SÉMINAIRE DE THÉORIE SPECTRALE ET GÉOMÉTRIE

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Séminaire de Théorie spectrale et géométrie, tome 6 (1987-1988), p. 81-89

<http://www.numdam.org/item?id=TSG_1987-1988__6__81_0>

Séminaire de théorie spectrale et géométrie
CHAMBÉRY-GRENOBLE
1987–1988 (81–89)

L^p PINCHING AND COMPACTNESS THEOREMS FOR COMPACT RIEMANNIAN MANIFOLDS

par *Deane YANG*

1. Introduction

The ultimate goal is to understand to what extent L^p constraints on the Riemann curvature, $p \geqslant \frac{n}{2}$, determine the geometry and topology of an n-dimensional Riemannian manifold. This is already relatively well-understood when $p = \infty$, *i.e.* under pointwise bounds on the curvature. Major results include the Berger-Klingenberg $\frac{1}{4}$–pinching theorem [CE], Gromov's theorem on almost flat manifolds [BK], the Cheeger finiteness theorem [P1], and the Gromov(–Greene–Wu–Peters) convergence theorem [GLP], [GW], [P2].

The natural question to ask is whether these L^∞ theorems can be extended to L^p theorems for $p \geqslant \frac{n}{2}$. In §3 I describe counterexample that shows the conclusions of the theorems do not necessarily hold if the pointwise constraints on curvature are simply replaced by L^p constraints.

The purpose of this note is to show that the L^∞ theorems cited above do generalize to corresponding L^p, $p > \frac{n}{2}$, theorems if a lower bound on the isoperimetric or Sobolev constant of the manifold is assumed. This was also observed by L.Z. Gao in [G2], where deeper $L^{n/2}$ pinching and compactness theorems may be found.

By applying the Sobolev inequality that is equivalent to the isoperimetric inequality and Moser iteration to Hamilton's Ricci flow, a Riemannian metric on a compact manifold can be regularized so that the smoothed metric has pointwise curvature bounds which depend on the L^p norm of the curvature of the original metric. From this we obtain the following L^p analogues to the L^∞ theorems listed earlier (see §2 for definitions and notations) :

THEOREM 1. — *Given $p_0 > 1$, $p > \frac{n}{2}$, χ, $C > 0$, there exists a constant $\delta(n, p_0, p, \chi, C) > 0$, such that any smooth, n-dimensional, compact, simply connected Riemannian manifold satisfying :*

$$C_S(M) > \chi \ , \quad v(M)^{-\frac{1}{p}} \|Rm\|_p < C \ , \quad \|K - 1\|_{p_0} \leqslant \delta(n, p, \chi, C) v(M)^{\frac{1}{p}}$$

is homeomorphic to the n-dimensional sphere.

THEOREM 2. — *Given* $p_0 > 1$, $p > \frac{n}{2}$, χ, $K > 0$, *there exists a constant* $\varepsilon(n, p_0, p, \chi, K) > 0$, *such that any smooth, n-dimensional, compact, simply connected Riemannian manifold satisfying:*

$$C_S(M) > \chi \ , \quad v(M)^{-\frac{1}{p}} \|Rm\|_p < K \ , \quad \|K\|_{p_0} \leqslant \varepsilon(n, p, \chi, K) d(M)^{-2} v(M)^{\frac{1}{2}}$$

is diffeomorphic to a nil–manifold.

THEOREM 3. — *Fix constants* V, D, χ, $K > 0$, *and* $p > \frac{n}{2}$. *Let* M_i, $i = 1, \ldots,$ *be a sequence of smooth n-dimensional compact Riemannian manifolds such that*

$$v(M_i) > V \ , \quad d(M_i) < D \ , \quad C_S(M_i) > \chi \ , \quad \|Rm\|_p < K v(M)^{\frac{1}{p}} .$$

Then the following hold:

1. *There is a finite number of diffeomorphism classes represented in the sequence.*

2. *There exist a subsequence that converges in Hausdorff distance to a smooth compact manifold with a* C^0 *Riemannian metric.*

Remarks. —

1. In the compactness theorem, the bounds on the volume, diameter, and Sobolev constant can be replaced by upper and lower bounds on volume and a lower bound on the injectivity radius. One simply applies Croke's curvature free estimates on the isoperimetric constant and on the volume of geodesic balls.
2. Theorem 3 implies, by standard arguments, pinching theorems where the pinching constant depends on all of the constants used in the theorem.

