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On filtrations related to purely discontinuous martingales

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Abstract

General martingale theory shows that every martingale can be decomposed into continuous and purely discontinuous parts. In this paper specify a filtration $\mathcal{F}_t$, for which the continuous part of the decomposition is 0 *a.s.* for any $\mathcal{F}_t$ martingale.

It is a well-known fact that every martingale can be decomposed into continuous and purely discontinuous parts. It is of interest to study the filtrations that do not support continuous martingales (i.e. those for which every continuous martingale with respect to that filtration is constant *a.s.*). In a previous work J. Jacod and A.V. Skorokhod (1994) [5] introduced the notion of jumping filtration. A filtration $\mathcal{F}_t$ is *jumping* if there is a sequence of increasing stopping times $\{T_n\}$ (we will call them loosely "jumps"), such that the σ-algebras $\mathcal{F}_t$ and $\mathcal{F}_{T_n}$ coincide up to the null sets on $\{T_n \leq t < T_{n+1}\}$. In other words $\mathcal{F}_t = \sigma\{A \cap \{T_n \leq t\} : A \in \mathcal{F}_{T_n}\}$. They proved that a σ-algebra is jumping iff it supports only martingales of bounded variation. Under more restrictive conditions we generalize their result to filtrations supporting only purely discontinuous martingales. As opposed to the jumping filtrations that support only martingales of locally bounded variation with finitely many jumps on finite intervals, our filtrations can support a martingale that has infinitely many jumps on a finite interval. An example of such a filtration is the natural filtration of an Azèma martingale (e.g. the filtration generated by the sign of Brownian motion) or a natural filtration of purely discontinuous Lévy process with infinitely many jumps on finite intervals.

To accommodate this change we replace the increasing sequence of stopping times with a countable set of totally inaccessible stopping times with disjoint graphs.

Unless stated otherwise we always assume that the filtration $\mathcal{F}_t$ is complete, right-continuous, quasi-left-continuous, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ *a.s.*, and the σ-algebra $\mathcal{F}_\infty$ is countably generated. All martingales are considered to be in their *càdlàg* version.

Let us introduce the following definitions.

Definition 1. A filtration is called purely discontinuous if any continuous adapted martingale is constant *a.s.*

*In this paper we present some of the results in my Ph.D. dissertation [3] completed under the supervision of Professor A. V. Skorokhod at Michigan State University

Definition 2. Let T be a countable collection of stopping times. Then we define

$$S_\tau = \inf\{v(\omega);\ v(\omega) > \tau(\omega), v \in T\},$$

where τ is a stopping time, and

$$A_t(\omega) = \sup\{\tau(\omega);\ \tau(\omega) < t, \tau \in T\},$$

where t is a deterministic time.

Note that S_τ is a stopping time, while A_t is not. The random variables S_t and A_t will be often referred to as "first jump after τ" and "last jump before t" respectively. This comes from the observation that if the set T is the set of all possible jumps of adapted martingales then any $\mathcal{F}_t$ martingale does not have jumps on the interval (τ, S_τ) and for all $\epsilon > 0$ there is an $\mathcal{F}_t$ adapted martingale that has at least one jump on $[S_\tau, S_\tau + \epsilon)$. An analogous statement is true for A_t.

Now we can state the main theorem of this article.

Theorem 1. *Let $\mathcal{F}_t$ be a purely discontinuous filtration. Then*

$$\mathcal{F}_t = \sigma\{A \cap \{\tau \le t\};\quad A \in \mathcal{F}_\tau,\ \tau \in T\}, \tag{1}$$

where T is a countable collection of totally inaccessible stopping times with disjoint graphs.

The intuitive meaning of this theorem is that the information contained in our filtration came only from jumps of the martingales. It is worth pointing out that this necessary condition is not sufficient. Two conceptually different examples of filtrations generated by times and sizes of jumps of martingales that allow a non-trivial continuous adapted martingale are given in [3]. At the same place we can find a useful but not very general sufficient condition for a filtration to be purely discontinuous. Well-known examples of purely discontinuous filtrations include natural filtration of purely discontinuous Lévy process and the smallest filtration that admits a sequence of independent non-negative random variables as stopping times.

