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CIRCLE-PACKING CONNECTIONS WITH RANDOM WALKS AND A FINITE VOLUME METHOD

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ABSTRACT. — Some recent results for random walks on planar graphs and approximation of solutions to the Dirichlet problem for planar domains, both based on properties of circle packings, are overviewed.

1. Introduction and preliminaries

The idea of this paper is to give a short overview of recent results regarding random walks for planar graphs and numerical approximation based on a finite volume method for planar domains, and explore some connections between these two areas. In a section about numerical approximation we are interested in solving a Dirichlet problem for triangular grids. In the case of random walks, we are interested in the type problem and the existence of Dirichlet finite harmonic functions for planar graphs. The results that will be discussed here have one thing in common, that is, they all have been proved using geometric objects called circle packings.

Circle packings can be described as follows: if $\mathbb{K}$ is a simplicial 2-complex which is simplicially isomorphic to a planar triangulation of a simply connected domain, then a collection $\mathcal{P} = \{C_{\mathcal{P}}(v)\}_{v \in \mathbb{K}^0}$ of circles in the plane is a *circle packing* for $\mathbb{K}$ iff for every edge uw in $\mathbb{K}$ the circles $C_{\mathcal{P}}(u)$ and $C_{\mathcal{P}}(w)$ are externally tangent; here $\mathbb{K}^0$ denotes the set of vertices of $\mathbb{K}$. For the purpose of this paper we add one more condition to the definition of circle packings, that all circles in circle packings have disjoint interiors. Figure 1 shows two different circle packings for the same simplicial complex.

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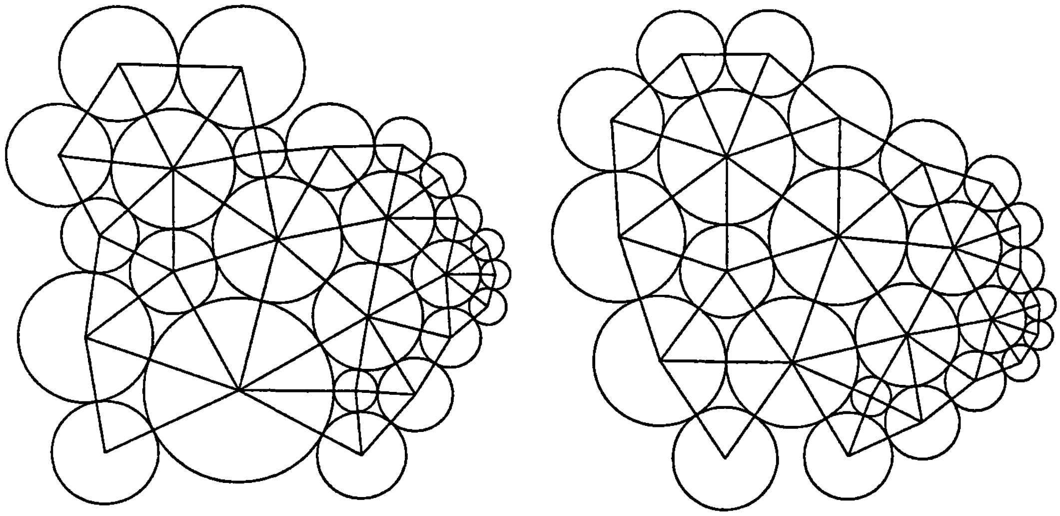

Figure 1
Different circle packings (and their underlying triangulations) for the same 2-complex

In general, a circle packing $\mathcal{P}$ can be identified with a pair $(\mathbb{K}, R_{\mathcal{P}})$, where $\mathbb{K}$ is the simplicial complex of $\mathcal{P}$ (which describes the tangency pattern of the packing) and $R_{\mathcal{P}} : \mathbb{K}^0 \to (0, \infty)$ is the radius function of $\mathcal{P}$ (which describes a metric of the packing), where $R_{\mathcal{P}}(v)$ is the radius of the circle in $\mathcal{P}$ associated with the vertex v. For more on circle packings and their properties the reader is referred to [BSt1],[BSt2], [BoSt], [CdV], [CdVMa], [D1], [HR], [HSc1], [HSc2], [RS], [St1, 2], and [Th1], [Th2].

2. Approximation

Circle packings have been used for approximation of Riemann mappings ([HR], [HSc3], [RS], [St1], [Th2]). The idea there was to use circle packings of very small mesh and Finite Riemann Mapping Theorem ([BSt1], [RS], [Th1], [Th2]) to get an approximation of classical Riemann mappings.