Acknowledgments. — The elliptic version of Moser iteration was used by L.Z. Gao in [G1] to study Einstein manifolds. When I learned of his work, I suggested to him that the parabolic version of his estimates might be useful in proving L^p pinching theorems. This paper contains the details of that simple observation.

More recently, in [G2] and [G3], Gao has studied the much more difficult and interesting case, when $p = \frac{n}{2}$. Although his arguments are much more complicated, estimates like the ones presented here are also proved and used in his paper. He also observes that the pinching and compactness theorems described here follow from the estimates. Moser iteration has also been applied to the Ricci flow independently by Min-Oo [M] to prove a pinching theorem for negative Einstein manifolds.

I would like to thank Ben Chow for helping me learn Moser iteration for a parabolic equation.

Part of this paper was written while I was visiting l'Institut des Hautes Études Scientifiques in Bures-sur-Yvette, France. I would like to thank the director, Marcel Berger, and the the staff for their hospitality. I was also partially supported by a Sloan Fellowship and by NSF grant DMS-86-01853.

I would also like to thank Sylvestre Gallot for observing that it suffices to use the "average L^p norm", $v(M)^{-\frac{1}{p}}\|f\|_p$, throughout.

2. Notation

Let M be a smooth compact n-dimensional Riemannian manifold. We shall denote the diameter of M by $d(M)$ and the volume by $v(M)$. The Riemann curvature tensor is Rm, and the Ricci tensor Rc. The sectional curvature function of M will be denoted K.

Given $\varepsilon > 0$ and $x \in M$, let $B_\varepsilon(x)$ denote the geodesic ball of radius ε centered at x.

All norms in this paper are defined with respect to the given Riemannian metric (which may vary with time t).

Suppose that the Riemannian metric of M depends on $0 \leqslant t \leqslant T$. Given $f \in C^\infty(M)$, denote

$$\|f\|_p = \left(\int_M |f|^p dV_g\right)^{1/p}$$

and

$$|||f|||_p = \left(\int_0^T \|f\|_p^p dt\right)^{1/p}$$

Define the Sobolev constant $C_S(M)$ of the manifold M to be the largest constant A^{-1} such that

$$\|f\|_{\frac{2n}{n-2}} \leqslant A\|\nabla f\|_2 + B\|f\|_2, \tag{0}$$

where $B = 2v(M)^{-\frac{1}{n}}$.

3. A counterexample

Let N be a compact $(n-1)$-dimensional flat manifold with volume 1 and $M = (-1,1) \times N$. Given $\varepsilon > 0$ and a positive integer k, consider the following metric on M :

$$g = dr^2 + (\varepsilon + r)^{2k} g_N.$$

A straightfoward calculation shows that the Riemann curvature always satisfies $|Rm| < k^2 r^{-2}$. Therefore, given any $p > 0$ and $k > (2p-1)/(n-1)$,

$$\|Rm\|_p \leqslant \frac{1}{k(n-1) - 2p + 1} < \infty.$$

In particular, the L^p norm of Rm stays bounded as ε approaches zero, and a singularity forms at $\varepsilon = 0$. By pasting this example into a given compact manifold, we obtain a contradiction to the statement obtained by replacing the the L^∞ bound on curvature in the Gromov convergence theorem with an L^p bound. On the other hand, there do not seem to be many ways in which a singularity can form with an L^p bound, $p > \frac{n}{2}$, on curvature. For this reason, the L^p analogue of Cheeger's finiteness theorem might still hold. Heuristically, there seems to be some connection between singularities with L^p bounded curvature and Cheeger-Gromov's construction of collapsing Riemannian manifolds.

In apparent contrast with $p > \frac{n}{2}$, it is possible to construct a Riemannian manifold with infinite topology near a point singularity and with $L^{\frac{n}{2}}$ bounded curvature.

4. Estimates for the heat equation

THEOREM 6. — *Let $q > n$, $\beta > 0$; and let f, b be smooth nonnegative functions on $M \times [0,T]$ which satisfy the following :*

$$\frac{\partial f}{\partial t} \leqslant \Delta f + bf \text{ on } M \times [0,T], \tag{1}$$

where Δ is the Laplace-Beltrami operator of the metric $g(t)$ and given $q > n$, b is assumed to satisfy :

$$\sup_{0 \leqslant t \leqslant T} \|b\|_{\frac{q}{2}} \leqslant \beta v(M)^{\frac{2}{q}}.$$

Assume that for each $0 \leqslant t \leqslant T$, the Sobolev inequality (0) holds for the metric $g(t)$ with fixed constants A, B.

