

SÉMINAIRE DE THÉORIE SPECTRALE ET GÉOMÉTRIE

KAZUYOSHI KIYOHARA

On Blaschke manifolds and harmonic manifolds

Séminaire de Théorie spectrale et géométrie, tome 6 (1987-1988), p. 33-37

http://www.numdam.org/item?id=TSG_1987-1988__6__33_0

© Séminaire de Théorie spectrale et géométrie (Chambéry-Grenoble), 1987-1988, tous droits réservés.

L'accès aux archives de la revue « Séminaire de Théorie spectrale et géométrie » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON BLASCHKE MANIFOLDS AND HARMONIC MANIFOLDS

by Kazuyoshi KIYOHARA

0. — A compact riemannian manifold M is called a Blaschke manifold if the diameter of M and the injectivity radius of M coincide. It is known that if M is a Blaschke manifold, then M is diffeomorphic to S^n or $\mathbf{R}P^n$, or $\pi_1(M) = \{0\}$ and $H^*(M, \mathbf{Z}) \cong$ the \mathbf{Z} -cohomology ring of $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathbf{C}aP^2$.

The main problem about Blaschke manifolds is to know if the following conjecture, the Blaschke conjecture, is true or not : if M is a Blaschke manifold, then it would be a compact rank one symmetric space.

There are classes of riemannian manifolds related to Blaschke manifolds. A riemannian manifold M is called a globally harmonic manifold if the determinant of $d(\exp_p)_x : T_p M \rightarrow T_{\exp_p x} M$ ($p \in M$, $x \in T_p M$) depends only on the norm $|x|$. A compact riemannian manifold is called a C_1 -manifold if all of its geodesics are closed and have the same length 1. The relation is as follows :

$$\begin{array}{l} \text{compact, simply connected,} \\ \text{globally harmonic} \end{array} \implies \text{Blaschke} \implies C_1 .$$

The following results are known :

1. (Green, Berger et al.). — If (S^n, g) is a Blaschke manifold, then it is isometric to the standard one.

2. (Green, Berger et al.). — If $(\mathbf{R}P^n, g)$ is a C_1 -manifold, then it is isometric to the standard one.

3. (Kiyohara). — Let P be one of the projective spaces $\mathbf{C}P^1$, $\mathbf{H}P^n$ ($n \geq 2$), $\mathbf{C}aP^2$, and let (P, g) be a C_π -manifold. If the metric g is sufficiently close to the standard C_π -metric g_0 , then (P, g) is isometric to the standard one (P, g_0) .

4. (Zoll, Weinstein). — There are non-standard C_1 -manifolds (S^n, g) for any dimension $n \geq 2$.

1. — From now on we assume M is a Blaschke manifold, $\pi_1(M) = \{0\}$, $H^*(M, \mathbf{Z}) \cong H^*(\mathbf{C}P^n, \mathbf{Z})$ ($\dim M = 2n$, $n \geq 2$), and the diameter of M is $\pi/2$. The followings are known about M :

1) For any $p \in M$ and any $q \in \text{Cut}(p)$ (the cut locus of p), the distance $d(p, q) = \pi/2$.

2) Every cut locus is a submanifold of codimension 2.

3) Let ρ be the bundle projection $TM \rightarrow M$, and let $\{\zeta_t\}$ be the geodesic flow on SM . Then $\rho \circ \zeta_{\pi/2} : S_p(M) \rightarrow \text{Cut}(p)$ is a fibre bundle whose fibres are great circles on $S_p M$.

4) For $p, q \in M$ with $d(p, q) = \pi/2$, we denote by $\Sigma(p, q)$ the union of geodesic orbits through p and q . Then $\Sigma(p, q)$ is a 2-dimensional submanifold diffeomorphic to S^2 .

Now we define a mapping $I : SM \rightarrow SM$ as follows: since $H_2(M, \mathbf{Z}) \cong \mathbf{Z}$, we fix a positive generator. Then on each $\Sigma(p, q)$ the orientation is determined. Hence we have an orientation on each fibre S^1 of the fibre bundle $\rho \circ \zeta_{\pi/2} : S_p M \rightarrow \text{Cut}(p)$, because the fibre S^1 over $q \in \text{Cut}(p)$ is nothing but the unit sphere of $T_p \Sigma(p, q)$. So $I : SM \rightarrow SM$ is defined by the conditions:

1) If $v \in S_p M$, then $Iv \in S_p M$ and $\rho(\zeta_{\pi/2} v) = \rho(\zeta_{\pi/2} Iv)$.

