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# RIGIDITY OF GROUP ACTIONS

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Let $\Gamma$ be a cocompact lattice of the real unimodular group $SL_{n+1}\,\mathbb{R}$, and denote by $A^0$ the restriction to $\Gamma \subset SL_{n+1}\,\mathbb{R}$ of the standard projective action of $SL_{n+1}\,\mathbb{R}$ on the $n$-sphere $S^n$.

THEOREM. — *The action $A^0$ is rigid in the following sense provided $n \geqslant 21$: If a smooth action $A$ of $\Gamma$ on $S^n$ is sufficiently close to $A^0$ in* $\mathrm{Hom}(\Gamma, \mathrm{Diff}(S^n))$, *the space of smooth actions of $\Gamma$ on $S^n$ endowed with an appropriate topology, then $A$ has to be smoothly conjugate to $A^0$.*

The theorem says that a small perturbation of the original action $A^0$ never changes the action in an essential manner, or, in other words, that a sufficiently small neighborhood of $A^0$ in $\mathrm{Hom}(\Gamma, \mathrm{Diff}(S^n))$ shrinks to a point under the projection of $\mathrm{Hom}(\Gamma, \mathrm{Diff}(S^n))$ onto $\mathrm{Hom}(\Gamma, \mathrm{Diff}(S^n))/(\text{smooth conjugacy})$, the moduli of the smooth actions of $\Gamma$ on $S^n$.

The core of the proof of the theorem is the following two problems, in which $F$ is a smooth vector bundle over a smooth manifold $N$ and $\Gamma$ is a discrete group acting on $F$ by vector bundle automorphisms. Note that the action of $\Gamma$ on the total space $F$ induces a smooth action on the base $N$.

PROBLEM 1. — *Does there exist an affine connection of $F$ that is $\Gamma$-invariant?*

PROBLEM 2. — *Is such a $\Gamma$-invariant affine connection flat?*

To establish our rigidity theorem we need to find a diffeomorphism of the sphere which is equivariant under a perturbed action $A$ and the original one $A^0$. This is *a priori* a nonlinear problem, for the unknown in this question belongs to the nonlinear space $\mathrm{Diff}(S^n)$. On the contrary, in the problems we have just mentioned we look for an affine

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connection satisfying certain conditions, instead of an equivariant diffeomorphism, which can be handled in a linear manner. After all, what we did here is a kind of "change of variables" by which we could translate the original nonlinear problem into a linear one. See [K] for details.

It is possible to describe the above problems in terms of some cohomology theory. Let $C^\infty(N, F)$ be the space of $C^\infty$-sections of the bundle $F$ over $N$. It naturally possesses the structure of $\Gamma$-module, and in consequence we are able to speak of $H^*(\Gamma; C^\infty(F, N))$, the Eilenberg-MacLane cohomology of $\Gamma$ with coefficients in it.

CLAIM 1. — *The bundle $F$ carries a $\Gamma$-invariant affine connection if*

$$H^1(\Gamma; C^\infty(N, T^*N \otimes F^* \otimes F)) = 0.$$

CLAIM 2. — *A $\Gamma$-invariant affine connection of $F$ has to be flat whenever*

$$H^0(\Gamma; C^\infty(N, T^*N \otimes T^*N \otimes F^* \otimes F)) = 0.$$

According to these claims, the problem boils down to the vanishing of the Eilenberg-MacLane cohomology.

