

SÉMINAIRE DE THÉORIE SPECTRALE ET GÉOMÉTRIE

BERNHARD LEEB

Compactness of moduli spaces of negatively curved metrics

Séminaire de Théorie spectrale et géométrie, tome 10 (1991-1992), p. 35-37

http://www.numdam.org/item?id=TSG_1991-1992__10__35_0

© Séminaire de Théorie spectrale et géométrie (Chambéry-Grenoble), 1991-1992, tous droits réservés.

L'accès aux archives de la revue « Séminaire de Théorie spectrale et géométrie » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

COMPACTNESS OF MODULI SPACES OF NEGATIVELY CURVED METRICS

par *Bernhard LEEB*

For a discrete group Γ , we define the space of d -dimensional *homotopy hyperbolic structures* $\mathcal{H}^d(\Gamma)$ to be the set

$$(1) \quad \text{Hom}_{d,f}(\Gamma, G)/G$$

of conjugacy classes of discrete and faithful representations $\Gamma \rightarrow G$, where G is the group of isometries of d -dimensional hyperbolic space \mathbb{H}^d . Elements of $\mathcal{H}^d(\Gamma)$ can be thought of as pairs (M, ϕ) , where M is a complete d -dimensional hyperbolic orbifold with fundamental group $\cong \Gamma$ and ϕ is an isomorphism of Γ to $\pi_1(M)$ defined up to conjugation. For example, if S is a closed surface, then $\mathcal{H}^2(\pi_1(S))$ is classical Teichmüller space. If Γ is finitely generated, the algebraic description (1) provides $\mathcal{H}^d(\Gamma)$ with a natural topology.

We are interested in the *question* when these moduli spaces are compact. Compactness of $\mathcal{H}^d(\Gamma)$ can be regarded as a rigidity phenomenon of Γ .

Here are some cases where $\mathcal{H}^d(\Gamma)$ is known to be compact:

- 1) If Γ is a lattice in $\text{Isom}(\mathbb{H}^3)$, then $\mathcal{H}^3(\Gamma)$ is finite as a consequence of Mostow's rigidity theorem.
- 2) Thurston proved in [Th] that $\mathcal{H}^3(\Gamma)$ is compact for a much larger class of groups, namely for fundamental groups of compact acylindrical 3-manifolds.
- 3) Morgan and Shalen [MS1,2] showed for the same groups Γ that $\mathcal{H}^d(\Gamma)$ is compact even for all $d \geq 3$.

The spaces $\mathcal{H}^d(\Gamma)$ are part of a bigger family: Define for any negatively curved symmetric space Y in an analogous way the moduli space $\mathcal{H}(\Gamma, Y)$ of *homotopy Y -structures* on Γ . We ask the same question of compactness for them. W. Goldman and J. Morgan showed [unpublished] that the method of Morgan and Shalen still works for

$\mathcal{H}(\Gamma, Y)$ if Y is a complex hyperbolic space, i.e. $\mathcal{H}(\Gamma, Y)$ is compact for the same class of discrete groups Γ as in 2) and 3).

The method of Morgan and Shalen goes as follows: They compactify $\mathcal{H}^d(\Gamma)$ for finitely generated groups Γ which are not virtually nilpotent by means of algebraic geometry and identify the ideal points of the compactification with small actions of Γ on metric trees without fixed points. (We shall give the necessary definitions below.) They prove that the groups Γ mentioned in 3) do not admit actions of this kind. Hence, there are no ideal points in the compactification and $\mathcal{H}^d(\Gamma)$ is therefore compact in this case.

Bestvina [B] describes a purely geometric way of constructing small actions of Γ on metric trees from unbounded sequences in $\mathcal{H}^d(\Gamma)$. This geometric point of view opens the way and suggests a generalization to variable curvature. A modification of Bestvina's construction can be carried out in nonpositive curvature: We construct limits of unbounded sequences of Γ -actions on Hadamard spaces, the analogues of Hadamard manifolds in the context of length spaces. In this general setting, the limits are no longer Γ -actions on metric trees but rather on "metric 1-complexes". However, for limits of Γ -actions on Hadamard spaces with negative curvature bounds of uniformly bounded ratio, the full conclusion of Bestvina in the case of constant negative curvature can be preserved, i.e. the limits are small Γ -actions on metric trees without fixed points. The case of variable negative curvature, in fact more generally of Gromov-hyperbolicity, has been treated by Paulin in [P].

