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THE GEOMETRY OF CLOSED HYPERSURFACES

by *Sebastião de ALMEIDA & Fabiano BRITO*

0. Introduction

Let M be an oriented hypersurface in a $(n+1)$ -dimensional Riemannian manifold W . We denote by $\lambda_1, \dots, \lambda_n$ the principal curvatures at a point $p \in M$. The r -th curvature κ_r of M at the point p is defined by

$$\kappa_r = \sum_{i_1 < \dots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}, \quad r = 1, \dots, n.$$

For a given real number a we consider the class of closed hypersurfaces

$$\mathcal{S}_r(a, W) = \{M \subset W : \kappa_r \equiv a\}$$

and denote by $\mathcal{S}_r^*(a, W)$. Curvature properties of closed hypersurfaces have been studied by several authors during the last year. From the work of Hsiung [9], Aleksandrov [1] and Ros [17] we know that if $M \in \mathcal{S}_i^*(a, \mathbf{R}^{n+1})$, $i \in \{1, 2, n\}$, then M is an embedded sphere. Hsiang, Teng and Yu [8] and Wente [18], constructed examples showing that $\mathcal{S}_1^*(a, \mathbf{R}^{n+1}) \subsetneq \mathcal{S}_1(a, \mathbf{R}^{n+1})$ when $n = 2k - 1$ and $n = 2$ respectively. Obviously $\mathcal{S}_1(0, \mathbf{R}^{n+1}) = \emptyset$. When $W = S^{n+1}$ the situation is quite different. For example for each integer $g \geq 0$, there is a compact surface of genus g in $\mathcal{S}_1^*(0, S^3)$, and if g is not prime this embedded surface is not unique. On the other hand given $M \in \mathcal{S}_1^*(0, S^3)$ there exists a diffeomorphism $f : S^3 \rightarrow S^3$ such that $M = \Sigma_g$ where Σ_g is a standardly embedded surface in S^3 (cf. [11], [12]). This unknottedness result was first proved by Lawson [12] assuming only that the 3-sphere S^3 has a positive Ricci curvature metric. The unknottedness result was extended to metrics of positive scalar curvature (cf. [3], [13]). Given any sequence of minimal surfaces $\Sigma_j \in \mathcal{S}_1^*(0, S^3)$, $j = 1, \dots, m$ we take connected sum at tiny disks away from the surface to produce a minimal embedding

$$\Sigma_1 \coprod \cdots \coprod \Sigma_m \in \mathcal{S}_1^*(0, S^3 \# \cdots \# S^3).$$

This produces disjointed minimal surfaces in a 3-sphere of positive scalar curvature. This is the only possibility topologically (cf. [3]). Chern, do Carmo, Kokayashy [6], Lawson [10], proved that if

$$M \in \mathcal{S}_1(0, S^{n+1}) \cap \mathcal{S}_2(a, S^{n+1})$$

with $a \geq -n/2$, then up to rotations of S^{n+1} , M is one of the minimal products of spheres $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$, $k = 0, \dots, [n/2]$. These products belong to $\mathcal{S}_i(a_i, S^{n+1})$. They are isoparametric hypersurfaces. For $n = 3$ we have the following results.

THEOREM ([2]). — *Let $M \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_2(a, S^4)$ such that $\kappa_3 \neq 0$. Then up to rotations of S^4 , M is the product $S^1 \times S^2$ embedded in S^4 .*

THEOREM ([4]). — *Let W be a 4-dimensional Riemannian manifold of constant curvature c and $M \in \mathcal{S}_1(0, W) \cap \mathcal{S}_3(a, W)$, $a \neq 0$. Then $c > 0$ and M is isoparametric.*

THEOREM ([4]). — *Let $M \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_3(a, S^4)$ such that the second fundamental form of M is never zero then M is either the isoparametric Clifford torus $S^1 \times S^2$ or M is boundary of the tube of a minimally immersed 2-dimensional $\Sigma \subset S^4$.*

From a compact minimal surface in S^4 one may construct a hypersurface $M_\Sigma \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_3(0, S^4)$. This involves mapping the unit normal sphere bundle of the minimal surface of S^4 , back into S^4 in the natural way. This process works only if the normal curvature K^\perp of Σ is nowhere zero. By a result of Tribuzy and Guadalupe [7] $\Sigma \cong S^2$. (cf. [4], [16]).

