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AN F_σ SEMIGROUP OF ZERO MEASURE WHICH CONTAINS A TRANSLATE OF EVERY COUNTABLE SET

by John A. HAIGHT (*)

In 1942, PICCARD [10] gave an example of a set of real numbers whose sum set has zero Lebesgue measure but whose difference set contains an interval. About thirty years later, various authors (CONNOLLY, JACKSON, WILLIAMSON and WOODALL) in a series of papers constructed F_σ sets $E \subset \underset{\sim}{R}$ such that $E - E$ contains an interval while $m((k)E) = 0$ for progressively larger values of k , where

$$(k)\, E = \{x_1 + x_2 + \dots + x_k \ ; \quad x_i \in E \quad , \quad 1 \leqslant i \leqslant k\} \ .$$

These authors' interest was in an approach to the construction of asymmetric Raikov systems, [5], defined as follows.

If G is a locally compact abelian group, a Raikov system is a family $\mathfrak{F}$ of F_σ subsets satisfying the following conditions :

(a) If F_1 , F_2 , ... $\in \mathfrak{F}$ then $\bigcup_{n=1}^{\infty} F_n \in \mathfrak{F}$.

(b) If $F_1 \subset F_2 \in \mathfrak{F}$ and F_1 is F_σ then $F_1 \in \mathfrak{F}$.

(c) If F_1 , $F_2 \in \mathfrak{F}$ then $F_1 + F_2 \in \mathfrak{F}$.

A Raikov system is said to be asymmetric if $A \in \mathfrak{F}$ does not necessarily imply $-A \in \mathfrak{F}$.

CONNOLLY and WILLIAMSON [3] noted that the existence in $\underset{\sim}{R}$ of an asymmetric Raikov system which was maximal among proper Raikov systems was equivalent to the existence of an F_σ semigroup of zero Lebesgue measure which is not contained in any proper subgroup of $\underset{\sim}{R}$, which in turn is equivalent to the existence of an F_σ set E such that $E - E = \underset{\sim}{R}$, but $m((k)\, E) = 0$ for $k = 1 , 2 , \dots$. I was able to solve this problem, although unfortunately the central idea was rather obscured by technical details. Recently, however, BROWN and MORAN [1] have simplified my proof. The results of this paper are a generalization of this simplification.

If $\mathcal{R}$ is a ring and α , $\beta \in \underset{\sim}{Q}$ and E , $F \subset \mathcal{R}$, we write

$$\alpha.E + \beta.F = \{\alpha.x + \beta.y \ ; \quad x \in E \ , \quad y \in F\} \ ;$$

if E is finite, $|E|$ denotes the number of elements in E . In this notation, the statement $E - E = \mathcal{R}$ is equivalent to the statement that, for every $F \subset \mathcal{R}$ such

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that $|F| \leqslant 2$, there is a $c \in \mathbb{R}$ such that $F + \{c\} \subset E$. (From now on, we shall write " c " instead of " $\{c\}$ ". This leads to the question : If $(k)\ E$ is "small", how "large" is the family of sets F that can be translated into E ?

For any $n \in \underset{\sim}{N}$, we write $I(n) = \{0, \dots, n-1\}$ and $\underset{\sim}{Z}(n)$ for the integers modulo n.

THEOREM 1. - <u>For all</u> $j, n \in \underset{\sim}{N}$ <u>and</u> $\varepsilon > 0$, <u>there is an</u> $N = N(j, n, \varepsilon)$ <u>such that, for any</u> $M \geqslant N$, <u>there is a subset</u> A <u>of</u> $\underset{\sim}{Z}(M)$ <u>such that</u>

(a) <u>If</u> $F \subset \underset{\sim}{Z}(M)$, $|F| \leqslant n$, <u>there is a</u> $c = c(F)$ <u>such that</u> $F + c \subset A$.

(b) $|(j)\ A|\ M^{-1} < \varepsilon$.