Before proceeding to the proof of the theorem we state a result directly implied by the proof of part γ of Theorem 2 (page 24) in [5].

Lemma 1. *The following is true under the assumptions of Theorem 1: Let $H \le S$ be two stopping times. If any $\mathcal{F}_t$ martingale is continuous on the interval (H, S) then $\mathcal{F}_t = \mathcal{F}_H$ on $\{H \le t < S\}$ (i.e. for every $A \in \mathcal{F}_t$ there is $A' \in \mathcal{F}_H$ such that $A \cap \{H \le t < S\} = A' \cap \{H \le t < S\}$ a.s.).*

Proof of Theorem 1. Since the filtration is quasi-left-continuous and $\mathcal{F}_\infty$ is a completion of a countably generated σ-algebra, it is known that there is a countable sequence T of totally inaccessible stopping times with mutually disjoint graphs that exhaust all possible jumps of martingales, i.e. if M is a martingale the graph $[\Delta M] \subset \bigcup_{\tau \in T}[\tau]$.

Let $\mathcal{G}_t = \sigma\{A \cap \{\tau \le t\}; A \in \mathcal{F}_\tau, \tau \in T\}$. It is easy to see that $\mathcal{G} \subset \mathcal{F}$. To prove Theorem 1 we need to prove $\{\mathcal{F}_t\} = \{\mathcal{G}_t\}$.

To begin with we prove that for any totally inaccessible $\mathcal{F}_t$ stopping time v, v is a $\mathcal{G}$ stopping time and the filtrations $\mathcal{G}_v = \mathcal{F}_v$ on the set $\{v < \infty\}$.

If $v \in T$, the assertion follows from the definition. Let us assume that $v \notin T$. It is known that $[v] = \bigcup_{\{\tau \in T\}} \{\tau = v\} \cap [\tau]$. Thus for any finite $t \in \mathbb{R}$, and for any $A \in \mathcal{F}_v$

$$A \cap \{v \leq t\} = \bigcup_{\{\tau \in T\}} A \cap \{\tau = v\} \cap \{\tau \leq t\} \in \mathcal{G}_t.$$

It follows that v is a stopping time with respect to $\mathcal{G}_t$ and $\mathcal{G}_v = \mathcal{F}_v$ on the set $\{v < \infty\}$. This restriction arises from the definition of $\mathcal{G}_\infty = \bigvee \mathcal{G}_t = \sigma\{A \cap \{\tau < \infty\}; A \in \mathcal{F}_\tau, \tau \in T\}$.

The following simple observations are valid for any sequence of stopping times $\{\tau_n\}$. If $\mathcal{F}_{\tau_n} = \mathcal{G}_{\tau_n}$ on $\{\tau_n < \infty\}$, the sequence τ_n is non-increasing and $\tau_n \to \tau$, then

$$\mathcal{F}_\tau = \bigwedge \mathcal{F}_{\tau_n} = \bigwedge \mathcal{G}_{\tau_n} = \mathcal{G}_\tau \text{ on } \bigcup \{\tau_n < \infty\} = \{\tau < \infty\}.$$

Similarly the sequence τ_n is non-decreasing and $\tau_n \to \tau$, then

$$\mathcal{F}_\tau = \bigvee \mathcal{F}_{\tau_n} = \bigvee \mathcal{G}_{\tau_n} \subset \mathcal{G}_\tau \subset \mathcal{F}_\tau \text{ on } \bigcap \{\tau_n < \infty\} \supset \{\tau < \infty\}.$$

The latter statement is true because the filtration $\mathcal{F}_t$ is quasi-left-continuous.

