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THE RIEMANN-ROCH THEOREM ON ALGEBRAIC CURVES

by *Herbert SCHRÖDER*

Abstract: We give an analytic proof of a Riemann-Roch type theorem on singular algebraic curves. Formulated as an index theorem for the Cauchy-Riemann operator twisted by a holomorphic vector bundle (over the regular part of the curve) we reduce at once to the case where the bundle is trivial and of rank one. Then the Cauchy-Riemann operator $\bar{\partial}$ can be transformed into an operator which near the singular set is a perturbation of a regular singular operator with operator coefficients, and to which we can apply the methods of J. Brüning and R.T. Seeley as developed further in [B]. The treatment in this special case is joint work with J. Brüning and N. Peyerimhoff, and will only be sketched here.

1. Introduction

The classical Riemann-Roch Theorem arose from trying to estimate the maximal number of linear independent meromorphic functions on a compact Riemann surface S which are subordinate to a given divisor $D = \sum m(p)p \in H_0(S, \mathbb{Z})$.

Denote by $M(D)$ the space of meromorphic functions which have a pole of order $\leq m(p)$ at p if $m(p) > 0$, and vanish of order $\geq -m(p)$ at p if $m(p) \leq 0$. By $M'(D)$ we denote the space of meromorphic one-forms with similar behavior but order of poles and zeros reversed. Then the Riemann-Roch theorem states that

$$\dim M(D) - \dim M'(D) = \chi(S) + \deg(D), \quad (1.1)$$

where $\deg(D) = \sum m(p)$, and where $\chi(S)$ is the arithmetic genus of S (which is half its Euler characteristic).

Patching poles and zeros in a trivial line bundle as prescribed by the divisor D one obtains a holomorphic line bundle $[D]$ over S whose holomorphic sections correspond to elements of $M(D)$, i.e. $M(D) \cong \ker \bar{\partial}_{[D]}$. $\bar{\partial}_{[D]}$ acts locally as $\partial/\partial\bar{z}$ on the coordinate function with respect to some frame. Similarly, $M'(D) \cong \ker \bar{\partial}'_{[D]}$ where $\bar{\partial}'_{[D]}$ is the formal adjoint with respect to some Hermitean metric on $[D]$ and the Kähler metric on S . Then $\bar{\partial}_{[D]}$ has a unique closed extension (as an operator in the corresponding

spaces of L^2 -sections) which, moreover, is Fredholm by ellipticity. In this setting (1.1) is equivalent to

$$\operatorname{ind} \bar{\partial}_{[D]} = \chi(S) + \int_S c_1([D]) \quad (1.2)$$

where $c_1([D]) \in H^2(S, \mathbb{Z})$ is the first Chern class of $[D]$, Poincaré dual to D by the residue theorem.

Whereas (1.1) is usually stated and proved using sheaf cohomology, cf. e.g. [GH], (1.2) can be proved in a purely analytic manner using the heat equation method, cf. [K]. Now it is easy enough to extend the sheaf theoretic version of the Riemann-Roch theorem to include singular algebraic curves and divisors which are supported off the singular locus. For the analytic version such an extension is not at all obvious since, at first glance, there is no natural Fredholm operator available. This and more will be supplied by our generalization of (1.2).

2. The twisted Cauchy-Riemann operator

Let C be an algebraic curve, i.e., a one-dimensional subvariety of $\mathbb{C}P^N$ which we assume to be irreducible. The desingularization (S, π) of C consists of a nonsingular curve S and an holomorphic map $\pi : S \rightarrow \mathbb{C}P^N$ with $\pi(S) = C$ and $\pi : S \setminus \pi^{-1}(\Sigma) \rightarrow C \setminus \Sigma$ biholomorphic. The singular locus $\Sigma \subset C$ is a finite set consisting of singular values of π , and of points p where $\# \pi^{-1}(p) \geq 2$. Moreover, let E be a continuous vector bundle of rank k over C , holomorphic over $C \setminus \Sigma$, and equipped with an Hermitean metric which is constant near the points of Σ . The metric on $C \setminus \Sigma$ is induced by the Fubini study metric of the ambient projective space. Then we consider

$$\bar{\partial}_E : \Omega_0^{0,0}(\dot{E}) \rightarrow \Omega_0^{0,1}(\dot{E})$$

acting on compactly supported sections of the bundle $\dot{E} = E|_{C \setminus \Sigma}$. We denote by $A^{0,i}(E)$ the completion of $\Omega_0^{0,i}(\dot{E})$, $i = 0, 1$, with respect to the inner product coming from the metrics.

Now we can state our main result.

