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HIROYUKI MINAKAWA

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## MILNOR-WOOD INEQUALITY FOR CRYSTALLOGRAPHIC GROUPS

*Hiroyuki MINAKAWA*

### 0. Introduction

Let  $H^2$  be the hyperbolic plane and  $\text{Isom}^+ H^2$  the isometry group of  $H^2$ . A 2-dimensional crystallographic group  $\Gamma$  is a cocompact discrete subgroup of  $\text{Isom}^+ H^2$ . As an abstract group,  $\Gamma$  is isomorphic to a unique group of the form

$$\Gamma(g; p_1, \dots, p_n) = \langle a_1, b_1, \dots, a_n, b_n, c_1, \dots, c_n \mid \\ c_i^{p_i} = 1 \ (i = 1, \dots, n), \\ c_1 \cdots c_n [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle$$

with  $g \geq 0$ ,  $p_i \geq 2$  and  $\chi(\Gamma(g; p_1, \dots, p_n)) < 0$ . Here  $\chi(\Gamma(g; p_1, \dots, p_n)) = 2 - 2g - \sum_{i=1}^n (p_i - 1)/p_i$  is the rational Euler characteristic of the group  $\Gamma(g; p_1, \dots, p_n)$ .

Let  $G^r$  be the group of all orientation preserving diffeomorphisms of class  $C^r$  ( $r = 0, 1, \dots, \infty$ ). For any homomorphism  $\phi : \Gamma \rightarrow G^r$ ,  $\Gamma$  acts on the trivial  $S^1$  bundle  $H^2 \times S^1$  through  $\phi$ . So we can construct a foliated Seifert bundle  $E_\phi = H^2 \times S^1 / \Gamma \rightarrow H^2 / \Gamma = \Sigma_g$  ( $g = \text{genus of } \Gamma$ ). We define the Euler number  $eu(\phi)$  of  $\phi$  by

$$\begin{aligned} eu(\phi) &= \text{the Euler number of Seifert bundle } E_\phi \rightarrow \Sigma_g \\ &= eu(E_\phi \rightarrow \Sigma_g). \end{aligned}$$

If  $\Gamma$  is a surface group, then we have the Milnor-Wood inequality

$$|eu(\phi)| \leq |\chi(\Sigma)| = |\chi(\Gamma)|.$$

Moreover, if  $\phi_i : \Gamma \rightarrow G^0$  ( $i = 1, 2$ ) both have the maximal Euler number  $eu(\phi_1) = eu(\phi_2) = \pm \chi(\Gamma)$ , then  $\phi_1$  is semi-conjugate to  $\phi_2$ .

In this paper, we shall consider a generalization of the Milnor-Wood inequality for homomorphisms from crystallographic groups to  $G^0$ , and we also prove that there exists a semi-conjugacy phenomenon in the case that the homomorphism has the maximal Euler number.

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## 1. Homological definition of Euler number

In this section, we give a homological definition of Euler number  $eu(\phi)$  first.

Let  $\Gamma = \Gamma(g; p_1, \dots, p_n)$  be a crystallographic group.  $\Gamma$  contains a finite index subgroup  $\Gamma_{g'}$  which is isomorphic to the fundamental group of a closed surface  $\Sigma_{g'}$ . So we have that the inclusion  $i : \Gamma_{g'} \rightarrow \Gamma$  induces an isomorphism

$$i_* : H_2(\Gamma_{g'}; Q) \rightarrow H_2(\Gamma; Q).$$

Since given presentation of  $\Gamma_{g'}$  determines an orientation of the closed surface  $\Sigma_{g'}$ , then there exists the fundamental class  $[\Gamma_{g'}] \in H_2(\Gamma_{g'}; Z) \cong H_2(\Sigma_{g'})$ . We use the notation  $[\Gamma_{g'}]_Q$  which is the image of  $[\Gamma_{g'}]$  by Bockstein homomorphism

$$H_2(\Gamma_{g'}; Z) \rightarrow H_2(\Gamma_{g'}; Q).$$

Now we define the fundamental class  $[\Gamma]$  of  $\Gamma$  by

$$[\Gamma] = i_*[\Gamma_{g'}]_Q / \text{index}(\Gamma; \Gamma_{g'}).$$

We can check easily that this definition does not depend on the choice of the finite index subgroup  $\Sigma_{g'}$ .