Recall Definition 2. We have defined:

$$S_\tau = \inf\{v_{\{v > \tau\}}; v \in T\},$$

and

$$A_t(\omega) = \sup\{\tau(\omega); \tau(\omega) < t, \tau \in T\}.$$

The random variable A_t is a $\mathcal{G}_t$ measurable random variable. This follows from the relation:

$$\{A_t \leq s\} = \{S_s \geq t\} \in \mathcal{G}_t.$$

Similarly S_τ is a $\mathcal{F}_t$ (resp. $\mathcal{G}_t$) stopping time if τ is an $\mathcal{F}_t$ (resp. $\mathcal{G}_t$) stopping time. Since T is a countable set, it follows from the previous statements, that if τ is an $\mathcal{G}_t$ stopping time $\mathcal{F}_{S_\tau} = \mathcal{G}_{S_\tau}$ on the set $\{S_\tau < \infty\}$.

To finish the proof it will be enough to prove $\mathcal{F}_{t_0} = \mathcal{G}_{t_0}$ for a fixed t_0. I will do it separately on three different $\mathcal{G}_{t_0}$ measurable sets.

Define the following $\mathcal{G}_{t_0}$ measurable sets:

$$\begin{aligned} B_1 &= \bigcup_{\tau \in T} \{A_{t_0} = \tau\}, \\ B_2 &= \{A_{t_0} = t_0\} \cup \{S_{t_0} = t_0\}, \\ B_3 &= \Omega \setminus (B_1 \cup B_2). \end{aligned}$$

It follows from the previous discussion that $\mathcal{F}_{t_0} = \mathcal{G}_{t_0}$ on the set B_2.

As mentioned before no $\mathcal{F}_t$ martingale has jumps between times τ and S_τ for any fixed $\mathcal{G}_t$ stopping time τ. It follows from Lemma 1 that if $\tau \in T$ and $B \in \mathcal{F}_{t_0}$ there is $B' \in \mathcal{F}_\tau = \mathcal{G}_\tau$ such that

$$B \cap \{\tau \leq t < S_\tau\} = B' \cap \{\tau \leq t < S_\tau\} \in \mathcal{G}_t.$$

Finally $B_2 \subset \bigcup_{\tau \in T} \{\tau \leq t < S_\tau\}$ implies $\mathcal{F}_{t_0} = \mathcal{G}_{t_0}$ on B_1.

To overcome problems associated with B_3 we will enlarge our filtrations. Define

$$\tilde{A} = \begin{cases} A_{t_0} & \text{on the set } B_3, \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\tilde{A}$ is $\mathcal{G}_{t_0}$ measurable random variable, and $\{\tilde{A} = \tau < \infty\} = \emptyset$ for all $\tau \in T$. Thus the graph $[\tilde{A}]$ is a subset of the left limit points of $\bigcup_{\tau \in T}[\tau]$. We further define[1]

$$\mathcal{H}_s = \sigma\{\tilde{A} \leq x;\, x \leq s\}, \quad \tilde{\mathcal{F}}_s = \mathcal{F}_s \vee \mathcal{H}_s, \quad \tilde{\mathcal{G}}_s = \mathcal{G}_s \vee \mathcal{H}_s.$$

The filtration was augmented just enough to make the random variable $\tilde{A}$ a stopping time.

Notice that if $\tau \in T$, then $\mathcal{F}_{\tau-} = \tilde{\mathcal{F}}_{\tau-}$. This follows from the following: Let $t < t_0$ and $B \in \tilde{\mathcal{F}}_t$,

$$B \cap \{\tau > t\} = (B \cap \{t < \tau \leq t_0\}) \cup (B \cap \{t_0 < \tau\}) \in \mathcal{F}_{\tau-},$$

since both $B \in \tilde{\mathcal{F}}_t \subset \mathcal{F}_{t_0}$ and $\{t < \tau \leq t_0\} \subset \{\tilde{A} > t\}$, which is an atom in the σ-algebra $\mathcal{H}_t$.