THEOREM. — $\bar{\partial}_E$ is closable as an operator with domain in $A^{0,0}(E)$ and range in $A^{0,1}(E)$. There is a one-to-one correspondence between closed extensions of $\bar{\partial}_E$ and subspaces W of the finite-dimensional space $W_0 = \mathcal{D}(\bar{\partial}_{E,\max})/\mathcal{D}(\bar{\partial}_{E,\min})$, $\bar{\partial}_{E,\max}$ and $\bar{\partial}_{E,\min}$ denoting the maximal and the minimal extension, respectively. Each closed extension $\bar{\partial}_{E,W}$ is a Fredholm operator with index

$$\operatorname{ind} \bar{\partial}_{E,W} = \operatorname{ind} \bar{\partial}_{E,\min} + \dim W \quad (2.1)$$

where

$$\operatorname{ind} \bar{\partial}_{E,\min} = k \chi(S) + \int_{C \setminus \Sigma} c_1(\dot{E}) \quad (2.2)$$

and

$$\dim W_0 = k \sum_{q \in \pi^{-1}(\Sigma)} (n(q) - 1). \quad (2.3)$$

Here, the numbers $n(q)$ are the multiplicities of the branches $C(q)$, $q \in \pi^{-1}(p)$, into which the curve decomposes near a singular point $p \in \Sigma$. It is well-known that by choosing a suitable holomorphic chart $\tilde{\varphi} : U(q) \rightarrow D_\epsilon = \{z \in \mathbb{C} \mid |z| < \epsilon\}$ of S at $q \in \pi^{-1}(p)$ one gets an $n(q)$ -fold covering

$$\pi \circ \tilde{\varphi}^{-1} : D_\epsilon^* = D_\epsilon \setminus \{0\} \rightarrow C(p) \setminus \{p\}$$

whose coordinate functions $P_j(z)$, $j = 0, \dots, N$ in $\mathbb{C}P^N$ are of the form

$$\begin{aligned} P_0(z) &= 1, \quad P_j(z) \equiv 0, \quad 1 \leq j \leq \ell - 1 \\ P_\ell(z) &= z^{n(q)} \end{aligned} \quad (2.4)$$

$P_j(z)$ a power series vanishing of order $> n(q)$, $j > \ell$

if $p = [1 : 0 : \dots : 0]$.

Example. — The cuspidal curve $C = \{[w_0 : w_1 : w_2] \in \mathbb{C}P^2 \mid w_0 w_2^2 = w_1^3\}$ has the singular locus $\Sigma = \{[1 : 0 : 0]\}$, and the desingularization $(\mathbb{C}P^1, \pi)$ with $\pi : \mathbb{C}P^1 \ni [z_0 : z_1] \mapsto [z_0^3 : z_0 z_1^2 : z_1^3] \in \mathbb{C}P^2$.

There are two closed extensions of $\bar{\partial}$ with

$$\text{ind } \bar{\partial}_{\min} = 1 \quad \text{and} \quad \text{ind } \bar{\partial}_{\max} = 2.$$

3. Sketch of proof

We can think of E as given by an open covering $(U_\alpha)_{\alpha \in A}$ of C and a cocycle $(g_{\alpha\beta})$ of holomorphic maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{C}),$$

where each U_α contains at most one singular point, and $U_\alpha \cap U_\beta \cap \Sigma = \emptyset$. Taking trivializing maps

$$\psi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{C}^k, \quad \alpha \in A,$$

and the standard basis $(e_j)_{1 \leq j \leq k}$ of $\mathbb{C}^k$, we obtain local frames

$$\zeta^\alpha = (\zeta_j^\alpha) = (\psi_\alpha^{-1}(\cdot, e_j)) \quad \text{on } U_\alpha$$

and continuous functions

$$h_{ij}^\alpha : U_\alpha \ni p \mapsto \langle \zeta_i^\alpha(p), \zeta_j^\alpha(p) \rangle_E \in \mathbb{C}$$

which we assume to be smooth in $U_\alpha \setminus \Sigma$, and constant near $U_\alpha \cap \Sigma$. Now it is well-known, cf. [GH] pp. 70ff, that the local expressions

$$\begin{aligned} \bar{\partial}_E f &= \bar{\partial}_E \left(\sum_{j=1}^k f_j \otimes \zeta_j^\alpha \right) = \sum_{j=1}^k \bar{\partial} f_j \otimes \zeta_j^\alpha \\ &= \sum_{j=1}^k \frac{\partial}{\partial \bar{z}} f d\bar{z} \otimes \zeta_j^\alpha, \quad f \in \Omega^{0,0}(U_\alpha \setminus \Sigma) \end{aligned} \quad (3.1)$$

extend to a globally defined operator

$$\bar{\partial}_E : \Omega^{0,0}(\dot{E}) \rightarrow \Omega^{0,1}(\dot{E}).$$

Moreover, there is a unique connection ∇_E on $\dot{E}$ compatible with the metric and with the complex structure.

