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SOME TOPICS IN SPECTRAL GEOMETRY

par *Nikolai S. NADIRASHVILI*

The first two theorems of this note express a quasisymmetry relation between the positive and the negative part of the eigenfunctions of the Laplace operator on a Riemannian manifold.

Let M be a two-dimensional compact real analytic Riemannian manifold, $u_1, u_2, \ldots$ the eigenfunctions of the Laplace operator on M,

$$\Delta u_i = \lambda_i u_i \ .$$

THEOREM 1. — *There exists a positive constant C which depends on M such that, for every $i = 1, 2, \ldots$*

$$\mathrm{vol}\{x \in M,\ u_i(x) > 0\} > C \ .$$

PROBLEM 1. — *Is the analytic condition in theorem 1 essential? Is it possible to prove theorem 1 for n-dimensional manifolds with $n > 2$, for example in the case $M = S^3$ with the standard Riemannian metric?*

Let M be an n-dimensional compact smooth Riemannian manifold, $u_1, u_2, \ldots$, the eigenfunctions of the Laplace operator on M.

THEOREM 2. — *There exists a positive constant C which depends only on n, an integer N which depends on M, such that, for all $i > N$,*

$$\frac{1}{C} < \frac{\sup_M u_i}{|\inf_M u_i|} < C \ .$$

In the article [1] we proved

THEOREM 3. — *The multiplicity of the first non zero frequency of a bounded and plane simply connected membrane with free boundary is not more than 3.*

In [1] we also proved that the condition that the membrane is simply connected in theorem 3 is essential. We built an example of membrane with three holes for which the multiplicity of the first non zero frequency is equal to 3.

PROBLEM 2. — *What is the sharp estimate for the multiplicity of the first non zero frequency of a plane membrane with free boundary which has one or two holes?*

Let us consider the problem of the vibrations of an elastic beam. The energy of the deformation of the homogeneous elastic beam will be written in the form

$$\int_0^1 |u''|^2 dx .$$

The energy of the deformation of a nonhomogeneous elastic beam will be written in the form

$$\int_0^1 |a(x)u''(x)|^2 dx ,$$

$a(x) > 0$. So, the main frequency of the vibrations of the nonhomogeneous elastic beam can be represented in a variational form as,

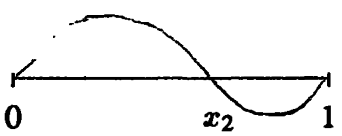
$$\inf_{u \in \overset{\circ}{W}{}_2^2[0,1]} \int_0^1 |a(x)u''(x)|^2 dx \Big/ \int_0^1 u^2(x) dx . \tag{1}$$

THEOREM 4. — *The first eigenfunction u_1 of the vibrations of a nonhomogeneous elastic beam with fixed ends has a constant sign on* $(0,1)$ *and, hence, the corresponding frequency is simple.*

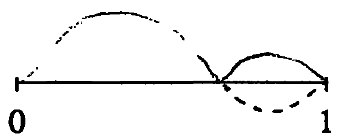
Proof. — Let us assume that the contrary holds. Thus we assume that there exists points $0 < x_1 < x_2 < x_3 < 1$ such that $u_1(x_1) > 0, u_1(x_2) = 0, u_1(x_3) < 0$. Denote

$$\tilde{u} = \begin{cases} u_1(x), & x \in [0, x_2], \\ -u_1(x), & x \in [x_2, 1]. \end{cases}$$

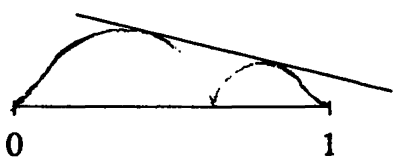
Let us take the common tangent line to the part of the graph of the function $\tilde{u}$ which is above the axis, on the segment $[0, x_2]$ and to the part of the graph which is above the axis on the segment $[x_2, 1]$, (see the pictures below) :



the function u_1



the function $\tilde{u}$



the function v

We built a new function v from the two parts of the graph of the function $\tilde{u}$ and the segment of the tangent line between them. It is clear that :

$$\int_0^1 |a(x)u_1''(x)|^2 dx > \int_0^1 |a(x)v''(x)|^2 dx,$$

$$\int_0^1 u_1^2(x)dx < \int_0^1 v^2(x)dx$$

and thus, u_1 cannot be a solution of the variational problem (1).

PROBLEM 3. — *Is it possible to have a sturm oscillating theory for the elastic nonhomogeneous beam similar to the one used for the string?*

Let us consider the problem of vibration of a plane plate with fixed boundaries,

$$\Delta\Delta u = \lambda u \text{ in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain.

The main frequency of a plate can be not simple. At this point the vibration of the plate is different from the vibration of a membrane with a fixed boundary.

THEOREM 5. — *There exists a bounded domain $\Omega \subset \mathbb{R}^2$ such, that the first eigenvalue of the problem* (2) *has multiplicity two.*

PROBLEM 4. — *Does there exist an a-priori estimate for the multiplicity of the first eigenvalue of the problem (2) which does not depend on the geometry of the domain Ω?*

Let L be a selfadjoint differential operator on $[0,1]$ of order $2n$ with constant coefficients. Let us consider the spectral problem :

$$Lu = \lambda u \text{ on } [0,1], \qquad \frac{\partial u^{(i)}}{\partial x^i}(0) = \frac{\partial u^{(i)}}{\partial x^i}(1) = 0, i = 0, 1, \ldots, n-1 \tag{3}$$

It is clear that the multiplicity of the eigenvalues of the problem (3) is $\leqslant n$.

PROBLEM 5. — *Is it possible to improve the last estimate?*

Bibliography

[1] NADIRASHVILI N.S. — *Multiple Eigenvalues of the Laplace operator*, Math USSR Sbornik, **61** n° 1 (1988), 225–238.

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