If $F_i \subset I(m)^n$, $i = 1, 2, \dots$, for some $m > 1$, we shall say that the set $C = \{\sum_{i=1}^{\infty} x_i\, m^{-i}\ ;\ x_i \in F_i\}$ is a Cantor set. If $|F_i| \leqslant r$, we shall say $C \in b(m, r)$. If $F_i = F$, $i = 1, 2, \dots$, we shall say that $C (\in b(m, |F|))$ is self-similar.

THEOREM 2. - <u>For any</u> $n, r, j \in \underset{\sim}{N}$ <u>and</u> $\varepsilon > 0$ <u>there is an</u> $N = N(j, r, \varepsilon)$ <u>and a self-similar Cantor set</u> $K \subset \underset{\sim}{R}^n$ <u>such that</u>

(a) <u>For any</u> $F \in b(N, r)$, <u>there is a</u> $d = d(F)$ <u>such that</u> $F + d \subset K$,

(b) $(j)K \in b(N, \varepsilon N)$.

<u>We note that</u> (a) <u>implies that if</u> F <u>is any finite set containing not more than</u> r <u>points, then there is a</u> c <u>such that</u> $F + c \subset K$.

THEOREM 3. - <u>There is an</u> F_σ <u>set</u> $E \subset \underset{\approx}{R}^n$ <u>such that</u>

(a) <u>If</u> $F \subset \underset{\sim}{R}^n$ <u>is a countable set, there is a</u> $c = c(F)$ <u>such that</u> $F + c \subset E$.

(b) <u>For any</u> $k \in \underset{\sim}{N}$, $m((k)\ E) = 0$.

CASSELS [2] proved that if $\lambda_1, \dots, \lambda_r$ are real numbers, there is a number α such that $\|(\alpha + \lambda_i)u\| > c/u$, $u \in \underset{\sim}{N}$, $i = 1, \dots, r$ ($\|x\|$ denotes the distance of x from the nearest integer to x, $c = c(r)$).

In our notation $\{\lambda_1 \dots \lambda_r\} + \alpha \in B_r$

$$B_r = \{x\ ;\ \|xu\| > c(r)\ ;\ u, u \in \underset{\sim}{N}\}.$$

Let $B = \bigcup_{r \in \underset{\sim}{N}} B_r$, then B is the set of "badly approximable numbers". It is well-known that $B = \bigcup_{n \in \underset{\sim}{N}} F(m)$ where $F(n)$ is the set of numbers whose continued fraction expansions have partial quotients $\leqslant m$ and that $m(B) = 0$. However, $(2)B = \underset{\sim}{R}$. Indeed M. HALL [7] proved that $(2)F(4) = \underset{\sim}{R}$ (more recently HLAVKA showed $(4)F(2) = \underset{\sim}{R}$). DAVENPORT and SCHMIDT ([4], [11]) extended Cassels' result in various ways. In particular, Schmidt's theorem implies that, for every countable $F \subset \underset{\sim}{R}^n$ there is an $\alpha \in \underset{\sim}{R}^n$ (actually many such α) such that $F + \alpha \subset B$ where B in $\underset{\sim}{R}^n$ is defined as

$$\{x = (x_1, \dots, x_n)\ ;\ \max\{\|ux_1\|, \dots, \|ux_n\|\} < c(x)\ u^{-1/n}\}.$$

Again $m(B) = 0$, and $(2)B = \underset{\sim}{R}^n$.

Proof of Theorems.

LEMMA 1. - For all $j , n \in \underset{\sim}{N}$ and $\epsilon > 0$, there is an $N = N(j , n , \epsilon)$ and a subset A of $\underset{\sim}{Z}(N)$ such that

(a) If $F \subset \underset{\sim}{Z}(N)$, $|F| \leqslant n$, there is a $c = c(F)$ such that $F + c \subset A$.

(b) $|(j)A|\ N^{-1} < \epsilon$.