## 2. Cohomological definition of the Euler number

Given a surface group  $\Gamma_g$  and a homomorphism  $\phi : \Gamma_g \rightarrow G^0$ , Euler number  $eu(\phi)$  is equal to the pairing

$$eu(\phi) = \langle \phi^* e, [\Gamma_g] \rangle.$$

Here,  $e \in H^2(G^0; Z)$  denotes the universal Euler class. The symbol  $e_Q$  denotes the rational universal Euler class which is the image of  $e$  by Bockstein homomorphism  $H^2(G^0; Z) \rightarrow H^2(G^0; Q)$ .

**PROPOSITION 2.1.** — *For any homomorphism  $\phi : \Gamma \rightarrow G^0$ , we have the formula*

$$eu(\phi) = \langle \phi^* e_Q, [\Gamma] \rangle.$$

In order to prove this proposition, we need the following lemma.

**LEMMA 2.2.** — *Let  $\pi_i : M_i \rightarrow \Sigma_i (i = 1, 2)$  be Seifert fibrations. Assume that there exist maps  $\tilde{h} : M_1 \rightarrow M_2$  and  $h : \Sigma_1 \rightarrow \Sigma_2$  such that  $\pi_2 \circ \tilde{h} = h \circ \pi_1$ ,  $\text{degree}(h) = k$  and  $\text{degree}(\tilde{h}|_{\text{regular fiber}}) = l$ . Then we have  $e(M_1 \rightarrow \Sigma_1) = (k/l)e(M_2 \rightarrow \Sigma_2)$ .*

*Proof of Proposition 2.1* We take a finite index subgroup  $\Gamma_{g'}$  of  $\Gamma$  which is isomorphic to  $\pi_1(\Sigma_{g'})$ . We put that  $k = \text{index}(\Gamma; \Gamma_{g'})$ . So there exist continuous maps

$\bar{h} : E_{\phi \circ i} \rightarrow E_\phi$  and  $h : \Sigma_{g'} \rightarrow \Sigma_g$  such that  $\pi_\phi \circ \bar{h} = h \circ \pi_{\phi \circ i}$ ,  $\text{degree}(h) = k$  and  $\text{degree}(\bar{h}|_{\text{regular fiber}}) = 1$ . By using the lemma above, we have

$$\begin{aligned} eu(\phi) &= eu(E_\phi \rightarrow \Sigma_g) \\ &= eu(E_{\phi \circ i} \rightarrow \Sigma_{g'})/k \\ &= eu(\phi \circ i)/k \\ &= \langle (\phi \circ i)^* e, [\Sigma_{g'}] \rangle / k \\ &= \langle (\phi \circ i)^* e_Q, [\Sigma_{g'}]_Q \rangle / k \\ &= \langle \phi^* e_Q, i_* [\Sigma_{g'}]_Q / k \rangle \\ &= \langle \phi^* e_Q, [\Gamma] \rangle. \end{aligned}$$

□

The same technique as in the proof of Proposition 2.1 gives us a generalization of the Milnor-Wood inequality for homomorphisms from crystallographic groups to  $G^0$ .

**THEOREM 2.3.** — *Let  $\Gamma$  be a crystallographic group. For any homomorphism  $\phi : \Gamma \rightarrow G^0$ , we have the following inequality*

$$|eu(\phi)| \leq |\chi(\Gamma)|.$$

*Proof.* We use the same notations as in the proof of Proposition 2.1. Then we have

$$|eu(\phi)| = |eu(\phi \circ i)/k| \leq |\chi(\Gamma_{g'})|/k = |\chi(\Gamma)|.$$

The last equality follows from the definition of the rational Euler characteristic  $\chi(\Gamma)$  (see [8]). □

### 3. Semi-conjugacy in maximal Euler numbers

Let  $\Gamma$  be a crystallographic group and  $T_1 H^2$  a unit tangent bundle of the hyperbolic plane  $H^2$ .  $\Gamma$  acts on  $T_1 H^2$ , since  $\Gamma$  acts on  $H^2$  isometrically. So we can construct a Seifert bundle  $E(\Gamma) = T_1 H^2 / \Gamma \rightarrow H^2 / \Gamma = \Sigma_g$  whose total holonomy homomorphism is the identity map  $\phi_\Gamma : \Gamma \rightarrow \Gamma \subset PSL(2, \mathbb{R})$ . We know that  $eu(\phi_\Gamma) = \chi(\Gamma)$ . The following theorem is a generalization of a theorem of S. Matsumoto to crystallographic groups. In [6], he proved this theorem for surface groups.