The following result is due to Terng and Peng.

THEOREM ([15]). — *Let $M \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_2(a, S^4)$ such that the principal curvatures are distinct. Then M is isoparametric.*

In this note we will consider immersions of M into $W = Q^4(c)$ where $Q^4(c)$ stands for \mathbf{H}^4 , \mathbf{R}^4 or S^4 . We will prove the following two theorems.

THEOREM 1. — *Let $M \in \mathcal{S}_1(H, Q^4) \cap \mathcal{S}_2(a, Q^4)$ with distinct principal curvatures and non-negative scalar curvature κ . Then $\kappa \equiv 0$ and M is isoparametric. In particular $c = 1$ and M is one of the hypersurfaces in the isoparametric family obtained from Cartan's example.*

THEOREM 2. — *Let $M \in \mathcal{S}_2(H, Q^4) \cap \mathcal{S}_3(K, Q^4)$ with $K \neq 0$, non-negative scalar curvature and distinct principal curvatures. Then $c = 1$, $\kappa \equiv 0$ and M is one isoparametric hypersurface in Cartan's family.*

Theorem 1 is a partial answer to the following question.

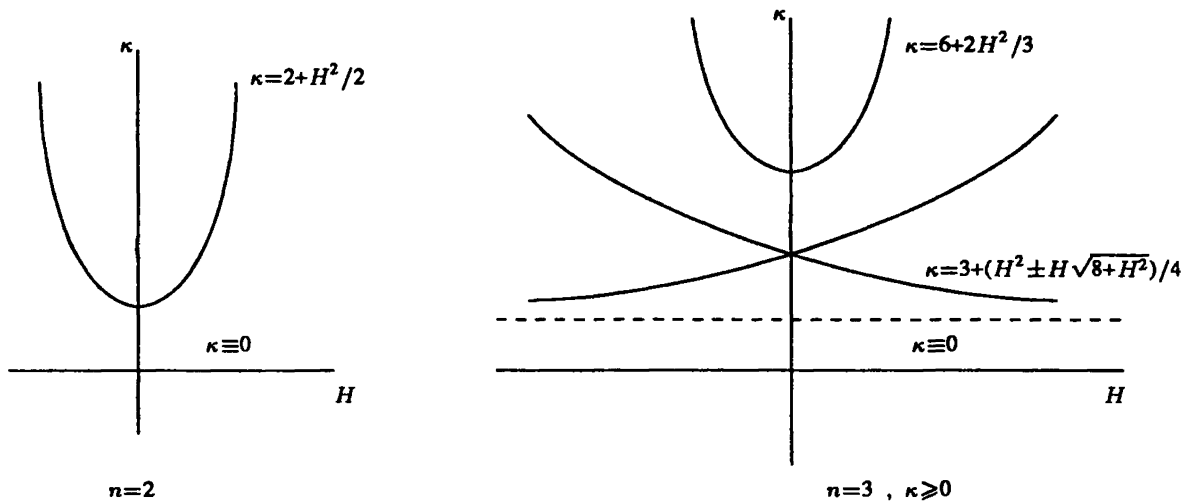
QUESTION. — *Let \mathcal{C} be the isoparametric hypersurfaces of S^4 . Is $\mathcal{S}_1(H, S^4) \cap \mathcal{S}_2(\kappa, S^4) = \mathcal{C}$?*

If the scalar curvature is non-negative the answer to the above question is positive.

The condition on the principal curvatures is superfluous. (cf. [5]). The case in which the scalar curvature is a negative constant still remains to be checked. This will be done in a succeeding paper. A more general problem would be to determine the set

$$\mathcal{S}_1(H, S^{n+1}) \cap \mathcal{S}_2(\kappa, S^{n+1}) .$$

When $n = 3$ a calculation shows that the scalar curvature of a hypersurface $M \in \cap_{i=1}^3 \mathcal{S}_i(a_i, S^4)$ is given by $\kappa = 0$, $\kappa = 3 + [H^2 \pm H(8 + H^2)^{1/2}]/4$ or $\kappa = 6 + 2H^2/3$ where $H = a_1$ is the mean curvature of M . The picture shows the possible values for the mean curvature (H) and scalar curvature (κ) when the dimension is two or three.