Given $p_0 > 1$, there exists a constant C which depends only on n, q, p_0, β, A, B, T such that for any $x \in M$, $0 < t \leqslant T$,

$$|f(x,t)| \leqslant C t^{-\frac{n}{2p_0}} v(M)^{-\frac{1}{p_0}} \|f_0\|_{p_0}, \tag{2}$$

where $f_0(x) = f(x,0)$.

Proof. — Throughout this discussion C is a positive constant that depends only on n, q, p_0, β, A, B, T and may change from line to line.

We shall assume that $v(M) = 1$. The estimates for arbitrary volume can be obtained by considering what happens when the metric is rescaled by a constant factor.

Integrating (1) by parts, we obtain :

$$\begin{aligned} p^{-1}\frac{\partial}{\partial t}\int_M f^p &\leqslant \int_M f^{p-1}\Delta f + bf^p \\ &\leqslant -\frac{4(p-1)}{p^2}\|\nabla f^{\frac{p}{2}}\|_2^2 + \|b\|_{\frac{q}{2}}\|f^{\frac{p}{2}}\|_{\frac{q}{q-2}}^2 \\ &\leqslant -\frac{4(p-1)}{p^2}\|\nabla f^{\frac{p}{2}}\|_2^2 + \beta\left(\varepsilon^{-\frac{n}{q}}\int_M f^p\right)^{1-\frac{n}{q}}\left(\varepsilon^{1-\frac{n}{q}}\int_M f^{p\frac{n}{n-2}}\right)^{\frac{n-2}{n}\frac{n}{q}}. \end{aligned}$$

Applying the Sobolev inequality, we obtain

$$\frac{1}{p}\frac{\partial}{\partial t}\int_M f^p + \frac{4(p-1)}{p^2}\int_M |\nabla f^{\frac{p}{2}}|^2 \leqslant \beta A^2\varepsilon^{1-\frac{n}{q}}\int_M |\nabla f^{\frac{p}{2}}|^2 + \beta(\varepsilon^{-\frac{n}{q}} + B^2\varepsilon^{1-\frac{n}{q}})\int_M f^p$$

Setting

$$\varepsilon = \left(\frac{3p-3}{p^2A^2\beta}\right)^{\frac{q}{q-n}},$$

we obtain the following basic estimate :

LEMMA 7. — *Given $p > p_0 > 1$ the following holds for $0 \leqslant t \leqslant T$:*

$$\frac{\partial}{\partial t}\int_M f^p + \int_M |\nabla f^{\frac{p}{2}}|^2 \leqslant Cp^{\frac{2n}{q-n}}\int_M f^p. \tag{3}$$

Now given $0 < \tau < \tau' < T$, let

$$\psi(t) = \begin{cases} 0 & 0 \leqslant t \leqslant \tau \\ (t-\tau)/(\tau'-\tau) & \tau \leqslant t \leqslant \tau' \\ 1 & \tau' \leqslant t \leqslant T \end{cases}$$

Multiplying (3) by ψ, we obtain

$$\frac{\partial}{\partial t}\left(\psi\int f^p\right) + \psi\int |\nabla f^{\frac{p}{2}}|^2 \leqslant (Cp^{\frac{2n}{q-n}}\psi + \psi')\int f^p.$$

Integrating this with respect to t we get

$$\int_t f^p + \int_{\tau'}^t\int |\nabla f^{\frac{p}{2}}|^2 \leqslant (Cp^{\frac{2n}{q-n}} + \frac{1}{\tau'-\tau})\int_\tau^T\int_M f^p\ , \quad \tau' \leqslant t \leqslant T. \tag{4}$$

Given $p \geqslant p_0$ and $0 \leqslant \tau \leqslant T$, denote

$$H(p,\tau) = \int_\tau^T\int_M f^p.$$

LEMMA 8. — *Given $p \geqslant p_0$ and $0 \leqslant \tau < \tau' \leqslant T$,*

$$H(p(1+\frac{2}{n}),\tau') \leqslant (Cp^{\frac{2n}{q-n}} + \frac{1}{\tau'-\tau})^{1+\frac{2}{n}} H(p,\tau)^{1+\frac{2}{n}},$$