**THEOREM 3.1.** — *Let  $\Gamma$  be as above. For given homomorphism  $\phi : \Gamma \rightarrow G^0$ , there exist a continuous degree one map  $h : S^1 \rightarrow S^1$  such that*

$$\phi_\Gamma(\gamma) \circ h = h \circ \phi(\gamma)$$

for any  $\gamma \in \Gamma$ .

By [5], it suffices to show that

$$\rho(\phi_\Gamma(\gamma)) = \rho(\phi(\gamma))$$

for any  $\gamma \in A$  which is a generating system of  $\Gamma$ . Here  $\rho(f) \in S^1$  is rotation number of  $f \in G^0$ . In order to show this, we need the following formula which is called Milnor's algorism.

LEMMA 3.2. — For any homomorphism  $\phi : \Gamma(g; p_1, \dots, p_n) \rightarrow G^0$  we can calculate the Euler number  $eu(\phi)$  as follows. We choose any lifts  $\widetilde{\phi}(a_i), \widetilde{\phi}(b_i), \widetilde{\phi}(c_i) \in G^0$ . Then, the number

$$\bar{\rho}([\widetilde{\phi}(a_1), \widetilde{\phi}(b_1)]) \circ \dots \circ [\widetilde{\phi}(a_g), \widetilde{\phi}(b_g)] \circ \widetilde{\phi}(c_1) \circ \dots \circ \widetilde{\phi}(c_n) + \sum_{i=1}^n \rho(\widetilde{\phi}(c_i))$$

does not depend on the choice of lifts. This number is equal to  $eu(\phi)$ .

Where,  $\bar{\rho}(\tilde{f})$  is the traslation number of  $\tilde{f}$ . We can prove the following lemma by Lemma 3.2 with [1], [4] and [7].

LEMMA 3.3. — For any homomorphism  $\phi : \Gamma(g; p_1, \dots, p_n) \rightarrow G^0$ , we have that

$$\rho(\phi(\gamma)) = \begin{cases} 0 & \text{if } \gamma = a_1, \dots, a_g, b_1, \dots, b_g \\ [1/p_i] & \text{if } \gamma = c_i (i = 1, \dots, n) \end{cases}$$

if  $eu(\phi) = \chi(\Gamma)$ .

## References

- [1] D. EISENBUD, U. HIRSCH and W. NEUMANN, *Transverse foliations of Seifert bundles and self homeomorphism of the circle*, Comment. Math. Helv. 56 (1981), 638–660.
- [2] E. GHYS, *Groupes d'homéomorphismes du cercle et cohomologie bornée*, Contemporary Math. 58, Part 3 (1987), 81–106.
- [3] M. JANKINS and W. NEUMANN, *Lectures on Seifert manifolds*, Univ. of Maryland (1981).
- [4] M. JANKINS and W. NEUMANN, *Rotational numbers of products of circle homeomorphisms*, MathSoc., vol.98, no.1 (1986), 163–168.
- [5] S. MATSUMOTO, *Some remarks on foliated  $S^1$  bundles*, Invent. Math. 90 (1987), 343–358.
- [6] R. NAIMI, *Foliations transverse to fibers of Seifert manifolds*, Comment. Math. Helv. 69 (1994), 155–162.
- [7] C.I.C. WALL, *Rational Euler characteristics*, Proc. Cambridge Philos. Soc. 57 (1961), 182–184.

Department of Mathematics  
Faculty of Science  
Hokkaido University  
Sapporo 060  
Japan  
e-mail : minakawa@math.hokudai.ac.jp