As a next step we want to prove that $\tilde{\mathcal{F}}_{\tilde{A}} = \tilde{\mathcal{G}}_{\tilde{A}}$ on the set $\{\tilde{A} < \infty\}$.

Observe that for any $B \in \mathcal{F}_\tau$, $\tau \in T$

$$B \cap \{\tau < \tilde{A}\} = \bigcup_{q \in \mathbb{Q}} B \cap \{\tau \leq q < \tilde{A}\} \in \tilde{\mathcal{G}}_{\tilde{A}-}.$$

Calculate on the set $\{\tilde{A} < \infty\}$

$$\begin{aligned} \tilde{\mathcal{F}}_{\tilde{A}-} &= \sigma\{B \cap \{\tilde{A} > t\}, \quad B \in \tilde{\mathcal{F}}_t\} \\ &= \sigma\{B \cap \{\tilde{A} > \tau > t\}, \quad \tau \in T, B \in \tilde{\mathcal{F}}_t\} \\ &= \sigma\{D \cap \{\tilde{A} > t\}, \quad D \in \mathcal{F}_{\tau-}\} \subset \tilde{\mathcal{G}}_{\tilde{A}-}. \end{aligned}$$

Since on the same set $\tilde{\mathcal{F}}_{\tilde{A}-} = \tilde{\mathcal{G}}_{\tilde{A}-} \subset \tilde{\mathcal{G}}_{\tilde{A}} \subset \tilde{\mathcal{F}}_{\tilde{A}}$, it is enough to prove $\tilde{\mathcal{F}}_{\tilde{A}-} = \tilde{\mathcal{F}}_{\tilde{A}}$.

To do this we will need to decompose the stopping time $\tilde{A}$ in a manner very similar to the decomposition to totally inaccessible and accessible parts. Let

$$p = \sup P\left(\bigcup_n \{\tilde{A}_{t_0} = \hat{\nu}_n < \infty\}\right),$$

where the suprema extends over all possible sequences of predictable $\mathcal{F}_t$ stopping times. Combining sequences such that the probability on the right-hand-side approaches p we construct a sequence $S = \{\nu_n\}$ of predictable $\mathcal{F}_t$ stopping times for which the suprema is attained. (Note that if $p = 0$ this sequence is empty.) Define

$$\tilde{A}_1 = \begin{cases} \tilde{A} & \text{on the set } \bigcup_{\nu \in S}\{\nu = \tilde{A}\}, \\ \infty & \text{otherwise;} \end{cases}$$

$$\tilde{A}_2 = \begin{cases} \infty & \text{on the set } \bigcup_{\nu \in S}\{\nu = \tilde{A}\}, \\ \tilde{A} & \text{otherwise.} \end{cases}$$

[1]The filtrations are enlarged to satisfy the usual conditions where necessary.

If $\nu \in S$ then the set $\{\nu = \tilde{A}\} \in \tilde{\mathcal{F}}_{\tilde{A}-} = \tilde{\mathcal{G}}_{\tilde{A}-}$, whence both $\tilde{A}_1$ and $\tilde{A}_2$ are $\tilde{\mathcal{G}}_t$ stopping times. Furthermore the fact that $\mathcal{F}_t$ is quasi-left-continuous implies that $\tilde{\mathcal{F}}_{\tilde{A}_1-} = \tilde{\mathcal{F}}_{\tilde{A}_1}$ on the set $\{\tilde{A}_1 < \infty\}$. We need to prove the same for $\tilde{A}_2$.

First we prove that $\tilde{A}_2$ is a totally inaccessible $\tilde{\mathcal{F}}_t$ stopping time. Notice that $P(\tilde{A}_2 = \tau < \infty) = 0$ for any $\mathcal{F}_t$ stopping time τ.