Proof. —

$$\begin{aligned}\int_{\tau'}^T\int_M f^{p(1+\frac{2}{n})} &\leqslant \int_{\tau'}^T\left(\int f^p\right)^{\frac{2}{n}}\left(\int f^{p\frac{n}{n-2}}\right)^{\frac{n-2}{n}} dt \\ &\leqslant \left(\sup_{\tau'\leqslant t\leqslant T}\int f^p\right)^{\frac{2}{n}}\int_{\tau'}^T A^2\int |\nabla f^{\frac{p}{2}}|^2 + B^2\int f^p dt\end{aligned}$$

Applying (4), we obtain the desired estimate. **Q.E.D.**

Now, denote

$$\nu = 1 + \frac{2}{n},\ \mu = \nu^{\frac{2n}{q-n}},$$

fix $0 < t < T$, and set

$$
\begin{aligned}
p_k &= p_0 \nu^k \\
\sigma_k &= \sum_{\substack{j=0\\k}} \mu^{-j} \\
\sigma'_k &= \sum_{\substack{j=0\\k}} j \mu^{-j} \\
\tau_k &= (1-\mu^{-k})t \\
\Phi_k &= H(p_k, \tau_k)^{\frac{1}{p_k}}
\end{aligned}
$$

Applying Lemma , we obtain

$$
\begin{aligned}
\Phi_{k+1} &= H(p_k(1+\frac{2}{n}), \tau_{k+1})^{\frac{1}{p_k(1+\nu)}} \\
&\leqslant \left(C p_0^{\frac{q}{q-n}} + \frac{C}{t}\right)^{\frac{1}{p_0(1+\nu)^k}} (1+\nu)^{\frac{2n}{q-n}\frac{k}{(1+\nu)^k}} \Phi_k^{\frac{1}{1+\nu}} \\
&\leqslant \left(C + \frac{1}{t}\right)^{\frac{\sigma_k}{p_0}} (1+\nu)^{\frac{2n}{q-n}\sigma'_k} \Phi_0 .
\end{aligned}
$$

Taking the limit $k \to \infty$,

$$
\begin{aligned}
|f(x,t)| &\leqslant \lim_{k\to\infty} \Phi_k \\
&= C t^{-\frac{1}{p_0}(1+\frac{n}{2})} |||f|||_{p_0}
\end{aligned}
$$

It only remains to estimate $|||f|||_{p_0}$ in terms of $\|f_0\|_{p_0}$. This is easy, since by (3)

$$\frac{\partial}{\partial t}\int_M f^{p_0} \leqslant C \int_M f^{p_0}.$$

Integrating this, we obtain

$$\int_t f^{p_0} \leqslant e^{Ct} \int_{t=0} f^{p_0}.$$

Integrating again,

$$|||f|||_{p_0} \leqslant C t^{\frac{1}{p_0}} \|f_0\|_{p_0}. \tag{6}$$

Q.E.D.

5. Smoothing a Riemannian metric

Applying the estimates of § 4 to Hamilton's Ricci flow, the following is straightforward :

THEOREM 9. — *Given constants $q > n$, χ, $K > 0$, there exists $T(n,q,\chi,K) > 0$ such that for any n-dimensional compact manifold M with a Riemannian metric g_0 satisfying $C_S(M) > \chi$ and $v(M)^{-\frac{2}{q}} \|Rm\|_{\frac{q}{2}} < K$, the Ricci flow*

$$\frac{\partial g}{\partial t} = -2Rc(g) + s(t)g \ , \quad g(0) = g_0 \ , \tag{7}$$

where

$$s(t) = v(M)^{-1}\int_M S(t)dV_g,\ S(t) = \text{ the scalar curvature of } g(t),$$

has a unique smooth solution $g(t)$ *for* $0 \leqslant t \leqslant T$ *satisfying the following estimates :*

$$C_S(g(t)) \geqslant \frac{1}{2}\chi \tag{8}$$

$$\|Rm(g(t))\|_{\frac{q}{2}} \leqslant 2Kv(M)^{\frac{2}{q}} \tag{9}$$

$$\|Rm(g(t)\|_\infty \leqslant C(n,q,\chi,K)t^{-\frac{n}{q}} \tag{10}$$

Proof. — First, observe that the flow fixes the volume of M. We can, as before, normalize the volume equal to 1.

We apply the following theorem of Hamilton (see [H]) :

THEOREM 10. — *Let* $g(t)$, $0 \leqslant t < T$, *be a solution to (7) on a compact manifold* M. *If the sectional curvature of* $g(t)$ *remains bounded as* $t \to T$, *then the solution extends smoothly beyond* T.