This motivates introducing moduli spaces of actions on Hadamard spaces of bounded negative curvature: Let Γ be a finitely generated group which is not virtually nilpotent. Fix a dimension $d \in \mathbb{N}$ and curvature bounds $-b^2 \leq -a^2 < 0$. Define the moduli space $\mathcal{M}(\Gamma, d, -b^2, -a^2)$ to be the space of properly discontinuous and effective actions of Γ by isometries on Hadamard spaces with these geometric data. We give the spaces $\mathcal{M}(\Gamma, \dots)$ a topology which is a variant of Gromov-Hausdorff topology and generalizes the algebraically defined topology of the moduli spaces $\mathcal{H}(\Gamma, Y)$ of homotopy Y -structures, i.e. for suitable values of $d, -b^2, -a^2$ they are closed topological subspaces of $\mathcal{M}(\Gamma, \dots)$. The $\mathcal{M}(\Gamma, \dots)$ turn out to be Hausdorff and locally compact.

By the generalization of Bestvina's construction, the $\mathcal{M}(\Gamma, \dots)$ are compact for groups Γ satisfying the following fixed point property: (The necessary definitions shall be given below.)

(F) *Every small action of Γ on a metric trees by isometries has a fixed point.*

Hence, we obtain for these groups Γ a simultaneous proof of the compactness of $\mathcal{H}(\Gamma, Y)$ for all negatively curved symmetric spaces Y . Another consequence of the compactness of the $\mathcal{M}(\Gamma, \dots)$ is: *If Γ satisfies (F), then the space of closed manifolds M with negative curvature bounded by $-b^2 \leq K_M \leq -a^2 < 0$ and fundamental group isomorphic to Γ is compact in the Gromov-Hausdorff topology.* This result can also be deduced in a straight-forward way by the methods in [G1,2].

We are led to the *question* of which discrete groups Γ satisfy (F). Let us recall two definitions: A *metric tree* is a length space which is 0-hyperbolic in the sense of Gromov. An action of Γ on a metric tree by isometries is called *small* if the stabilizers of arcs are small, i.e. if they do not contain non-abelian free subgroups.

As mentioned above, Morgan and Shalen showed that fundamental groups of compact acylindrical 3-manifolds have property (F). We give an argument proving property (F) for groups which contain enough subgroups for which we know property (F) already and which interfere with each other sufficiently. A concrete example to which this method applies are arithmetic lattices in $SO(n, 1)$, $n \geq 3$, defined by quadratic forms with coefficients in algebraic number fields. This method also yields examples of discrete groups which are not finitely generated and satisfy property (F).

References

- [B] BESTVINA M. — *Degenerations of the hyperbolic space*, Duke Math. J. **56** n° 1 (1988), 143–161.
- [G1] GROMOV M. — *Manifolds of negative curvature*, J. of Diff. Geom. **13** (1978), 223–230.
- [G2] GROMOV M. — *Hyperbolic manifolds according to Thurston and Jorgensen*, in: Springer LNM **842** (1981), 40–53.
- [MS1] MORGAN J. and SHALEN P. — *Valuations, trees and degenerations of hyperbolic structures 1*, Ann. of Math. **120** (1984), 401–476.
- [MS2] MORGAN J. and SHALEN P. — *Degenerations of hyperbolic structures 2, Measured laminations in 3-manifolds*, Ann. of Math. **127** (1988), 403–456.
- [P] PAULIN F. — *Topologie de Gromov équivariante, structures hyperboliques et arbres réels*, Inv. Math. **94** (1988), 53–80.
- [Th] THURSTON W. — *Hyperbolic structures on 3-manifolds 1*, Ann. of Math. **124** (1986), 203–246.
- [1] LEEB B. — *Compactness of moduli spaces of negatively curved metrics*, in preparation.
- [2] LEEB B. — *Examples of discrete groups without small actions on metric trees*, in preparation.

Bernhard LEEB
 University of Maryland
 Department of Mathematics
 College Park MD 20742 (USA)