If $x , y \in \underset{\sim}{R}^n$ for some $n \geqslant 1$, write

$$x . y = x_1\, y_1 + \dots + x_n\, y_n \ .$$

LEMMA 2. - Given $M , j , n \in \underset{\sim}{N}$ and $\epsilon > 0$, there is an $N = N(M , j , n , \epsilon)$ such that for each $j \in I(N)^n$, there is $c = c(j) \in I(N)$, such that if $c^* = (c, \cdots, c)$ and

$E = E(M , N) = \{x ;\ x \in I(N) ,\ x \equiv g.(j + c^*) \mod N ,\ g \in I(M)^n ,\ j \in I(N)^n\}$,

then $|(j)\ E|\ N^{-1} < \epsilon$.

If $M \geqslant 2$ this is stronger than lemma 1. For if $F = \{j_1 , \dots , j_n\}$, then $j = (j_1 , \dots , j_n) \in I(N)^n$ and, taking $g(i) = (0 , \dots , 1$ (i^{th} place)$, \dots , 0)$, we have $g(i) \in I(M)^n$, and so $g(i)(j + c^*) \in E$, that is $j_i + c \in E$, $i = 1, \cdots, n$.

Proof of Lemma 2. - Fix n . We use induction on j . Assume the lemma holds for some j and for all M, ϵ .

Given M , ϵ, let $T = N(2M , j , n , \epsilon/2)$. Let

$$G = \{g ;\ g : F \longrightarrow I(M)^n \setminus \{(0 , \dots , 0)\} ,\ F \subset I(T)^n ,\ 0 < |F| \leqslant j + 1\} \ .$$

Let $\ell = |G|$, and let $\emptyset : I(\ell) \longrightarrow G$ be a bijection.

Now choose primes $P_0 < P_1 < \dots < P_{\ell-1}$ such that

(a) $P_0 > nMT$,

(b) $\sum_{0 \leqslant i \leqslant \ell-1} \frac{1}{P_i} < \epsilon/2$.

Let $P = P_0 \times \dots \times P_{\ell-1}$ and let $S = PT$.

For $j \in I(S)^n$, let $j' \in I(T)^n$, where $j'_i \equiv j_i \mod T$, we choose $c(f)$ to satisfy (1) and (2)

(1) $c(j) \equiv c(j') \mod T$

(2) $\emptyset(r)(j')(j + c^*(j)) \equiv 0 \mod P_r$

if $j' \in \operatorname{dom} \emptyset(r)$, otherwise let $c(j) \equiv 0 \mod P_r$, $r = 0 , \dots , \ell - 1$.

We need to show that (2) is possible. If $j' \in \operatorname{dom} \emptyset(r)$, then $\emptyset(r)(j') = \lambda$ say, where $\lambda \in I(M)^n \setminus \{(0 , \dots , 0)\}$ so

$$\phi(r)(j')(j + c^*(j)) = \lambda(j + c^*) = \lambda_1 j_1 + \dots + \lambda_n j_n + c(j)(\lambda_1 + \dots + \lambda_n) .$$

Now $0 < \lambda_1 + \dots + \lambda_n \leqslant nM$ so condition (b) ensures that we can find $c(j)$ so that the above equation is congruent to zero mod P_r.

Now if $x \in (j + 1) E(M , S)$,

$$x \equiv \sum_{i=1}^{j+1} \lambda^{(i)}(j^{(i)} + c^*(j^{(i)})) \bmod S$$

for some $\lambda^{(i)} \in I(M)^n$ and $j^{(i)} \in I(S)^n$, $i = 1 , \dots , j + 1$.

We have two cases :

Case 1. – $j^{(s)\prime} = j^{(t)\prime}$ for some $s \neq t$. Then

$$x' = \sum_{i=1}^{j+1} \lambda^{(i)\prime}(j^{(i)\prime} + c^*(j^{(i)\prime}))$$

$$= \sum_{\substack{1 \leqslant i \leqslant j+1,\\ i \neq s,t}} \lambda^{(i)\prime}(j^{(i)\prime} + c^*(j^{(i)\prime})) + (\lambda^{(s)\prime} + \lambda^{(t)\prime})(j^{(s)\prime} + c^*(j^{(s)\prime}))$$

so $x \in (j) E(2M , T) + T\mathbb{Z}$.