2) $\langle v, Iv \rangle = 0$.

3) $\{v, Iv\}$ is positive in this order.

We extend the mapping I to $TM \setminus \{0\}$ -section homogeneously, and let $I_{*v} : T_{\rho(v)} M \rightarrow T_{\rho(v)} M$ be the differential of $I|_{T_{\rho(v)} M \setminus \{0\}}$ at v . From the definition the mapping I satisfies $I \circ I = (-1)$ identity. So it looks like an almost complex structure, and we have the following

PROPOSITION A. — Assume $I_{*v}^2 + 1 = 0$ for all $v \in SM$. Then $I : T_p M \setminus \{0\} \rightarrow T_p M \setminus \{0\}$ can be extended to a linear mapping on $T_p M$ for every $p \in M$, i.e. I is an almost complex structure and it is integrable. Therefore (M, I) is a hermitian manifold. Moreover each cut locus is a complex submanifold and is holomorphically isomorphic to $\mathbf{C}P^{n-1}$.

PROPOSITION B. — Assume $\dim M = 4$. If $I_{*v}^2 + 1 = 0$ for all $v \in SM$ and if every cut locus is minimal, then M is isometric to $(\mathbf{C}P^2, g_0)$.

LEMMA C. — If M is moreover globally harmonic, then $(I_{*v}^2 + 1)^{n-1} = 0$ for every $v \in SM$ and every cut locus is minimal ($\dim M = 2n$).

COROLLARY D. — If $\dim M = 4$ and M is globally harmonic, then M is isometric to $(\mathbf{C}P^2, g_0)$.

Remarque. — This corollary is already known by a different method. See [1].

2. — For the proof of propositions we need some lemmas.

LEMMA 1. — *There is a Jacobi field $Y(t)$ along the geodesic $\gamma_v(t) = \rho(\zeta_t v)$ such that*

$$\begin{bmatrix} Y(0) \\ Y'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ Iv \end{bmatrix} \quad , \quad \begin{bmatrix} Y(\pi/2) \\ Y'(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ -I\bar{v} \end{bmatrix} \quad , \quad \bar{v} = \zeta_{\pi/2} v .$$

Moreover if a Jacobi field $X(t)$ along $\gamma_v(t)$ satisfies $X(0) = X(\pi/2) = 0$ then $X(t)$ is a constant multiple of $Y(t)$.

For $X, Y \in T_p M$, $Y \neq 0$, we put $\nabla_X I \cdot Y = \nabla_{\partial/\partial t}(IY_t)|_{t=0}$, where we take a curve $c(t)$ in M such that $c'(0) = X$, and Y_t is the parallel displacement of Y along $c(t)$. $\nabla_X I \cdot Y$ is linear in X , but not necessarily in Y .

LEMMA 2. — *Let $Y(t)$ be a periodic Jacobi field along the geodesic $\gamma_v(t)$, $v \in SM$. Then we have a periodic Jacobi field $Z(t)$ along the geodesic $\gamma_{e^{sI}v}(t)$ ($e^{sI}v = v \cos s + Iv \sin s$) such that*

$$\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} Y(0) \\ (\cos s + \sin s I_{*v})Y'(0) + \sin s(\nabla I \cdot v)Y(0) \end{bmatrix}$$

$$\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} Y(\pi/2) \\ (\cos s + \sin s I_{*v})Y'(\pi/2) - \sin s(\nabla I \cdot \bar{v})Y(\pi/2) \end{bmatrix} ,$$

where $(\nabla I \cdot v)Y(0) = \nabla_{Y(0)} I \cdot v$, etc.

LEMMA 3. —

1) *There are Jacobi fields $Y_1(t)$, $Y_2(t)$ along $\gamma_v(t)$ such that*

$$\begin{bmatrix} Y_1(0) \\ Y_1'(0) \end{bmatrix} = \begin{bmatrix} Iv \\ -\nabla_v I \cdot v \end{bmatrix} \quad , \quad \begin{bmatrix} Y_1(\pi/2) \\ Y_1'(\pi/2) \end{bmatrix} = \begin{bmatrix} -I\bar{v} \\ \nabla_{\bar{v}} I\bar{v} \end{bmatrix}$$

$$\begin{bmatrix} Y_2(0) \\ Y_2'(0) \end{bmatrix} = \begin{bmatrix} 2\nabla_v I \cdot v \\ R(Iv, v)v - \nabla_v^2 I \cdot v \end{bmatrix} \quad , \quad \begin{bmatrix} Y_2(\pi/2) \\ Y_2'(\pi/2) \end{bmatrix} = \begin{bmatrix} -2\nabla_{\bar{v}} I \cdot \bar{v} \\ -R(I\bar{v}, \bar{v})\bar{v} + \nabla_{\bar{v}}^2 I \cdot \bar{v} \end{bmatrix} .$$

2) $\nabla_{e^{sI}v} I \cdot d^{sI}v = \nabla_v I \cdot v$.