Let τ be a $\tilde{\mathcal{F}}_t$ stopping time. I will prove that there is a $\mathcal{F}_t$ stopping time τ', such that $\tau = \tau'$ on the set $\{\tau < \tilde{A}_2\}$. Denote $C_s = \{\tau > s\} \cap \{\tilde{A}_2 > s\} = \{\tau \wedge \tilde{A}_2 > s\}$. Since $\{\tilde{A}_2 > s\}$ is an atom in $\mathcal{H}_s$, there is $\bar{D}_s \in \mathcal{F}_s$ such that $C_s = \bar{D}_s \cap \{\tilde{A}_2 > s\}$. We define

$$D_s = \bigcup_{\substack{q_1 > s \\ q_1 \in \mathbb{Q}}} \bigcap_{\substack{q_2 \leq q_1 \\ q_2 \in \mathbb{Q}}} \bar{D}_{q_2}.$$

The right-continuity of the filtrations involved gives $D_s \in \mathcal{F}_s$, and the definition of C_s gives $C_s = D_s \cap \{\tilde{A}_2 > s\}$. Define $\tau'(\omega) = \sup\{t : \omega \in D_t\}$. It is a $\mathcal{F}_t$ stopping time and $\tau = \tau'$ on the set $\{\tau < \tilde{A}_2\}$. The fact that $\tilde{A}_2$ is a totally inaccessible $\tilde{\mathcal{F}}_t$ stopping time follows directly.

At this point we will use a rather unusual feature of our enlargement: Notice

$$\{\tilde{A}_2 \leq s\} = \{\tilde{A}_2 \leq t\} \cap \{A_t \leq s\} \quad \text{for } s \leq t < t_0.$$

However $\{A_t \leq s\} \in \mathcal{F}_t$. This implies that the σ-algebra $\mathcal{F}_t$ was augmented only by 1 set. More precisely

$$\tilde{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\{\tilde{A}_2 \leq t\}) \ a.s. \tag{2}$$

Let $\mathcal{Z}$ be the set of all non-negative, integrable, $\mathcal{F}_\infty$ measurable random variables, and $Z \in \mathcal{Z}$. Simple algebra and equation (2) show that the martingale

$$\eta_t^Z = E[Z|\tilde{\mathcal{F}}_t] = \xi_1^Z(t) 1_{\{\tilde{A}_2 \leq t\}} + \xi_2^Z(t) 1_{\{\tilde{A}_2 > t\}}, \tag{3}$$

where

$$\xi_1^Z(t) = \frac{E[Z 1_{\{\tilde{A}_2 \leq t\}} | \mathcal{F}_t]}{P[\tilde{A}_2 \leq t | \mathcal{F}_t]}, \quad \text{and} \quad \xi_2^Z(t) = \frac{E[Z 1_{\{\tilde{A}_2 > t\}} | \mathcal{F}_t]}{P[\tilde{A}_2 > t | \mathcal{F}_t]}.$$

Observe that the process $E[Z 1_{\{\tilde{A}_2 \leq t\}} | \mathcal{F}_t]$ is a submartingale and the function $t \to E[Z 1_{\{\tilde{A}_2 \leq t\}}]$ is continuous, hence there is a *càdlàg* modification of $E[Z 1_{\{\tilde{A}_2 \leq t\}} | \mathcal{F}_t]$. Thus we can assume without loss of generality that the processes $\xi_1^Z(t)$ and $\xi_2^Z(t)$ are $\mathcal{F}_t$ adapted *càdlàg* processes. This immediately implies that

$$\tilde{\mathcal{F}}_{\tilde{A}_2} = \sigma(\eta_{\tilde{A}_2}^Z, \ Z \in \mathcal{Z}) = \sigma(\xi_1^Z(\tilde{A}_2-), \ Z \in \mathcal{Z}) \subset \tilde{\mathcal{F}}_{\tilde{A}_2-} \quad \text{on the set} \{\tilde{A}_2 < \infty\},$$

since any $\mathcal{F}_t$ adapted *càdlàg* process is $a.s.$ continuous at the time $\tilde{A}_2$. Combining the results for $\tilde{A}_1$ and $\tilde{A}_2$ we conclude that $\tilde{\mathcal{F}}_{\tilde{A}} = \tilde{\mathcal{G}}_{\tilde{A}}$. on the set $\{\tilde{A} < \infty\}$.