Case 2. – $j^{(s)\prime} = j^{(t)\prime}$ only if $s = t$. Then there is an r , $0 \leqslant r \leqslant \ell - 1$, such that

$$\phi(r)(j^{(i)\prime}) = \lambda^{(i)} \quad \text{for } i = 1 , \dots , j + 1 .$$

So

$$x = \sum_{i=1}^{j+1} \phi(r)(j^{(i)\prime})(j^{(i)} + c^*(j^{(i)})) \in P_r \mathbb{Z}$$

Thus we have

$$(j + 1) E(M , S) + S.\mathbb{Z} \subset (\cup_{r=0}^{\ell-1} P_r.\mathbb{Z}) \cup ((j) E(2M , T) + T.\mathbb{Z}) + S\mathbb{Z} ,$$

so $|(j + 1) E(M , S)| S^{-1} < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

We modify the argument for $j = 0$: take $T = 1$ and $G = I(M)^n \setminus \{(0 , \dots , 0)\}$ and use (2) (only) to define $< (j)$. Then if $x \in E(M , S)$,

$$x = \lambda(j + c^*(j)) = \phi(r)(j + c^*(j)) \quad \text{for some } r .$$

So $x \in \cup_{0 \leqslant r \leqslant \ell-1} P_r.\mathbb{Z}$.

LEMMA 3. – <u>For any</u> j , $n \in \mathbb{N}$ <u>and</u> $\varepsilon > 0$, <u>there is an</u> $N \in \mathbb{N}$ <u>and an</u> $E \subset \mathbb{T}$ (<u>where</u> $\mathbb{T}$ <u>is</u> $\mathbb{R}$ modulo 1) <u>such that</u>

(1) E <u>consists of a finite union of closed intervals</u>.

(2) <u>If</u> $G \subset \mathbb{T}$ <u>and</u> $|G| \leqslant n$, <u>there is a</u> $d = d(G)$ <u>such that</u> $G + d \subset E$.

(3) $m((j) E + r^{-1}[0 , 5j]) < \varepsilon$ <u>for all</u> $r > N$.

The difference between Lemma 1 and Theorem 1 is that the modulus of Lemma 1 has to be the product of very large primes. We need Lemma 3 to show that, rather surprisingly, any sufficiently large modulus will do.

Proof of Lemma 3 (assuming Lemma 1). - Let $\underline{T} = N(j , n , \varepsilon/6j)$, where N is as in Lemma 1. Let $E = N^{-1} A + [0 , N^{-1})$. We show that E has the required properties

$$(j)E + r^{-1} [0 , 5j))$$
$$= (j) N^{-1} A + [0 , jN^{-1}) + [0 , 5jr^{-1}) \subset (j) N^{-1} A + [0 , 6jN^{-1}) \quad ,$$

so $m((j) E + r^{-1}[0 , 5j)) \leqslant |(j) A| \times 6j/N < \varepsilon$.

Now G can be covered by at most n intervals of the form $[i/N , (i + 1)/N)$, $i \in \underset{\sim}{Z}$ so

$$G \subset N^{-1} F + [0 , N^{-1}) \text{ for some } F \subset \underset{\sim}{Z}(N) \text{ where } |F| \leqslant n .$$

If $c = c(F)$, we have

$$G + c/N \subset N^{-1}(F + c) + [0 , N^{-1}) \subset N^{-1} A + [0 , N^{-1}) = E ,$$

which gives (2), taking $d = c/N$.