The next task is to introduce a new cohomology theory which is formally equivalent to the Eilenberg-MacLane cohomology but is easier to handle than the Eilenberg-MacLane cohomology from geometric and analytic viewpoints. From now on, assume that $\Gamma$ is isomorphic to the fundamental group of some closed smooth manifold $M$. Denote by $\widetilde{M}$ the universal cover of $M$, on which $\Gamma \cong \pi_1(M)$ acts by the deck transformations. We are then able to form the quotient $N^\times = \Gamma \backslash (\widetilde{M} \times N)$ of the product $\widetilde{M} \times N$ by the diagonal action of $\Gamma$ on it. Since the projection $\widetilde{M} \times N \to \widetilde{M}$ is $\Gamma$-equivariant, it gives rise to a fiber bundle $N^\times = \Gamma \backslash (\widetilde{M} \times N) \to M = \Gamma \backslash \widetilde{M}$ with fibers diffeomorphic to $N$. Moreover, since the action of $\Gamma$ on $\widetilde{M} \times N$ maps a "horizontal" submanifold $\widetilde{M} \times \{y\}$ ($y \in N$) to a horizontal one, the foliation of $\widetilde{M} \times N$ by those horizontal submanifolds induces a foliation $\mathscr{H}$ of $N^\times = \Gamma \backslash (\widetilde{M} \times N)$ whose leaves are transverse to the fibers of the fibering $N^\times \to M$. In short, we obtain a so-called foliated bundle $(N^\times, \mathscr{H}) \to M$. Similarly, the quotient $F^\times = \Gamma \backslash (\widetilde{M} \times F)$ turns out to be a vector bundle over $N^\times = \Gamma \backslash (\widetilde{M} \times N)$. A section of $\bigwedge^p T^*\mathscr{H} \otimes F^\times$ is to be called a tangential $p$-form of the foliated manifold $(N^\times, \mathscr{H})$ taking values in the bundle $F^\times$. Denote by $\Omega^p = \Omega^p(N^\times, \mathscr{H}; F^\times)$ the space of smooth sections of the bundle $\bigwedge^p T^*\mathscr{H} \otimes F^\times$. Moreover the bundle $F^\times$ has an additional structure. To see that, note that the bundle $F^\times \to N^\times$ is covered by the vector bundle $\widetilde{M} \times F \to \widetilde{M} \times N$, the restriction of which to each horizontal submanifold $\widetilde{M} \times \{y\} \subset \widetilde{M} \times N$ ($y \in N$) has a natural trivialization $\widetilde{M} \times F|_{\widetilde{M} \times \{y\}} = (\widetilde{M} \times \{y\}) \times F_y$ with $F_y$ being the fiber of the bundle $F \to N$ over $y \in N$. The trivialization clearly yields a *flat* affine connection of the vector bundle $\widetilde{M} \times F$ over $\widetilde{M} \times N$ *along* the foliation of $\widetilde{M} \times N$ by the horizontal submanifolds $\widetilde{M} \times \{y\}$, which is readily seen to be $\Gamma$-invariant. We consequently obtain a flat affine connection $D$ of the bundle $F^\times = \Gamma \backslash (\widetilde{M} \times F) \to N^\times = \Gamma \backslash (\widetilde{M} \times N)$ along the horizontal foliation $\mathscr{H}$ of $N^\times$, which enables us to introduce the *tangential exterior derivative*

$d_{\mathcal{H}} : \Omega^p \to \Omega^{p+1}$ by

$$(d_{\mathcal{H}}\omega)(X_0, \cdots, X_p) = \sum_q D_{X_q}\omega(X_0, \cdots \hat{X}_q \cdots, X_p) + \sum_{q<r} \omega([X_q, X_r], X_0, \cdots \hat{X}_q \cdots \hat{X}_r \cdots, X_p)$$

for $\omega \in \Omega^p$, where $X_0, \cdots, X_p$ are tangential vector fields of $(N^\times, \mathcal{H})$, i.e., sections of the bundle $T\mathcal{H}$ over $N^\times$. The flatness of $D$ immediately implies $d_{\mathcal{H}}^2 = 0$, and in consequence we are able to think of the cohomology $H^*(N^\times, \mathcal{H}; F^\times) = H^*(\{\Omega(N^\times, \mathcal{H}; F^\times), d_{\mathcal{H}}\})$ which we call the *tangential de Rham cohomology* of the foliated manifold $(N^\times, \mathcal{H})$ *with coefficients in* the bundle $F^\times$. It is purely elementary to show

LEMMA.

$$H^*(\Gamma; C^\infty(N, F)) \cong H^*(N^\times, \mathcal{H}; F^\times) \quad \textit{for } * = 0, 1 .$$

Due to the lemma, the remaining task for us is to establish a vanishing theorem for the tangential de Rham cohomology. Bochner's trick is presumably the most powerful machinery to show the vanishing. This is indeed the case with our problem. We can prove a vanishing theorem for the tangential de Rham cohomology by means of Bochner's trick. However, in order to apply Bochner's trick we have to introduce a relevant laplacian. The laplacian that is appropriate in our setup has to be a tangential one since so is our exterior derivative. In other words, we need to deal with a tangential laplacian in which only derivatives in the direction tangent to the foliation $\mathcal{H}$ are performed. Although it is an elliptic operator restricted to each leaf, it is degenerate on the whole space $N^\times$. This causes serious problems. The hardest one among them is transverse regularity. Let $\Delta_{\mathcal{H}}$ be our tangential laplacian. If $\Delta_{\mathcal{H}}$ were elliptic, the smoothness of $\Delta_{\mathcal{H}}\omega$ ($\omega \in \Omega^p$) would imply that of $\omega$ itself. But we cannot expect this in our problem. Nevertheless, we can overcome this difficulty by applying stochastic calculus, in particular a tangential diffusion on the foliated manifold $(N^\times, \mathcal{H})$, under some assumption of the foliation $\mathcal{H}$. We refer to [K] for detailed accounts.

## Reference

[K] M. KANAI, *A new approach to the rigidity of discrete group actions*, GAFA **6**(1996), 943-1056.

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