LEMMA 4. — *Let $Y(t)$ be a periodic Jacobi field along $\gamma_v(t)$. Then there is a periodic Jacobi field $Z(t)$ along $\gamma_v(t)$ such that*

$$\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} {}^t I_{*v} Y(0) \\ I_{*v} Y'(0) + (\nabla I \cdot v - {}^t \nabla I \cdot v)Y(0) + \{(Y(0), \nabla_v I \cdot v) + (Y'(0), Iv)\}v \end{bmatrix}$$

$$\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} {}^t I_{*\bar{v}} Y(\pi/2) \\ -I_{*\bar{v}} Y'(\pi/2) - (\nabla I\bar{v} - {}^t \nabla I\bar{v})Y(\pi/2) - \{(Y(\pi/2), \nabla_{\bar{v}} I\bar{v}) + (Y'(\pi/2), I\bar{v})\}\bar{v} \end{bmatrix} .$$

3. Proof of Proposition A. — Fix $p \in M$ and consider the S^1 -principal bundle $\rho \circ \zeta_{\pi/2} : S_p M \rightarrow \text{Cut}(p)$, where the S^1 -action is given by e^{sI} , $0 \leq s \leq 2\pi$.

We define a 1-form ω on $S_p M$ by

$$\omega(X) = \langle X, Iv \rangle, \quad X \in T_v(S_p M) = \{Y \in T_p M \mid \langle v, Y \rangle = 0\}.$$

As is easily seen, ω is a connection form, *i.e.* invariant under the S^1 -action. We have

$$d\omega(X, Y) = \langle (I_{*v} - {}^t I_{*v})X, Y \rangle.$$

So there is a unique closed 2-form Ω on $\text{Cut}(p)$ such that $(\rho \circ \zeta_{\pi/2})^* \Omega = d\omega$. We can see that $[(1/2\pi)\Omega]$ is a generator of $H^2(\text{Cut}(p), \mathbf{Z}) \cong \mathbf{Z}$. Therefore

$$(1/2\pi)^{n-1} \int_{\text{Cut}(p)} \Omega^{n-1} = 1$$

under a proper orientation of $\text{Cut}(p)$, and thus

$$\int_{S_p M} \omega \wedge (d\omega)^{n-1} = (2\pi)^n.$$

Now put $J_v = I_{*v} - {}^t I_{*v}$, $S_v = I_{*v} + {}^t I_{*v}$. Then $2I_{*v} = J_v + S_v$ and

$$I_{*v}^2 + 1 = 0 \iff J_v^2 + S_v^2 + 4 + J_v S_v + S_v J_v = 0. \quad (\#)$$

Let e_1, \dots, e_{2n-2} be an orthonormal basis of the orthogonal complement to $\mathbf{R}v + \mathbf{R}Iv$ in $T_p M$ such that $J_v e_{2i-1} = \lambda_i e_{2i}$, $J_v e_{2i} = -\lambda_i e_{2i-1}$, $\lambda_i \geq 0$, $i = 1, \dots, n-1$. By (#) we have

$$-\lambda_i^2 + |S_v e_{2i}|^2 + 4 = 0.$$

Hence $\lambda_i \geq 2$, and $\lambda_i = 2$ for every i if and only if $S_v = 0$. Then

$$(\omega \wedge (d\omega)^{n-1})(Iv, e_1, \dots, e_{2n-2}) = (n-1)! \prod_{i=1}^{n-1} \lambda_i \geq 2^{n-1} (n-1)!,$$

and the equality holds if and only if $S_v = I_{*v} + {}^t I_{*v} = 0$. Therefore we have

$$(2\pi)^n = \int_{S_p M} \omega \wedge (d\omega)^{n-1} \geq 2^{n-1} (n-1)! \text{vol}(S_p M).$$