Here we are interested in applying the circle packing approach in solving the following Dirichlet problem:

Suppose Ω is a Jordan domain, ϕ is a continuous function on $\partial\Omega$, and $f \in L^2(\Omega)$ (i.e., f is a square-itegrable function over the set Ω). Find $U : \overline{\Omega} \to \mathrm{R}$ such that

$$-\Delta U = f \text{ in } \Omega \quad \text{and} \quad u = \phi \text{ on } \partial\Omega. \tag{$*$}$$

Equivalently, the above problem can be stated without differential condition: $-\Delta\, U = f$ but replaced by the following integral condition:

$$-\frac{1}{|\mathcal{V}|}\int_{\partial\mathcal{V}} \nabla U \cdot \overrightarrow{\eta}\, ds = \frac{1}{|\mathcal{V}|}\int_{\mathcal{V}} f\, dx \qquad **)$$

for every subset $\mathcal{V} \subset \overline{\Omega}$ with Lipschitz boundary, where $\overrightarrow{\eta}$ denotes the outward unit normal vector on $\partial\mathcal{V}$. In the sequence we will show how the solution to the Dirichlet problem can be approximated using circle packings and piecewise linear functions given by a finite volume condition similar to the above integral condition. For the proofs of the results presented below the reader should see [D3] and [D4].

Let $\mathcal{P} = \{\mathcal{C}_{\mathcal{P}}(v)\}_{v\in\mathbb{K}^0}$ be a circle packing for a complex $\mathbb{K}$. We denote by $\mathrm{T}(\mathcal{P})$ the triangulation induced by $\mathcal{P}$, i.e. it is the triangulation given by connecting centers of tangent circles in $\mathcal{P}$ by line segments. >From our definition of circle packings it follows that $\mathrm{T}(\mathcal{P})$ and $\mathbb{K}$ are isomorphic 2-complexes. We write $z_w(\mathcal{P})$ for the center of the circle in $\mathcal{P}$ associated with the vertex w in $\mathbb{K}^0$. By $\mathcal{V}_w(\mathcal{P})$ we denote the volume given by $\mathcal{P}$ associated with the vertex w in $\mathbb{K}^0$; it is the polygon circumscribed on $\mathcal{C}_{\mathcal{P}}(w)$ with its edges perpendicular to the edges of $\mathrm{T}(\mathcal{P})$ coming out of the vertex w (Figure 2(a)). (A slight modification of this definition is required for boundary vertices in which case volumes are polygons bounded by perpendicular edges and boundary edges as on the picture.) If $z_{uw}(\mathcal{P})$ is the edge in $\mathrm{T}(\mathcal{P})$ with end points $z_u(\mathcal{P})$ and $z_w(\mathcal{P})$, then the common side of volumes $\mathcal{V}_u(\mathcal{P})$ and $\mathcal{V}_w(\mathcal{P})$ will be denoted $z^*_{uw}(\mathcal{P})$ (see Figure 2(b)). It will also be helpful to use $IT^0(\mathcal{P})$ and $\partial T^0(\mathcal{P})$ to denote sets of interior and boundary vertices of $\mathrm{T}(\mathcal{P})$, respectively, and write $u \sim w$ to indicate that vertices u and w are adjacent.