Let $[0, T_{\max})$ be a maximal time interval on which (7) has a smooth solution and such that (8), (9), and (10) hold. The curvature tensor Rm satisfies the following equation (see [H]):

$$\frac{\partial Rm}{\partial t} = \Delta_g Rm + Q(Rm) + \frac{2}{n}s(t)Rm, \tag{11}$$

where $Q(Rm)$ is a tensor that is quadratic in Rm. From this it follows that

$$p^{-1}\frac{\partial}{\partial t}\int_M |Rm|^p dV_g \leqslant \int_M |Rm|^{p-1}\Delta|Rm|dV_g + c(n)\int_M |Rm|^{p+1}dV_g. \tag{12}$$

We can therefore apply (6) and (2) to obtain the first two estimates of the following :

LEMMA 11. — *Let*

$$h(t) = \sup_{0\leqslant\tau\leqslant t}\|Rm(g(\tau))\|_{\frac{q}{2}}$$

$$k(t) = \|Rm(g_t)\|_\infty$$

$$f(t) = C_S(g(t))$$

Then for $0 \leqslant t < T_{\max}$,

$$h(t) \leqslant e^{c(n,q,\chi)t}K \tag{13}$$

$$k(t) \leqslant C(n,q,K,\chi)t^{-\frac{n}{q}} \tag{14}$$

$$f(t) \geqslant \chi e^{-c(n)\int_0^t k(\tau)d\tau} \tag{15}$$

Proof of (15). — Given $u \in C^\infty(M)$, define

$$E_t[u] = \frac{\int_M |\nabla u|^2_{g(t)} + u^2 dV_{g(t)}}{\left(\int_M |u|^{\frac{2n}{n-2}} dV_{g(t)}\right)^{\frac{n-2}{n}}}.$$

A straightfoward calculation shows that

$$\frac{d}{dt}E_t[u] \geqslant -c(n)\|Rc(g_t)\|_\infty E_t[u].$$

Integrating this differential inequality proves (15).

From (13), (14), (15), it follows that there exists $T(n,q,\chi,K) > 0$ such that if $t < \min(T(n,q,\chi,K), T_{\max})$, then strict inequalities hold in (8), (9) and k is bounded. In particular, if $T_{\max} < T(n,q,\chi,K)$, then by Theorem 10, the solution to (11) can be extended beyond $T_{\max}$ with (8), (9), and (10) still holding. This would contradict the maximality of $T_{\max}$. Therefore, $T(n,q,\chi,K) \leqslant T_{\max}$. **Q.E.D.**

6. Proof of pinching and compactness theorems

Theorems 1 and 2 are now easily proved. Assume that a compact Riemannian manifold M satisfies the assumptions of Theorem 1, with δ small. Let $g(t)$ be the unique solution to (7). Now a straightforward calculation shows that the function

$$f = |R_{ijkl} - (g_{ik}g_{jl} - g_{il}g_{jk})| = |K - 1|$$

satisfies the parabolic inequality

$$\frac{\partial f}{\partial t} \leqslant \Delta f + c|Rm|f.$$

Therefore, if δ is chosen sufficiently small, the metric $g(T)$ is a smooth Riemannian metric satisfying the pinching condition

$$\frac{1}{4} < K \leqslant 1.$$

By the Berger-Klingenberg theorem, M must therefore be homeomorphic to the sphere. The proof of Theorem 2 is exactly the same, except that the Gromov almost flat theorem is used.

Theorem 3 is almost as easy. Given a sequence of Riemannian manifolds M_i satisfying the assumptions of the theorem, Hamilton's Ricci flow can be solved on each manifold and the metrics $g_i(0)$ replaced by $g_i(T)$. The value T is given by Theorem 10 and is independent of i. The new metrics still have a lower bound for volume and an upper bound for diameter, but now have a pointwise upper bound for curvature. The finiteness statement now follows directly from the Cheeger finiteness theorem.

To prove convergence, observe that for each k, the solution of the Ricci flow $g_k(t)$ satisfies an estimate of the form

$$\|g_k(t_1) - g_k(t_2)\|_\infty < C|t_2 - t_1|^\alpha,$$

where $0 < \alpha < 1$, and C is independent of k and t. Letting $k \to \infty$ on a subsequence, it follows that for positive time, the limiting metrics $g_\infty(t)$ satisfy the same estimate. Therefore, as $t \to 0$, it converges uniformly to a continuous metric.

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