But $\text{vol}(S_p M)$ is just $2\pi^n / (n-1)!$. So the equality holds in the above inequality. Hence we have $S_v = I_{*v} + {}^t I_{*v} = 0$ for any $v \in SM$. Since $I_{*v}^2 + 1 = 0$, it follows that ${}^t I_{*v} I_{*v} = 1$. This implies that the mapping $I : S_p M \rightarrow S_p M$ is an isometry, and therefore the restriction of a linear orthogonal transformation of $T_p M$. Hence I is extended as a tensor field of type (1,1) with $I^2 = -1$, *i.e.* an almost complex structure on M , and (M, I) is an almost hermitian manifold.

By using the square of the endomorphisms on the space of Jacobi fields in Lemma 4, one gets

$$\langle \{I(\nabla I \cdot v - {}^t \nabla I \cdot v) - (\nabla I \cdot v - {}^t \nabla I \cdot v)I\} X, Y \rangle = 0, \quad X, Y \perp v, Iv.$$

Moreover Lemma 3 (2) gives

$$\langle \{I(\nabla I \cdot v + {}^t \nabla I \cdot v) - (\nabla I \cdot v + {}^t \nabla I \cdot v)I\} X, Y \rangle = 0, \quad X, Y \perp v, Iv.$$

These formula gives

$$\nabla_{IX}I = I\nabla_X I$$

for any vector X . By this it is easy to see that the Nijenhuis' tensor vanishes, and (M, I) turns out to be a hermitian manifold.

It is now clear that each cut locus is a complex submanifold of M and the S^1 -fibration $\rho \circ \zeta_{\pi/2} : S_p M \rightarrow \text{Cut}(p)$ is nothing but the standard Hopf fibration : $S^{2n-1} \rightarrow CP^{n-1}$. Hence the last statement of the proposition follows.

Proof of Proposition B. — For $v \in SM$ we define the symmetric endomorphism Φ_v of $T_{\rho(v)}M$ by $\Phi_v v = \Phi_v I v = 0$ and

$$\langle \Phi_v X, Y \rangle = -\langle h(X, Y), v \rangle, \quad X, Y \in T_{\rho(v)}M, \quad X, Y \perp v, I v,$$

where h is the second fundamental form of $\text{Cut}(\rho\zeta_{\pi/2}v)$ in M at $\rho(v)$. If we take a curve $c(t)$ in $\text{Cut}(\rho\zeta_{\pi/2}v)$ such that $c'(0) = X$, and a normal vector field v_t to $\text{Cut}(\rho\zeta_{\pi/2}v)$ along $c(t)$, we have

$$\langle \Phi_v X, Y \rangle = \langle \nabla_{\partial/\partial t} v_t |_{t=0}, Y \rangle.$$

So the following lemma is clear.

LEMMA 5. — $\Phi_{Iv}X = I\Phi_v X + (\nabla_X I)v$, $X \in T_{\rho(v)}M$, $X \perp v, Iv$.

Since every cut locus is minimal, it follows that $\text{tr } \Phi_v = 0$ for any $v \in Sm$, tr being the trace. Hence in view of Lemma 5 one gets

$$\text{tr } (\nabla I)v = 0.$$

This together with the formula $\nabla_{IX}I = I\nabla_X I$, shown in the proof of Proposition A, implies that $\nabla I = 0$, i.e. (M, I) is kählerian.

By applying Lemma 1 to the Jacobi field Y_2 in Lemma 3,

$$R(Iv, v)v = c(v)Iv, \quad v \in SM,$$

where c is a function on SM satisfying $c(\zeta_{\pi/2}v) = c(v)$. As is easily seen, $c(v)$ is pointwise constant, i.e. if v_1 and v_2 are based at the same point on M , then $c(v_1) = c(v_2)$. Using the fact that for any two points p and q on M , there is a point m such that $d(p, m) = d(q, m) = \pi/2$, we see the constancy of $c(v)$.

Since (M, I) is kählerian and has constant holomorphic sectional curvature, it must be holomorphically isometric to (CP^2, g_0) .

Lemma C is an immediate consequence of Lemma 4.

Reference

- [1] BESSE A. — *Manifolds all of whose geodesics are closed*, Springer, 1978.

Kazuyoshi KIYOHARA
 Max-Planck-Institut für Mathematik
 Gottfried-Claren-Strasse 26
 D - 5300 BONN 3
 (R.F.A.)