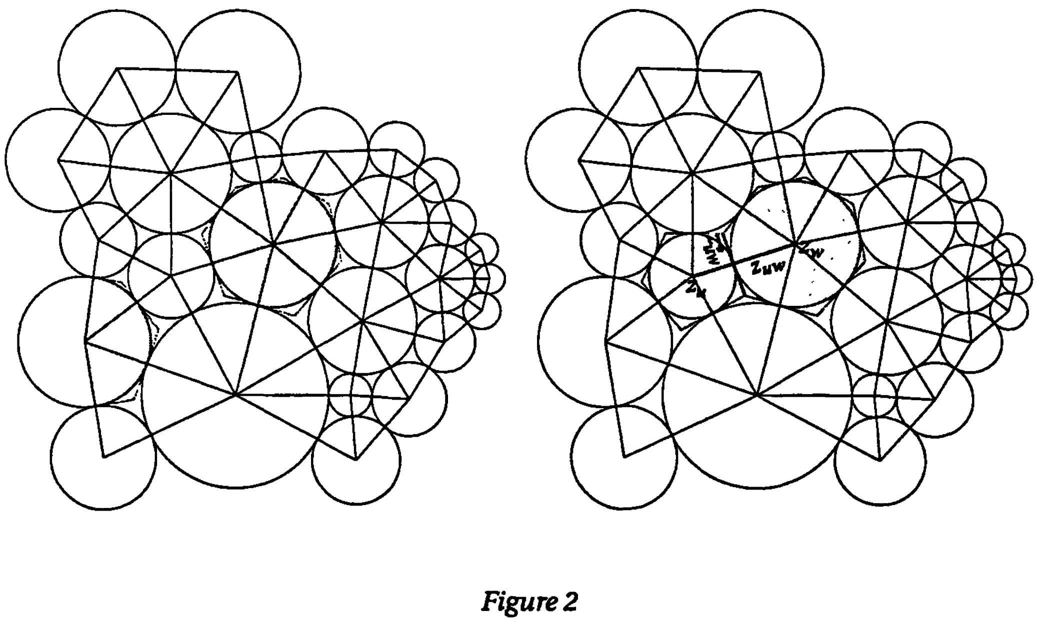

Figure 2
(a) Volumes induced by a circle packing, (b) Corresponding edges and sides

Notation. — If there is no reason for confusion, we will generally drop the index of a packing from the notation, e.g. by writing z_w instead of $z_w(\mathcal{P})$.

We can now define a discrete Dirichlet problem:

Suppose that $\mathrm{T}(\mathcal{P})$ is finite and simply connected, $F : IT^0(\mathcal{P}) \to \mathbf{R}$, and $\Phi : \partial T^0(\mathcal{P}) \to \mathbf{R}$. Find $U : T^0(\mathcal{P}) \to \mathbf{R}$ such that

$$\begin{cases} \dfrac{1}{|\mathcal{V}_u|} \displaystyle\sum_{w \sim u} \dfrac{|z^*_{uw}|}{|z_{uw}|} (U(z_u) - U(z_w)) = F(z_u) & \text{for } z_u \in IT^0 \\ U(z) = \Phi(z) & \text{for } z \in \partial T^0, \end{cases} \tag{$\star$}$$

where $|\cdot|$ denotes the Euclidean length (area) of a segment (respectively, of a volume).

Remark 1.

1. It can easily be verified, using Green's theorem, that (**) leads exactly to the condition $(\star)$ when (i) $\overline{\Omega} := \mathrm{T}(\mathcal{P})$, (ii) U is linear on each triangle in $\mathrm{T}(\mathcal{P})$, (iii) $F(z_w) := \frac{1}{|\mathcal{V}_w|} \int_{\mathcal{V}_w} f\,dx$, and (iv) volumes $\mathcal{V}$'s in (**) are only volumes $\mathcal{V}_w$'s given by $\mathcal{P}$.

2. The discrete problem $(\star)$ is always solvable and Maximum Principle holds for solutions of $(\star)$ (see [D4]).

Suppose that $\bar{\phi}$ is a continuous extension of the function ϕ and is defined inside Ω in some neighborhood of $\partial\Omega$. For example, when $\partial\Omega$ is C^2 (i.e., 2-times continuously differentiable curve) then there exists ϵ such that when $\mathrm{dist}(z, \partial\Omega) < \epsilon$ then there is a unique point $z_\partial \in \partial\Omega$ with $\mathrm{dist}(z, z_\partial) = \mathrm{dist}(z, \partial\Omega)$, and $\bar{\phi}$ can be defined by a projection, i.e. $\bar{\phi}(z) := \phi(z_\partial)$. Suppose further that that $\mathcal{P}$ is a circle packing contained in Ω and that $\bar{\phi}$ is defined on $\partial T^0(\mathcal{P})$. We introduce the corresponding approximate solution $U_{\mathcal{P}}$ of the classical Dirichlet problem (*) as the solution of $(\star)$ with $\Phi(z) := \bar{\phi}(z)$ for $z \in \partial T^0(\mathcal{P})$ and $F(z_w) := \frac{1}{|\mathcal{V}_{\underline{w}}|} \int_{\mathcal{V}_w} f\,dx$ for $z_w \in IT^0(\mathcal{P})$.

Using circle packings of fine mesh one obtains the following result (see [D4]).

THEOREM 1. — *Let Ω be a Jordan domain. Suppose that $\{\mathcal{P}_n\}$ is a collection of finite circle packings contained in Ω such that*

(1) radii of circles in $\mathcal{P}_n$ go to 0 as $n \to \infty$,

(2) there is a constant $d > 0$ such that no vertex in $\mathrm{T}_n := \mathrm{T}(\mathcal{P}_n)$ has more than d neighbours for all n, and

(3) triangulations T_n exhaust Ω.