Using this connection we represent the first Chern class $c_1(\dot{E})$ by a closed differential form with compact support in $C \setminus \Sigma$. With respect to the frame ζ^α we get the connection form

$$\omega^\alpha = \partial \log h^\alpha, \quad h^\alpha = (h_{ij}^\alpha)_{i,j},$$

of ∇_E on U_α , and since the connection form of the induced connection $\nabla_{\Lambda^k \dot{E}}$ on $\Lambda^k \dot{E}$,

$$\tilde{\omega}^\alpha = \partial \log \det h^\alpha,$$

is a globally defined one-form, and since

$$c_1(\nabla_E) = c_1(\nabla_{\Lambda^k \dot{E}}),$$

(cf. [GH] p. 414) we obtain

$$\begin{aligned} c_1(\nabla_E) &= \frac{i}{2\pi} (d\tilde{\omega}^\alpha - \tilde{\omega}^\alpha \wedge \tilde{\omega}^\alpha) \\ &= \frac{i}{2\pi} \bar{\partial} \partial \log \det h^\alpha. \end{aligned}$$

In particular, $c_1(\nabla_E) = 0$ near Σ , hence $\int_{C \setminus \Sigma} c_1(\nabla_E)$ is well defined and does not depend on the choice of the Hermitean metric provided the metric is constant near Σ .

Next we choose a local parametrization

$$\psi = \pi \circ \tilde{\varphi}^{-1} : D_\varepsilon^* \rightarrow C(q)^*, \quad \tilde{\varphi}^{-1}(0) = q \in \pi^{-1}(p),$$

of some branch $C(q)$ of C , where $C(q)^* = C(q) \setminus \{p\}$. We may assume that $C(q) \subset U_\alpha$ for some α and that $h^\alpha|_{C(q)}$ is given by a constant diagonal matrix

$$\Lambda^2 = \text{diag}(\lambda_1^2, \dots, \lambda_k^2).$$

If we pull back the Fubini-Study metric on $C(q)^*$ to D_ε^* , using (2.4) we obtain the metric

$$g(z) = \text{Re}(h(z)dz \otimes d\bar{z}),$$

where

$$h(z) = n(q)^2 |z|^{2n(q)-2} + O(|z|^{2n(q)}).$$

Then we define unitary transformations

$$\begin{aligned} \Phi_0 : \Omega_0^{0,0}(D_\varepsilon^* \times \mathbb{C}^k) \ni f &\mapsto \psi_\alpha^{-1} \circ (h^{-1/2} \Lambda^{-1} f) \circ \psi^{-1} \in \Omega_0^{0,0}(E|_{C(q)^*}) \\ \Phi_1 : \Omega_0^{0,1}(E|_{C(q)^*}) \ni d\bar{z} \otimes f &\mapsto \Lambda \psi_\alpha^{-1} \circ f \circ \psi \in \Omega_0^{0,0}(D_\varepsilon^* \times \mathbb{C}^k), \end{aligned} \tag{3.2}$$

which intertwine $\bar{\partial}_E$ with

$$\Phi_1 \bar{\partial}_E \Phi_0 : f \mapsto \bar{\partial}_{C^*} h^{-1/2} f,$$

or with

$$T = B_1(x) \partial_x + x^{-1} (S_0 \otimes I_{C^*} + S_1(x) \otimes I_{C^*}), \quad (3.3)$$

if we use polar coordinates (r, φ) and $x = r^{n(q)}$.

Here B_1 and S_1 are operator-valued C^∞ -maps, B_1 is a "small" perturbation of I , and S_1 a small perturbation of 0, cf. [BPS]. S_0 is the essentially self-adjoint first order differential operator

$$S_0 = \frac{i}{n(q)} \partial_\varphi - \frac{1}{2}$$

acting in $L^2(S^1)$. So up to these perturbations we end up with a regular singular operator as considered by J. Brüning and R.T. Seeley. We continue to work over a single branch $C(q)$ but note that finally one has to consider the union U of all such branches and add up the various S_0 's to get

$$\bigoplus_{q \in \pi^{-1}(\Sigma)} \left(\frac{i}{n(q)} \partial_\varphi - \frac{1}{2} \right) \quad \text{acting in} \quad \bigoplus_{q \in \pi^{-1}(\Sigma)} L^2(S^1).$$