Proof of Theorem 1 (assuming Lemma 3). - Since E is a finite union of closed intervals,

$$E = \bigcup_{i=1}^{k}(x_i + [0 , \delta_i]) \text{ for some } x_1 , \dots , x_k \text{ and } \delta_1 , \dots , \delta_k .$$

If $M \geqslant N$, then

$$E + [0 , M^{-1}) = \bigcup_{i=1}^{k}([x_i , (x_i + 1/M)) + [0 , \delta_i])$$
$$\supset \bigcup_{i=1}^{k}(y_i M^{-1} + [0 , \delta_i]) \text{ for } y_i = [Mx_i + 1) , \; i = 1 , \dots , k$$
$$\supset \bigcup_{i=1}^{k}(y_i M^{-1} + [0 , z_i M^{-1}]) \text{ where } z_i = [M\delta_i] , \; i = 1 , \dots , k$$
$$= \bigcup_{i=1}^{k}(y_i M^{-1} + M^{-1} I(z_i) + [0 , M^{-1}))$$
$$= M^{-1} B + [0 , M^{-1}) \quad \dots \qquad (A)$$

where $B = {}^{k}_{i=1}(y_i + I(z_i))$.

Now

$$M^{-1} B + [0 , 3/M) = \bigcup_{i=1}^{k}(y_i M^{-1} + M^{-1}[0 , z_i]) + [0 , 2/M)$$
$$= \bigcup_{i=1}^{k}([y_i M^{-1} , (y_i + 1) M^{-1}] + M^{-1}[0 , z_i + 1])$$
$$\supset \bigcup_{i=1}^{k}(x_i + 1/M + [0 , \delta_i])$$
$$= E + 1/M \quad \dots \qquad (B)$$

Now suppose $F \subset \underset{\sim}{Z}(M)$, $|F| \leqslant n$. Then there is, by lemma 3 , a d such that $M^{-1} F + d \subset E$. So

$$E + [0 \, , \, 2/M) \supset M^{-1} F + d + [0 \, , \, 2/M)$$

$$\supset M^{-1} F + [d \, , \, d + 1/M) + [0 \, , \, 1/M)$$

$$\supset M^{-1} F + c/M + [0 \, , \, 1/M) \qquad (C)$$

where $c = [Md + 1]$. But by (B),

$$M^{-1} B + [0 \, , \, 5/M) \supset E + 1/M + [0 \, , \, 2/M)$$

$$\supset M^{-1}(F + c + 1) + [0 \, , \, 1/M) \quad , \text{ by (C)} .$$

This is equivalent to

$$M^{-1}(B + I(5)) + [0 \, , \, 1/M) \supset M^{-1}(F + c + 1) + [0 \, , \, 1/M)$$

which implies that $F + c \subset B + I(5) - 1$.

We take $A = B + I(5) - 1 \pmod{M}$. Then

$$|(j) \, A| \, M^{-1} = m((j) \, M^{-1} A + [0 \, , \, M^{-1}))$$

$$= m((j)(M^{-1} B + M^{-1} I(5)) + [0 \, , \, M^{-1}))$$

$$< m((j)(E + [0 \, , \, M^{-1}) + M^{-1} I(5)))$$

$$= m((j) \, E + M^{-1} [0 \, , \, 5j)) < \varepsilon \text{ , by Lemma 3.}$$

<u>Proof of Theorem 2</u>. - We work in $\underset{\sim}{T}^n$ ($\underset{\sim}{R}^n$ modulo 1), for convenience. Choose $N(j \, , \, r \, , \, \varepsilon/2j)$ to be the same as in Theorem 1. If $C \in b(N \, , \, r)$ then, from the definition

$$C = \sum_{i=1}^{\infty} N^{-i} F_i \quad \text{for some } F_1 \, , \, F_2 \, , \, \ldots \in I(N)^n .$$