Denote by U the solution of the Dirichlet problem: $-\Delta U = f$ in Ω and $U = \phi$ on $\partial\Omega$, where $f \in L^2(\Omega)$ and ϕ is a continuous function on $\partial\Omega$. Suppose $\bar{\phi}$ is a continuous extension of ϕ to some neighborhood of $\partial\Omega$. Write U_n for the corresponding discrete solutions for packings $\mathcal{P}_n$. Then $\|u - u_n\|_{L^2(\widetilde{\mathrm{T}_n})} \to 0$ as $n \to \infty$, where $\widetilde{\mathrm{T}_n}$ is the triangulation *obtained from T_n by disregarding all boundary triangles of T_n. Furthermore, if $\{\mathcal{P}_n\}$ is a quasiuniform family then $U_n \to U$ almost uniformly in Ω as $n \to \infty$.*

Remark 2.

1. Regarding terms used in the above statement, $|| \cdot ||_{L^2(A)}$ stands for the L^2-norm on the set A. Also, an almost uniform convergence in a domain means the uniform convergence on any compact subset of the domain. And, a sequence $\{\mathcal{P}_n\}$ is a quasiuniform family of packings if there is a constant κ such that for all n the ratio of radii of any two circles in any packing $\mathcal{P}_n$ is less than κ.

2. A proof of the above result is given in [D4]. It is a consequence of properties of circle packings and results from Sections 3 and 4 of [D4] for convergence of discrete solutions given by the finite volume method for general triangular grids.

One of the key results in [D4] is Thm. 3.5 which is essentially due to [C]/[CMM]. While visiting Ecole Normale Supérieure de Lyon, the author have learned that this result has also been recently independently proven in [EGM]. (For some other related results the reader is referred to [BR], [Ha], and [Hn].)

3. Using Discrete Riemann Mapping Theorem (see [Th1], [BSt1], [RS]) for circle packings and the above theorem, one can transfer Dirichlet problem for arbitrary Jordan domain to a standard domain such as the unit disk (see Cor. 5.4 in [D4]).

3. Random Walks

Let G be a graph with the sets of vertices and edges denoted by G^0 and G^1, respectively. Suppose that $\Pi : G^0 \times G^0 \to [0, \infty)$ is a function such that:

(1) $\Pi(u, w) = 0$ if u and w are not neighbors,

(2) $\sum_{w \sim v} \Pi(v, w) = 1$ for every interior vertex v of G.

Then the pair (G, Π) is a *random walk* on G and Π is called the *transition probability*. Moreover, if there exists a function $\Gamma : G^0 \times G^0 \to (0, \infty)$ such that $\Gamma(u, v) = \Gamma(v, u)$ and

$$\Pi(u, v) = \frac{\Gamma(u, v)}{\sum_{w \sim u} \Gamma(u, w)}$$

for every adjacent vertices u and v, then Π is said to be a *reversible* random walk and $\Gamma(\cdot, \cdot)$ is called the *conductance* function. If $\Gamma \equiv 1$ for all edges then the random walk induced by Γ will be called *simple*. We recall also that G is said to be *recurrent* if the simple random walk on G is recurrent (i.e., the random walk with probability 1 always returns to its starting point), and it is called *transient* otherwise (see [DoSn], [S], [W]). Finally, a function f :

$G^0 \to \mathbf{R}$ is called *harmonic* for (G, Π) if $f(u) = \sum_{w\sim u} \Pi(u,w)f(w)$, and it is called *Dirichlet finite* for (G, Π) given by conductance Γ if $\sum_{w\sim u} (f(u) - f(w))^2\Gamma(u,w) < \infty$.

Assume now that G is simplicially isomorphic to a planar triangulation and $\mathcal{P}$ is a circle packing for G. We will introduce a random walk on G induced by $\mathcal{P}$. First, we define the conductance $\Gamma_{\mathcal{P}} : G^0 \times G^0 \to (0, \infty)$ as follows

$$\Gamma_{\mathcal{P}}(u,w) = \frac{|z^*_{uw}(\mathcal{P})|}{|z_{uw}(\mathcal{P})|}.$$

Notice that $\Gamma_{\mathcal{P}}$ is well defined by being symmetric. Moreover, it is inversely proportional to the distance between vertices $z_u(\mathcal{P})$ and $z_w(\mathcal{P})$ in the triangulation $\mathrm{T}(\mathcal{P})$ (which, as we recall, is isomorphic to G), and it is directly proportional to the "flux" going through the common side (i.e., edge) of the volumes $\mathcal{V}_u(\mathcal{P})$ and $\mathcal{V}_w(\mathcal{P})$, where the flux is measured as the length (i.e., 1-dimentional area) of the side. We denote by $\Pi_{\mathcal{P}}$ the random walk on G given by the conductance $\Gamma_{\mathcal{P}}$.