In §1 we give an integral formula involving the mean (H), scalar (κ) and Gauss–Kronecker (K) curvatures for immersed hypersurfaces $M \subset Q^4(c)$. The proofs of theorem 1 and 2 are given in §2 and §3 respectively.

1. A formula involving the curvatures

In this section we will prove an integral formula involving the curvature invariants H, κ, K for immersed hypersurfaces $M \subset Q^4(c)$ with distinct principal curvatures. For this we choose a local orthonormal frame field ℓ_A in $Q^4(c)$ such that when restricted to $M, \ell_1, \ell_2, \ell_3$ give principal directions. We denote by θ_A the dual coframe and write the structure equations of $Q^4(c)$ as

$$d\theta_A = - \sum \theta_{AB} \wedge \theta_B, \quad \theta_{AB} + \theta_{BA} = 0 \tag{1}$$

$$d\theta_{AB} = - \sum_c \theta_{AC} \wedge \theta_{CB} + c\theta_A \wedge \theta_B \tag{2}$$

In general we have $\theta_{Ai} = \sum h_{ij}\theta_j, h_{ij} = h_{ji}$. In our case the second fundamental form

$$h = \sum h_{ij}\theta_i\theta_j \tag{3}$$

is diagonalized, *i.e.* $h_{ij} = \lambda_i \delta_{ij}$.

As in [5] we let φ be the exterior 2-form on M given by

$$\varphi = \theta_{12} \wedge \theta_3 - \theta_{13} \wedge \theta_2 + \theta_{23} \wedge \theta_1. \quad (4)$$

Taking exterior derivative of (4) we obtain

$$\begin{aligned} d\varphi &= d\theta_{12} \wedge \theta_3 - d\theta_{13} \wedge \theta_2 + d\theta_{23} \wedge \theta_1 - \theta_{12} \wedge d\theta_3 + \theta_{13} \wedge d\theta_2 - \theta_{23} \wedge d\theta_1 \\ &= -[\theta_{13} \wedge \theta_{32} - (c + \lambda_1 \lambda_2) \theta_1 \wedge \theta_2] \wedge \theta_3 + [\theta_{12} \wedge \theta_{23} - (c + \lambda_1 \lambda_3) \theta_1 \wedge \theta_3] \wedge \theta_2 \\ &\quad - [\theta_{21} \wedge \theta_{13} - (c + \lambda_2 \lambda_3) \theta_2 \wedge \theta_3] \wedge \theta_1 + [\theta_{12} \wedge [\theta_{31} \wedge \theta_1 + \theta_{32} \wedge \theta_2] \\ &\quad - \theta_{13} \wedge [\theta_{21} \wedge \theta_1 + \theta_{23} \wedge \theta_3] + \theta_{23} \wedge [\theta_{12} \wedge \theta_2 + \theta_{13} \wedge \theta_3] \end{aligned}$$

After the simplifications we obtain

$$d\varphi = \frac{\kappa}{2} \theta_1 \wedge \theta_2 \wedge \theta_3 + \theta_{13} \wedge \theta_{32} \wedge \theta_3 + \theta_{12} \wedge \theta_{32} \wedge \theta_2 + \theta_{21} \wedge \theta_{13} \wedge \theta_1 \quad (5)$$

We will now compute the right hand side of (5) in terms of H, κ, K . For this we need the covariant derivative, Dh , of h . Recall that the covariant derivatives h_{ijk} of h are given by

$$\sum h_{ijk} \theta_k = dh_{ij} - \sum_m h_{im} \theta_{mj} - \sum h_{mj} \theta_{mi} \quad (6)$$

