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COLLAPSING AND SOUL THEOREM IN THREE-DIMENSION

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In this article, we present a recent joint work with Takashi Shioya [SY]. We study the topology of three-dimensional Riemannian manifolds with a lower sectional curvature bound collapsing to an Alexandrov space of lower dimension.

For a positive integer n and $D > 0$, we denote by $\mathcal{M}(n, \mathcal{D})$ the isometry classes of closed n-dimensional Riemannian manifolds M with sectional curvature $\geqslant -1$ and diameter $\leqslant D$. The Gromov precompactness theorem [G] implies that for any infinite sequence $\{M_i\}$ in $\mathcal{M}(n, \mathcal{D})$ there is a convergent subsequence of M_i, with respect to the Gromov-Hausdorff distance, whose limit is a compact Alexandrov space X of dimension $\leqslant n$ and curvature $\geqslant -1$. We now assume that M_i itself converges to X and consider the following:

PROBLEM 1. — *Describe the topology of M_i by using the geometry and topology of X.*

In some extremal cases, the following results are known: Perelman [P1, P2] proved that if $\dim X = n$, then M_i is homeomorphic to X. Fukaya and Yamaguchi [FY] proved that if $\dim X = 0$, then $\pi_1(M_i)$ contains a nilpotent subgroup of finite index. Therefore, it suffices to consider only the cases $1 \leqslant \dim X \leqslant n - 1$ for which there are no general results known. For the special case when the singularity is small in a sense, we have the fibration theorem; there exists a locally trivial fibration $M_i \to X$ with almost nonnegatively curved fibre ([Y1], [Y2]). It is also announced in [P3] that if X has no serious singularities (precisely called extremal subsets), there is a homotopical fibration $M_i \to X$. When $1 \leqslant \dim X \leqslant n - 1$ and X has serious singularities, any solution to Problem 1 was

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not known. In the work [SY], we completely solve the problem in the case when $n = 3$ and $\dim X = 1, 2$. In [FY], it was obtained that if $n = 3$ and $\dim X = 0$, then some finite cover of M_i is homeomorphic to either a homotopy sphere, $S^1 \times S^2$, T^3, or a nilmanifold.

We first consider the case $\dim X = 2$. It is known [BGP] that X is a topological manifold possibly with boundary.

From now on, we always assume that all the M_i are orientable.

THEOREM 1. — *If $\dim X = 2$ and X has empty boundary, then M_i is homeomorphic to a Seifert fibred space over X, for which any singular fibre over a point $x \in X$ has type (μ, ν) satisfying $\mu \leqslant 2\pi / L(\Sigma_x)$.*

Here $L(\Sigma_x)$ is the length of the space of directions Σ_x at x.

We describe the situation of Theorem 1 in more detail. It is easy to see that for each $\epsilon > 0$, the set $S_\epsilon(X)$ of singular points of X at which the spaces of directions have lengths less than $2\pi - \epsilon$ is finite. Note that one can apply the fibration theorem on a compact domain X_ϵ of X away from $X - S_\epsilon(X)$ for a small ϵ. Thus a compact domain $M_i(\epsilon)$ of M_i converging to X_ϵ has the structure of S^1-fibration over X_ϵ. We proved that each component of the closure of $M_i - M_i(\epsilon)$ is homeomorphic to $S^1 \times D^2$ and it has a Seifert structure which is compatible to the S^1-bundle structure over $M_i(\epsilon)$.

THEOREM 2. — *If $\dim X = 2$ and X has non-empty boundary, then there is a Seifert fibred space* $\mathrm{Seif}_i(X)$ *over X such that*

1. *M_i is homeomorphic to the union $\mathrm{Seif}_i(X) \cup (\partial X \times D^2)$ glued along their boundaries;*
2. *any singular fibre of $\mathrm{Seif}_i(X)$ over a point $x \in \mathrm{int}\, X$ has type (μ, ν) satisfying $\mu \leqslant 2\pi / L(\Sigma_x)$.*

Applying Theorem 1 to a compact domain M_i' of M_i converging to a compact domain X' away from the boundary of X, we see that M_i' is homeomorphic to a Seifert fibred space, $\mathrm{Seif}_i(X)$, over X'. An essential part in the proof of Theorem 2 is to prove that $M_i - M_i'$ is homeomorphic to $\partial X \times D^2$.