The next result relates the existence and type of infinite packings with the type of underlying tangency graphs.

THEOREM 2. — *Let $\mathbb{K}$ be a simplicial 2-complex isomorphic to a planar triangulation without boundary (i.e., every vertex in the complex is an interior vertex).*

(a) If the simple random walk on the 1-skeleton $\mathbb{K}^1$ of $\mathbb{K}$ is recurrent then there exists a circle packing $\mathcal{P}$ for $\mathbb{K}$ such that $\mathrm{T}\mathcal{P})$ is a triangulation of the plane. Conversely, if $\mathbb{K}$ is of bounded degree and the simple random walk on its 1-skeleton is transient, then there exists a circle packing $\mathcal{Q}$ for $\mathbb{K}$ such that $\mathrm{T}(\mathcal{Q})$ is a triangulation of the unit disk.

(b) If $\mathbb{K}$ is of bounded degree and transient then there exist nontrivial Dirichlet finite harmonic functions for the simple random walk on $\mathbb{K}^1$.

Remark 3.

1. The above results have originally been proved, in case of the part (a), in [HSc2] and also in [Mc], and in case of the part (b), in [De]. It should also be noted that in [BeSc] it was shown that the result of the part (b) is true not only for triangulations but for general planar graphs.

2. A simple proof of the above theorem is given in [D3]. It is based on a beautiful and intriguing uniformization result of [HSc1] that states, roughly speaking, that for each simply connected planar triangulation there exists a circle packing that "fills in" the plane or the unit disk, but not both, and that such a packing is unique up to Möbius transformations preserving the underlying space.

We will now discuss connections between harmonic measure of a plane domain and exit probabilities of random walks on circle packings that fill out the domain. Recall that if Ω is a domain in the plane, A is a subset of $\partial\Omega$, and z is a point in Ω, then the probability $M_\Omega(z, A)$ that a Brownian particle, after staring at the point z, will hit the boundary $\partial\Omega$ for the first time and such a hit will be at some point of the set A is called the *exit probability* from z through A in Ω. One can similarly introduce the exit probability for finite graphs and circle packings. For this, suppose that $\mathcal{P}$ is a finite circle packing and $\mathrm{T}(\mathcal{P})$ is the underlying triangulation given by $\mathcal{P}$. If A is a subset of boundary vertices of $\mathrm{T}(\mathcal{P})$ and z is an interior vertex of $\mathrm{T}(\mathcal{P})$ then we define the exit probability $M_\mathcal{P}(z, A)$ from z through A in $\mathrm{T}(\mathcal{P})$ as the probability of the random walk $\Pi_\mathcal{P}$, originated at z, reaching for the first time the boundary of $\mathrm{T}(\mathcal{P})$ at some vertex in A.

From approximation results in the earlier section and properties of harmonic functions and harmonic measure in the classical setting we obtain the following result (see [D4] for a proof).

THEOREM 3. — *Let Ω be a Jordan domain with C^2 boundary. Let γ be an arc in $\partial\Omega$. Suppose $\mathcal{P}_n$ is a quasiuniform sequence of circle packings as in Theorem 1. Write γ_n for the set $\{z \in \partial T_n^0 : z_\partial \in \gamma\}$, where, as before, z_∂ denotes the nearest point on $\partial\Omega$ to the point z, and $T_n = T(\mathcal{P}_n)$. Then, for any compact subset ω of Ω,*

$$\lim_{n\to\infty} \sup_{z\in\omega} |M_{\mathcal{P}_n}(z, \gamma_n) - M(z, \gamma)| = 0.$$

Remark 4.

1. As we pointed out earlier, because $\partial\Omega$ is of class C^2, there exists a neighborhood of $\partial\Omega$ such that any point from this neighborhood has exactly one point in $\partial\Omega$ that is closest to it. In particular, we get that "the nearest point" is well defined.

2. The condition on the smoothness of $\partial\Omega$ can be weakened; for example, it can be assumed that $\partial\Omega$ is piecewise C^2 without any loss in the conclusion of the above result. However, as the boundary of Ω gets more bizarre then there arises a problem of establishing good correspondence between (arcs of) $\partial\Omega$ and (arcs of) boundaries of polygonal domains $\mathrm{T}(\mathcal{P}_n)$.

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Figure 1 was created using `CirclePack` [St3].

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