Write $C_i = N^{-i}(F_i + N \cdot \underset{\sim}{Z}^n) + N^{-i} [0 \, , \, 1)^n$. Then $C = \bigcap_{i=1}^{\infty} C_i$. Also, since $|F_i| \leqslant r$,

$$F_i \subset G_i^{(1)} \times \ldots \times G_i^{(n)} \text{ for some } G_i^{(k)} \subset I(N)^n \, , \quad |G_i^{(k)}| < r \, , \quad 1 \leqslant k \leqslant n .$$

Applying Theorem 1 for each $G_i^{(k)}$, we have a $c(G_i^{(k)}) \in I(N)$ such that

$$G_i^{(k)} + c(G_i^{(k)}) + N \, \underset{\sim}{Z} \subset A + N \, \underset{\sim}{Z} .$$

So if $c_i = c_i(F_i) = (c(G_i^{(1)}) \, , \, \ldots \, , \, c(G_i^{(n)}))$,

$$F_i + c_i + N \, \underset{\sim}{Z}^n \subset A^n + N \, \underset{\sim}{Z}^n .$$

Let $d = \sum_{i=1}^{\infty} c_i \, N^{-i}$. Then

$$C_i + d = N^{-i}(F_i + c_i + N.\underset{\sim}{Z}^n) + N^{-1}[0\ ,\ 1)^n + \sum_{\ell > i} c_\ell\ N^{-\ell}$$

$$\subset N^{-i}(A^n + N.\underset{\sim}{Z}^n) + N^{-i}[0\ ,\ 2)^n$$

$$= N^{-i}(A + \{0\ ,\ 1\} + N.\underset{\sim}{Z})^n + N^{-i}[0\ ,\ 1)\ .$$

So, if $B \equiv A + \{0\ ,\ 1\}$ mod N , $B \subset I(N)$, and $K = \{\sum_{i=1}^{\infty} x_i\ N^{-i}\ ;\ x_i \in B^n\}$, we have $C + d \subset K$. Since K is a closed perfect self-similar Cantor set, we need only show (b) . But

$$K = \bigcap_{i=1}^{\infty}(N^{-i}(B + N.\underset{\sim}{Z})^n + N^{-i}[0\ ,\ 1)^n)$$

so

$$(j)\ K = \bigcap_{i=1}^{\infty}(N^{-i}((j)\ B + N.\underset{\sim}{Z})^n + N^{-i}[0\ ,\ j\ )^n)$$

$$= \bigcap_{i=1}^{\infty}(N^{-i}((j)\ B + I(j) + N.\underset{\sim}{Z})^n + N^{-i}[0\ ,\ 1\ )^n)\ .$$

Now

$$(j)B + I(j) \equiv (j)\ A + I(2j)\ ,\ \bmod N$$

and

$$(j)\ A + I(2j)| \leqslant 2j|(j)\ A| < \epsilon\ .$$

<u>Proof of Theorem 3</u>. - Again we work in $\underset{\sim}{T}^n$. If $F \subset \underset{\sim}{T}^n$ is any countable set, let $x_1\ ,\ x_2\ ,\ \ldots$ be an enumeration of F . In the notation of Theorem 1, let $n_i = N(i\ ,\ i\ ,\ i^{-1})$, $A_i = A(n_i)$ (so that $|(i)\ A_i| \leqslant i^{-1}\ n_i$) and let $M_i = n_1 \times n_2 \times \ldots \times n_i$.

Assuming that we have chosen $c_k \in I(M_k)^n$, $0 \leqslant k \leqslant i-1$, we choose C_i as follows.

There is an $F_i \subset I(n_i)^n$, $|F_i| \leqslant i$ such that

$$M_{i-1}\{x_1\ ,\ \ldots\ ,\ x_i\} + \underset{\sim}{Z}^n \subset n_i^{-1}\ F_i + n_i^{-1}[0\ ,\ 1)^n + \underset{\sim}{Z}^n$$

(this is just a way of saying that we need at most i intervals to contain i points).