To prove the first part of the theorem one has to construct a parametrix. For the unperturbed operator

$$T_0 = \partial_x + x^{-1} S_0 = \bigoplus_{s \in \text{spec } S_0} (\partial_x + x^{-1} s)$$

a "boundary" parametrix can be defined using the parametrices for the ordinary differential operators on the right-hand side

$$P_{0,s} u(x) = \int_0^x \left(\frac{y}{x} \right)^s u(y) dy, \quad s > -1/2$$

$$P_{1,s} u(x) = \int_x^1 \left(\frac{y}{x} \right)^s u(y) dy, \quad s < 1/2.$$

Combined with an "interior" parametrix on $C \setminus U$ which is guaranteed by ellipticity one obtains a global parametrix. It turns out that this gives also a parametrix for the perturbed operator we started from. If $(-1/2, 1/2) \cap \text{spec } S_0 \neq \emptyset$ one has several choices leading to different closed extensions, i.e. imposing different boundary conditions. Moreover, one has

$$W_0 \cong \bigoplus_{|s| < 1/2} \ker(S_0 - s).$$

The index formula for $\bar{\partial}_{\min}$ will be deduced by the heat equation method. On S one has, putting $D_+ = \bar{\partial}^* \bar{\partial}$, $D_- = \bar{\partial} \bar{\partial}^*$,

$$\text{ind } \bar{\partial} = \dim \ker D_+ - \dim \ker D_-$$

$$= \sum_{\lambda} e^{-\lambda t} [\dim \ker (D_+ - \lambda) - \dim \ker (D_- - \lambda)], \quad t > 0$$

$$= \text{tr}(e^{-t D_+} - e^{-t D_-}), \quad t > 0$$

$$= \int_S \omega_{\bar{\partial}}$$

while in our case we get an interior contribution

$$\int_{C \setminus U} \omega_{\bar{\partial}} = \frac{1}{2} \int_{C \setminus U} c_1(T'C \setminus U)$$

($T'C \setminus U$ denotes the holomorphic tangent bundle over $C \setminus U$), and a boundary contribution

$$-\frac{1}{2}(\eta(S_0) + \dim \ker S_0) - \sum_{-1/2 < s < 0} \dim \ker (S_0 - s),$$

cf. [BPS]. We get

$$\begin{aligned} \int_{C \setminus U} c_1(T'(C \setminus U)) &= \chi_e(C \setminus U) + \frac{1}{2\pi} \sum_q \int_{\partial C(q)} \kappa_g ds \\ &= \chi_e(C \setminus U) + \sum_q n(q) \\ &= \chi_e(C) - \#\Sigma + \sum_q n(q) \\ &= \chi_e(S) - \#\pi^{-1}(\Sigma) + \sum_q n(q) \end{aligned}$$

by the Gauß–Bonnet theorem (for surfaces with boundary). Here χ_e denotes the Euler characteristic, and κ_g the geodesic curvature of the boundary component $\partial C(q)$, $q \in \pi^{-1}(\Sigma)$. Since $\text{spec } S_0 = \bigcup_q (1/n(q)\mathbb{Z} - 1/2)$, we have $\eta(S_0) = 0$ and counting eigenvalues leads to a boundary contribution

$$-\frac{1}{2} \sum_q (n(q) - 1),$$

i.e.

$$\text{ind } \bar{\partial}_{\min} = \frac{1}{2} \chi_e(s) = \chi(s).$$

4. The Riemann–Roch Theorem

In this final section we state our generalization of the Riemann–Roch theorem.

Starting from a divisor D with support in $C \setminus \Sigma$ one defines a line bundle $[D]$ in the usual way, cf. [GH], pp. 133ff. Then $c_1([D])$ is compactly supported and dual to D , i.e.

$$\int_{C \setminus \Sigma} c_1([D]) = \deg D.$$

Just like in the theorem of Section 2 one gets several Riemann–Roch formulas depending on the boundary conditions which can be imposed on the meromorphic functions. We only consider the one that corresponds to the closure of $\bar{\partial}_{[D]}$.

By [B], the domain of $\bar{\partial}_{[D],\min}^*$ consists of functions whose transformations u according to (3.2) satisfy the following conditions near Σ

$$u \in L_{\text{loc}}^2((0, \varepsilon], H^1(S^1)) \cap H_{\text{loc}}^1((0, \varepsilon], L^2(S^1)) \cap L^2((0, \varepsilon), L^2(S^1)) \quad (4.1a)$$

$$Tu \in L^2((0, \varepsilon), L^2(S^1)) \quad (4.1b)$$

$$\|u(x)\|_{L^2(S^1)} = o(|x \log x|^{1/2}), \quad x \rightarrow 0. \quad (4.1c)$$

THEOREM. — *Let $M(D)$ be the space of meromorphic functions specified in Section 1 and obeying (4.1), and $M'(D)$ the space of meromorphic one-forms which are in the domain of $\bar{\partial}_{[D],\min}^*$ near Σ . Then we have*

$$\dim M(D) - \dim M'(D) = \chi(S) + \deg(D).$$

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