As a corollary of Theorem 2, we have the following

COROLLARY 1. — *Let M_i, X, and $\mathrm{Seif}_i(X)$ be as in Theorem 2, and let g and k denote the genus of X and the number of components of ∂X respectively. Then we have the following factorization:*

$$M_i \simeq S^3 \# \underbrace{S^2 \times S^1 \# \cdots \# S^2 \times S^1}_{f(g,k)} \# L(\mu_1, \nu_1) \# \cdots \# L(\mu_\ell, \nu_\ell),$$

where

$$f(g,k) = \begin{cases} 2g + k - 1 & \text{if } X \text{ is orientable,} \\ g + k - 1 & \text{if } X \text{ is non-orientable,} \end{cases}$$

and (μ_j, ν_j), $1 \leqslant j \leqslant \ell$, *denote the orbit types of singular fibres of* $\mathrm{Seif}_i(X)$.

In particular, the set of homeomorphism classes in the sequence $\{M_i\}$ *is finite.*

Note that a regular fibre of $\mathrm{Seif}_i(X)$ represents the trivial element in $\pi_1(M_i)$. This is the essential reason for the factorization above.

Let us next consider the case when $\dim X = 1$, i.e., X is isometric to either a circle or a closed interval. When X is isometric to a circle, we observe from the fibration theorem ([Y1]) that M_i is a fibre bundle over S^1 whose fibre is S^2 or T^2. The rest in our study is the case when X is isometric to a closed interval.

THEOREM 3. — *If X is isometric to a closed interval, then M_i is homeomorphic to the gluing of U and V along their boundaries, where U and V are respectively either D^3, the twisted product $P^2 \tilde{\times} I$, $S^1 \times D^2$ or $\mathrm{M\ddot{o}} \tilde{\times} S^1$, the twisted product of Möbius band and circle.*

More precisely, we have a decomposition $M_i = B_i \cup A_i \cup C_i$, where A_i is a compact domain converging to a subinterval of X away from the endpoints and B_i and C_i are the components of the closure of the complement of A_i. It follows from the fibration theorem that A_i is homeomorphic to $F_i \times I$, where F_i is either S^2 or T^2. We proved that if $F_i \simeq S^2$ (resp. $F_i \simeq T^2$), then B_i and C_i are homeomorphic to either D^3 or $P^2 \tilde{\times} I$ (resp. $S^1 \times D^2$ or $\mathrm{M\ddot{o}} \tilde{\times} S^1$).

Theorem 1 is the most basic result; for the proof of Theorem 2, we need Theorem 1 and for the proof of Theorem 3, we need both Theorems 1 and 2. Here we give an idea of the proof of Theorem 1.

Let p be an (essential) singular point of the two-dimensional Alexandrov surface X without boundary, and $p_i \in M_i$ be such that $p_i \to p$ under the convergence $M_i \to X$. We have to show that for a fixed small $r > 0$, $B(p_i, r)$ is homeomorphic to $S^1 \times D^2$ for large i. From the fibration theorem applied to the convergence, $\partial B(p_i, r) \to \partial B(p, r)$, we see that $\partial B(p_i, r)$ is homeomorphic to T^2. To get the topological information of collapsed balls $B(p_i, r)$, we use a rescaling argument of the metrics with a suitable choice of base point. In general, we need to retake a base point different from p_i but close to p_i. Let $\hat{p}_i$ be a point of $B(p_i, r)$ which attains a maximum of the averaged distance function from the points of $\partial B(p_i, r)$. Let $\delta_i \to 0$ be the maximum distance from $\hat{p}_i$ to the critical point set of $\hat{p}_i$. Then we can prove that the limit (Y, y_0) of a subsequence of $(\frac{1}{\delta_i} M_i, \hat{p}_i)$ has dimension three; namely, no collapsing occurs in this situation. Note that

1. Y is three-dimensional complete open Alexandrov space with nonnegative curvature without boundary which is a topological manifold;

2. we can recover the topology of $B(p_i, r)$ from that of Y.

Now we need a geometry on Y to classify such spaces. Let us denote by S the soul of Y.

THEOREM 4. — *Y is homeomorphic to the normal bundle of S.*

This is a natural extension of the soul theorem ([CG]) to Alexandrov spaces in three-dimension. It should also be noted that the above result does not hold when Y is not a topological manifold and S is a point. Actually we have a more general classification of three-dimensional complete open Alexandrov spaces with nonnegative curvature without boundary which are not necessarily topological manifolds.

Finally, together with the boundary information $\partial B(p_i, r) \simeq T^2$, the above soul theorem implies that S has dimension one and that $B(p_i, r)$ is homeomorphic to $S^1 \times D^2$, as required.

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