As in the proof of Theorem 2, we may choose c_i so that

$$F_i + c_i + n_i\ \underset{\sim}{Z}^n \subset A_i^n + n_i\ \underset{\sim}{Z}^n\ ,$$

so

$$(1)\quad \{x_1\ ,\ \ldots\ ,\ x_i\} + c_i\ M_i^{-1} + M_{i-1}^{-1}\ \underset{\sim}{Z}^n \subset M_i^{-1}\ A^n + M_i^{-1}[0\ ,\ 1)^n + M_{i-1}^{-1}\ \underset{\sim}{Z}^n \ldots$$

For $k \in \underset{\sim}{N}$, $E \subset \underset{\sim}{T}^n$, $m(k^{-1}\ E + k^{-1}\ \underset{\sim}{Z}^n) = m(E)$, so if

$$C_i = M_i^{-1}\ A_i^n + M_{i-1}^{-1}\ \underset{\sim}{Z}^n + M_i^{-1}[0\ ,\ 2)^n\ ,$$

$$(i)\ C_i \subset M_{i-1}^{-1}((i)(m_i^{-1}(A_i^n + [0\ ,\ 2)^n)) + M_{i-1}^{-1}\ \underset{\sim}{Z}^n$$

and so

$$(2) \qquad m((i)\ C_i) = m((i)\ (m_i^{-1}\ A_i^n + m_i^{-1}\ [0\ ,\ 2)^n)$$
$$\leqslant |(i)\ A_i^n| \times 2^n/m_i^n < 2^n/i^n\ .$$

Let $c = \sum_{i=1}^{\infty} c_i\ M_i^{-1}$, and suppose x_i is any member of F . Then, if $k \geqslant i$,

$$\begin{aligned} x_i + c &= x_i + \textstyle\sum_{j=1}^{k-1} c_j\ M_j^{-1} + \sum_{j=k}^{\infty} c_j\ M_j^{-1} \\ &\subset x_i + \textstyle\sum_{j=k}^{\infty} c_j\ M_j^{-1} + M_{k-1}^{-1}\ \underset{\sim}{Z}^n \\ &= x_i + c_k\ M_k^{-1} + M_{k-1}^{-1}\ \underset{\sim}{Z}^n + \textstyle\sum_{j=k+1}^{\infty} c_j\ M_j^{-1} \\ &\subset \{x_1\ ,\ \dots\ ,\ x_k\} + c_k\ M_k^{-1} + M_{k-1}^{-1}\ \underset{\sim}{Z}^n + M_k^{-1}\ [0\ ,\ 1)^n \\ &= C_k \quad (\text{from } (1))\ . \end{aligned}$$

So $x_i + c \in \bigcap_{k=i}^{\infty} C_k$. If we put $K = \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} C_k$, K is F_σ and $F + c \subset K$.

We complete the proof by showing that, for any k , $m((k)\ K) = 0$. For suppose $y \in (k)\ K$. Then $y = y_1 + \dots + y_k$ for some $y_1\ ,\ \dots\ ,\ y_k \in K$. Now for any $z \in K$, there is a smallest $i = i(z)$ such that $z \in \bigcap_{j=i}^{\infty} C_j$. Let $m = \max\{i(y_1), \cdots, i(y_k)\}$, then $y \in (k)\ \bigcap_{j=m}^{\infty} C_i$. So

$$(3) \qquad (k)\ K \subset \bigcup_{m=1}^{\infty} (k)\ \bigcap_{j=m}^{\infty} C_j\ .$$

Now for any m , if ℓ is any number $\geqslant m$, k , then

$$m((k)\ \bigcap_{j=m}^{\infty} C_j) \leqslant m(k)\ C_\ell) < m((\ell)\ C_\ell) < (2/\ell)^n\ ,\ \text{from } (2)\ ,$$

so $m((k) \bigcap_{j=m}^{\infty} C_j) = 0$ since ℓ was arbitrarily large.

Substituting in (3), we see that $(k)\ K$ is the countable union of sets of measure zero and so is of measure zero.

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