Exterior differentiating equation (6) we obtain

$$\sum h_{ijk} \theta_k \wedge \theta_j = 0. \quad (7)$$

From equation (7) and the symmetry of h we conclude that the covariant derivatives h_{ijk} are symmetric in any two of their indices. Observe that in our case $h_{ij} = \lambda_i \delta_{ij}$. Therefore

$$h_{iik} = dh_{ii}(\ell_k). \quad (8)$$

Equivalently

$$h_{iik} = \langle \nabla \lambda_i, \ell_k \rangle. \quad (9)$$

If $i \neq j$ we obtain

$$h_{ijk} = (h_{ij} - h_{ii}) \theta_{ij}(\ell_k) \quad (10)$$

this gives

$$\theta_{ij} = \sum \frac{h_{ijk}}{\lambda_j - \lambda_i} \theta_k. \quad (11)$$

Using equations (9) and (11) we will compute the exterior derivative of φ . From (5), (11) and the symmetry of h_{ijk} we obtain

$$\begin{aligned} d\varphi &= \frac{\kappa}{2} \theta_1 \wedge \theta_2 \wedge \theta_3 + \frac{1}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)} [h_{131} \theta_1 + h_{123} \theta_2] \wedge [h_{321} \theta_1 + h_{322} \theta_2] \wedge \theta_3 \\ &\quad + \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} [h_{121} \theta_1 + h_{123} \theta_3] \wedge [h_{321} \theta_1 + h_{322} \theta_3] \wedge \theta_2 \quad (12) \\ &\quad + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)} [h_{212} \theta_2 + h_{213} \theta_3] \wedge [h_{132} \theta_2 + h_{133} \theta_3] \wedge \theta_1 \end{aligned}$$

Let us denote by dM the volume form $\theta_1 \wedge \theta_2 \wedge \theta_3$ and by W the product $\prod_{i < j} (\lambda_i - \lambda_j)$. With this notation we have

$$d\varphi = adM \quad (13)$$

where

$$\begin{aligned} a &= \frac{\kappa}{2} + \frac{\lambda_2 - \lambda_1}{W} [h_{113}h_{223} - h_{123}^2] + \frac{\lambda_1 - \lambda_3}{W} [h_{112}h_{332} - h_{123}^2] \\ &\quad + \frac{\lambda_3 - \lambda_2}{W} [h_{221}h_{331} - h_{123}^2] \\ &= \frac{\kappa}{2} + \frac{\lambda_2 - \lambda_1}{W} h_{113}h_{223} + \frac{\lambda_1 - \lambda_3}{W} h_{112}h_{332} + \frac{\lambda_3 - \lambda_2}{W} h_{221}h_{331} . \end{aligned} \quad (14)$$

Recall that the principal curvatures λ_i , $i = 1, 2, 3$ satisfy the polynomial equation $p(\lambda, x) = \prod(\lambda - \lambda_i) = 0$, $x \in M$. In our case

$$p(\lambda - x) = \lambda^3 - H\lambda^2 + \frac{\kappa - 6c}{2}\lambda - K . \quad (15)$$

Differentiating the equation $p(\lambda_i, x) = 0$, $i = 1, 2, 3$ we obtain

$$0 = \frac{\partial P}{\partial \lambda}(\lambda_i, x)d\lambda_i - \alpha_i \quad (16)$$

$$\alpha_i = \lambda_i^2 dH - \frac{1}{2}\lambda_i d\kappa + dK . \quad (17)$$

This gives the following identities :

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)d\lambda_1 = \alpha_1 \quad (18)$$

$$(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)d\lambda_2 = \alpha_2 \quad (19)$$

$$(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)d\lambda_3 = \alpha_3 . \quad (20)$$

An easy computation shows that

$$a = \frac{\kappa}{2} + \frac{1}{W^2} [\alpha_1(\ell_3)\alpha_2(\ell_3) + \alpha_1(\ell_2)\alpha_3(\ell_2) + \alpha_2(\ell_1)\alpha_3(\ell_1)] . \quad (21)$$

