)  $\text{Trace}(T_r) = (n - r)S_r$
- (2)  $S'_r = \text{Trace}(B'T_{r-1})$ , the derivative being taken with respect to the parameter  $t$  of the deformation.
- (3) seen as  $(0, 2)$  tensors, the  $T_r$  are divergence free.

By (0) and (2), we have, at each point  $p \in M$ :

$$S'_r(0) = - \sum_{i=1}^n \langle \nabla_{T_{r-1}(e_i)} N'(o), e_i \rangle - \sum_{i=1}^n \langle D_{BT_{r-1}(e_i)} \xi, e_i \rangle \tag{4}$$

where the  $e_i$  form an orthonormal basis of  $T_pM$ . Now, the operator  $BT_{r-1}$  being symmetric (it is a polynomial in the symmetric operator  $B$ ) we may choose an orthonormal basis  $e_1 \dots e_n$  of eigenvectors of  $BT_{r-1}$ . Moreover, since the deformation is isometric, we have:  $\langle D_u \xi, u \rangle = 0$ , for every  $u \in TM$ . Therefore the second sum in the right hand side of (2) vanishes. It follows from (3) (see also [Ro] for a direct proof) that:

For any vector field  $V$  on  $M$ :

$$\text{Trace}(u \rightarrow \nabla_{T_k(u)} V) = \text{Trace}(u \rightarrow \nabla_u T_k(V))$$

Therefore formula (4) becomes simply:

$$S'_r(0) = - \text{div}_M(T_{r-1}(N'(o)))$$

The result now follows from the divergence theorem. ■

*Remarks.*

1) To treat the case of the mean curvature ( $r = 1$ ) there is no need to introduce the Newton transformations.

2) We have proved the theorem using only the fact that the variation is infinitesimally isometric, that is, its variation vector field satisfies  $\langle D_u \xi, u \rangle = 0$ , for every  $u \in TM$ .

3) The cases  $r$  even and  $r = n$  are obvious. Indeed, for  $r$  even,  $H_r$  is intrinsic as it follows from Gauss equation (see e.g. [Lo]). For  $r = n$ ,  $\int_M H_n dA$  is a homotopic invariant since  $H_n$  is the determinant of the Gauss mapping of  $\phi$  and therefore  $\int_M H_n dA = \text{deg}(N)\omega_n$ ,  $\text{deg}(N)$  being the degree of the Gauss mapping and  $\omega_n$  the volume of the canonical unit  $n$ -sphere. Furthermore, if  $n$  is even, then in fact,  $\int_M H_n dA = \frac{1}{2}\omega_n \chi(M)$ , where  $\chi(M)$  is the Euler characteristic of  $M$ . On the other hand, it can be checked on the standard example (and its obvious generalizations to higher dimensions) of a topological sphere which admits two isometric and non-congruent embeddings in  $\mathbb{R}^3$  (see [Spi], p. 307) that the total mean curvatures of odd order  $r$ , are extrinsic quantities.

4) In a general ambient space with curvature tensor  $R$ , formulae (0) and (4) become:

$$\langle B'(o)u, v \rangle = - \langle \nabla_u N'(o), v \rangle - \langle D_{Bu} \xi, v \rangle + R(\xi, u, N, v), \quad u, v \in TM$$

$$S'_r(0) = - \sum_{i=1}^n \langle \nabla_{T_{r-1}(e_i)} N'(o), e_i \rangle - \sum_{i=1}^n \langle D_{BT_{r-1}(e_i)} \xi, e_i \rangle + R(\xi, T_{r-1}(e_i), N, e_i)$$

It therefore follows that the theorem is true, for  $r = 1$ , in Ricci-flat manifolds.

5) Consider the case of a general ambient space form  $M_K^{n+1}$ , of constant curvature  $K$ . Call  $V$  the volume enclosed by the (closed and oriented) hypersurface  $M$ . Our theorem extends and states that  $KV - \int_M H_1 dA$  and  $\int_M H_r$ , for  $r \geq 2$ , are invariant under isometric deformations of  $\Sigma$ . Details can be found in [Sch-So].

6) A polyhedral version of our theorem is proved in [Sch-So]. It is obtained as a corollary of Schläfli formulae of higher order generalizing the classical Schläfli one.

Our next result is a consequence of the theorem. An open question is to decide whether the volume enclosed by a hypersurface in  $\mathbb{R}^{n+1}$  is invariant under isometric deformations (cf. [Sab]). The analogous statement for closed polyhedra in  $\mathbb{R}^3$  was proved recently by I. Sabitov (see e.g. [CSW] and the references therein). The following corollary (see also [Ri-Sch]) suggests that this should also be true for regular hypersurfaces. Denote, by  $\phi^\epsilon(M)$ , the parallel hypersurface to  $\phi(M)$  at (the algebraic) distance  $\epsilon$  from  $\phi(M)$ , that is  $\phi^\epsilon(p) = \phi(p) + \epsilon N(p)$ ;  $p \in M$  (it is an immersion for  $\epsilon$  small enough). Denote by  $V^\epsilon$  the (algebraic) volume enclosed between  $\phi$  and  $\phi^\epsilon$ . Note that (see the proof), in contrast with the volume of a tube around  $\phi(M)$ , it is an extrinsic quantity. We then have the following:

**COROLLARY.** — *The volume  $V^\epsilon$  ( $\epsilon$  small enough) is invariant under isometric deformations of the immersion  $\phi$ .*

*Proof.* — We have:

$$V^\epsilon = \int_{[0, \epsilon] \times M} \Phi^* dV$$

where  $dV$  is the volume form on  $\mathbb{R}^{n+1}$  and  $\Phi : [0, \epsilon] \times M \rightarrow \mathbb{R}^{n+1}$  is given by:  $\Phi(s, p) = \phi(p) + \epsilon N(p)$ . It is known (and can easily be checked) that:

$$\Phi^* dV = \prod_{i=1}^{i=n} (1 - sk_i) dA ds.$$

Therefore:

$$V^\epsilon = \int_0^\epsilon \int_M \prod_{i=1}^{i=n} (1 - sk_i) dA ds = \sum_{r=1}^{r=n} (-1)^r \frac{\epsilon^{r+1}}{r+1} \int_M S_r dA$$

and is hence invariant by the theorem. ■

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