Lemma 1 implies that $\mathcal{F}_t$ is constant on any interval $[s, S_s)$, i.e. for any $t > s$ and $B \in \mathcal{F}_t$ we have $B' \in \mathcal{F}_s$ such that $B \cap \{S_s > t\} = B' \cap \{S_s > t\}$. A similar statement is true for σ-algebra $\mathcal{H}_s$ — notice that either $\tilde{A} < s$ or $\tilde{A} > S_s$ — and consequently for $\tilde{\mathcal{F}}_s$.

To finish the proof we will closely follow the proof that appears in section 2 (page 22) of [5]. Let M_t be any uniformly integrable $\tilde{\mathcal{F}}_t$ martingale such that M_t is 0 on $[0, \tilde{A}]$ and constant on $[t_0, \infty)$. To prove that M_t is 0 on $[0, \infty)$, we define $M_t^s = M_{t \wedge S_s} - M_{t \wedge s}$.

Note that $\{S_s < t_0\} \subset \{S_s < \tilde{A}\}$, and therefore the martingale $M_t^s \equiv 0$ on the set $\{S_s < t_0\}$, so $M_t^s = M_t^s 1_{\{t_0 \le S_s\}}$. The statement established in the previous paragraph implies that for any $t > s$ there is a $\tilde{\mathcal{F}}_s$ measurable random variable N_t such that $N_t = M_t^s$ on the set $\{s \le t < S_s\}$. Call G a regular version of the law of the pair $(S_s, M_{S_s}^s)$ conditional on $\tilde{\mathcal{F}}_s$, and $G''(t) = G((t,\infty] \times \mathbb{R} \cap [t_0,\infty] \times \mathbb{R}))$. If $t \ge s$, we have the following string of *a.s.* equalities (see [5] for justification):

$$(4) \quad N_t G''(t) = E[N_t 1_{\{t<S_s\}} 1_{\{t_0 \le S_s\}} | \tilde{\mathcal{F}}_s] = E[M_t^s 1_{\{t_0 \le S_s\}} 1_{\{t<S_s\}} | \tilde{\mathcal{F}}_s]$$
$$= E[M_t^s 1_{\{t<S_s\}} | \tilde{\mathcal{F}}_s] = E[M_{S_s}^s 1_{\{t<S_s\}} | \tilde{\mathcal{F}}_s] = \int x \, 1_{\{u>t\}} G(du, dx).$$

The functions $G''(t)$ and $\int x \, 1_{\{u>t\}} G(du, dx)$ (taken as a function of t) are *a.s.* constant on the interval $[0, t_0)$. The fact that the second function is constant follows from $G((t, t_0) \times (\mathbb{R} \setminus \{0\})) = 0$ *a.s.* To conclude that $M_t^s = 0$ *a.s.* on the interval $[0, t_0]$ notice that the set $\{G''(0) = 0\}$ is $\mathcal{F}_s$ measurable, and more important $\{G''(0) = 0\} \subset \{S_s < t_0\}$. (The continuity of M^s at the point t_0 is implied by $\tilde{\mathcal{F}}_{t-} \supset \mathcal{F}_{t-} = \mathcal{F}_t = \tilde{\mathcal{F}}_t$.) Since s was arbitrary, we get $M_t = 0$ for $t \in [0, t_0]$. From here we finally obtain $\mathcal{F}_{t_0} = \tilde{\mathcal{F}}_{\tilde{A}} = \tilde{\mathcal{G}}_{\tilde{A}} \subset \mathcal{G}_{t_0}$ on the set $\{\tilde{A} < \infty\} = B_3$.

The proof is now complete, since if $A \in \mathcal{F}_{t_0}$ then

$$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_1) \in \mathcal{G}_{t_0}.$$

□

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