Given $p \in M$ we may regard the second fundamental form $h_p(v, w)$ as a linear map $A : T_p M \rightarrow T_p M$ given by

$$h(v, w) = \langle A(v), w \rangle . \quad (22)$$

We then consider the symmetric operator L given by

$$L = \frac{1}{2}(H^2 - S)I - HA + A^2 . \quad (23)$$

Where I is the identity operator. The operator L is diagonalizable with respect to the orthonormal frame field ℓ_1, ℓ_2, ℓ_3 . Its associated matrix is the following

$$L \cong \begin{pmatrix} \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix} \quad (24)$$

A straightforward computation gives

$$\begin{aligned}
 a = & \frac{\kappa}{2} + \frac{1}{W^2} \left[\langle L(\nabla H), L(\nabla H) \rangle + \frac{1}{4} \langle L(\nabla \kappa), \nabla \kappa \rangle - \frac{H}{2} \langle L(\nabla H), L(\nabla \kappa) \rangle \right. \\
 & + \frac{K}{2} \langle \nabla H, \nabla \kappa \rangle + S \langle \nabla H, \nabla K \rangle - \frac{H}{2} \langle \nabla \kappa, \nabla K \rangle \\
 & \left. + |\nabla K|^2 - \langle A(\nabla H), A(\nabla K) \rangle + \frac{1}{2} \langle A(\nabla \kappa), \nabla K \rangle \right].
 \end{aligned} \tag{26}$$

Since M is compact without boundary we apply Stokes's theorem to obtain

$$\int_M d\varphi = 0 \tag{27}$$

This gives the following

THEOREM. — *Let $M^3 \subset Q^4(c)$ be a closed immersed 3-manifold in a space form $Q^4(c)$. Suppose M is orientable and its principal curvatures are all distinct on M . Then we have the following integral formula*

$$\begin{aligned}
 0 = & \int \left\{ \frac{\kappa}{2} + \frac{1}{W^2} \left[\langle L^2(\nabla H), \nabla H \rangle + \frac{1}{4} \langle L(\nabla \kappa), \nabla \kappa \rangle - \frac{H}{2} \langle L^2(\nabla H), \nabla \kappa \rangle \right. \right. \\
 & + \frac{K}{2} \langle \nabla H, \nabla \kappa \rangle + S \langle \nabla H, \nabla K \rangle - \frac{H}{2} \langle \nabla \kappa, \nabla K \rangle \\
 & \left. \left. + |\nabla K|^2 + \langle A(\nabla H), A(\nabla K) \rangle + \frac{1}{2} \langle A(\nabla \kappa), \nabla K \rangle \right] \right\} dV.
 \end{aligned}$$

2. Proof of theorem 1

With the assumptions of theorem 1 we have H and κ constant. Therefore the integral formula of §1 becomes

$$0 = \int \left\{ \frac{\kappa}{2} + \frac{1}{W^2} |\nabla K|^2 \right\} dV.$$

Since $\kappa \geq 0$ by assumption, then $\kappa \equiv 0$ and $|\nabla K| = 0$. Therefore M is a scalar flat isoparametric hypersurface. It is well known (cf. [14]) that if $c \leq 0$ the number of distinct principal curvatures of an isoparametric hypersurface in $Q^4(c)$ is ≤ 2 . We may conclude that $c = 1$. The only possibility left is that M be one of the hypersurfaces in the isoparametric family obtained from Cartan's example.

3. Proof of theorem 2

In theorem 2 we assume that K and κ are constant. Therefore applying the integral formula of §1 we get

$$\begin{aligned} 0 &= \int_M \left\{ \frac{\kappa}{2} + \frac{1}{W^2} \langle L(\nabla H), L(\nabla H) \rangle \right\} dV \\ &= \int \left\{ \frac{\kappa}{2} + \frac{1}{W^2} [\lambda_1^2 \lambda_2^2 (\ell_3 H)^2 + \lambda_2^2 \lambda_3^2 (\ell_1 H)^2 + \lambda_3^2 \lambda_1^2 (\ell_2 H)^2] \right\} dV . \end{aligned}$$

Since $\kappa \geq 0$ and $K \neq 0$ we obtain that $\kappa \equiv 0$ and $\nabla H \equiv 0$. Then M is isoparametric and the theorem follows as in the proof of theorem 1.

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