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TOPOLOGICAL INVARIANTS OF REAL ALGEBRAIC CURVES

par Benedict H. GROSS

Let X be an absolutely irreductible projective smooth curve over the field $\mathbb{R}$ of real numbers. One can associate to X three topological invariants :

$$(1)\qquad \begin{cases} g = \text{genus of the Riemann surface } X(\mathbb{C}) \\ n = \text{number of components of the real locus } X(\mathbb{R}) \\ a = \begin{cases} 0 & \text{if } X(\mathbb{R}) \text{ disconnects } X(\mathbb{C}) \\ 1 & \text{if } X(\mathbb{C})-X(\mathbb{R}) \text{ is connected.} \end{cases} \end{cases}$$

These invariants satisfy the following relations, which follow from the classification of compact 2-manifolds :

$$(2)\qquad \begin{cases} g \geq 0 \ . \\ 0 \leq n \leq g+1 \ . \\ \text{If } n=0 \text{ then } a=1 \ . \\ \text{If } n=g+1 \text{ then } a=0 \ . \\ \text{If } a=0 \text{ then } n \equiv g+1 \pmod 2 \ . \end{cases}$$

Klein showed that any triple (g,n,a) satisfying the relations (2) actually occurs as the invariants of a real algebraic curve X , and that the moduli of all real curves with a fixed triple of invariants is a connected real analytic space of dimension g for $g \leq 1$ and $3g-3$ for $g \geq 2$.

The invariant g can also be described as the algebraic genus of the function field $\mathbb{R}(X)$. Here we give an algebraic description of the invariants n

and a in the case where $n > 0$ (so X has a real point). Consider the finite set S of real theta-characteristics on X ; this is partitioned into even and odd characteristics, and one can show that :

$$(3) \qquad \begin{cases} \text{Card } S = 2^{g+n-1} \\ \text{Card } S_{\text{odd}} = 2^{g-1}(2^{n-1}+a-1) \ . \end{cases}$$

For example, any smooth quintic $X \subset \mathbb{P}^2$ with one real component has $a = 1$, as S contains an odd characteristic which is cut out by lines.

For more details, see the forthcoming article by the author and J. Harris in the Annals of the Ecole Normale Supérieure.

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