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ARE SQUARED BESSEL BRIDGES INFINITELY DIVISIBLE ?

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Abstract: Consider a squared Bessel bridge between two positive values x and y . If x or y is equal to 0, then this process is infinitely divisible. In the case when both x and y are strictly positive, Pitman and Yor conjectured in [P-Y] that the process is not infinitely divisible. We show here that it is not infinitely decomposable in the sense of Shiga and Watanabe [S-W].

1 - Introduction

Let $\mathcal{C}$ be the canonical space $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ and $\mathcal{F}$ be the σ -field, $\sigma\{\omega \rightarrow \omega(s) = X_s(\omega); s \geq 0\}$. For $d \geq 0$ and $x \geq 0$, let $\mathcal{Q}_x^d$ be the distribution on $(\mathcal{C}, \mathcal{F})$ of the square of a Bessel process with dimension d starting from $\sqrt{x}$. In [S-W], Shiga and Watanabe have established the following important additivity property:

$$\mathcal{Q}_x^d \oplus \mathcal{Q}_{x'}^{d'} = \mathcal{Q}_{x+x'}^{d+d'} \quad (1)$$

where, for P and Q two probabilities on $(\mathcal{C}, \mathcal{F})$, $P \oplus Q$ denotes the distribution of $(X_t + Y_t, t \geq 0)$ with $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ two independent processes respectively P and Q distributed.

An immediate consequence of the above additivity property is that squared Bessel processes are infinitely divisible. Indeed, we have for any $n \in \mathbb{N}$:

$$\mathcal{Q}_x^d = \mathcal{Q}_{x/n}^{d/n} \oplus \mathcal{Q}_{x/n}^{d/n} \oplus \dots \oplus \mathcal{Q}_{x/n}^{d/n}$$

But we also have the following stronger property :

$$\mathcal{Q}_x^d = \mathcal{Q}_{x_1}^{d/n} \oplus \mathcal{Q}_{x_2}^{d/n} \oplus \dots \oplus \mathcal{Q}_{x_n}^{d/n}$$

for any sequence $(x_i)_{1 \leq i \leq n}$ such that : $\sum_{i=1}^n x_i = x$.

Shiga and Watanabe have introduced this last property as the property of infinite decomposability. More precisely, let $\mathcal{IP} = \{P_x, x \in \mathbb{R}_+\}$ be a system of probabilities on $(\mathcal{C}, \mathcal{F})$ such that :

- for any $B \in \mathcal{F}$, $x \longrightarrow P_x(B)$ is measurable
- for every $x \in \mathbb{R}_+$, $P_x(X(0) = x) = 1$.

We denote by $\mathcal{P}$ the set of such systems $\mathcal{IP}$. Shiga and Watanabe set in [S-W] the definition below.

Definition 1.1 : Let $\mathbb{P}$ be an element of $\mathcal{P}$. $\mathbb{P}$ is said to be infinitely decomposable if for any $n \in \mathbb{N}^*$, there exists $\mathbb{P}^{(n)} \in \mathcal{P}$ such that for any $x \in \mathbb{R}$:

$$\mathbb{P}_x = \mathbb{P}_{x_1}^{(n)} \oplus \mathbb{P}_{x_2}^{(n)} \oplus \dots \oplus \mathbb{P}_{x_n}^{(n)}$$

for any sequence $(x_i)_{1 \leq i \leq n}$ such that : $\sum_{i=1}^n x_i = x$.

The distribution under $\mathbb{Q}_x^d$ of $(X_s, 0 \leq s \leq t)$ given that $(X_t = y)$ has been clearly defined by Pitman and Yor in [P-Y] for $d, x, y, t \geq 0$. This distribution represents the law of the d-dimensional squared Bessel bridge from x to y over time t . Without loss of generality, we will choose $t = 1$ and write $\mathbb{Q}_{x \rightarrow y}^d$ for the law of the d-dimensional squared Bessel bridge from x to y over time 1.

Thanks to the additivity property (1) , we have :

$$\mathbb{Q}_{x \rightarrow 0}^d \oplus \mathbb{Q}_{x' \rightarrow 0}^{d'} = \mathbb{Q}_{x+x' \rightarrow 0}^{d+d'}$$

which gives immediately the infinite decomposability of $\{\mathbb{Q}_{x \rightarrow 0}^d, x \in \mathbb{R}_+\}$ for any $d \geq 0$.

We are going to prove that for $y > 0$, $\{\mathbb{Q}_{x \rightarrow y}^d, x \in \mathbb{R}_+\}$ is not infinitely decomposable. Actually, we will show that $\{\mathbb{Q}_{x \rightarrow y}^d, x \in \mathbb{R}_+\}$ is not even "2-decomposable". Namely, we have the following property :

Theorem 1.2 : For $y > 0$ and $d \geq 0$, there is no couple $(\mathbb{P}, \tilde{\mathbb{P}})$ of $\mathcal{P} \times \mathcal{P}$ such that for any $(x, x_1, x_2) \in \mathbb{R}_+^3$ verifying : $x_1 + x_2 = x$, we have :

$$\mathbb{Q}_{x \rightarrow y}^d = \mathbb{P}_{x_1} \oplus \tilde{\mathbb{P}}_{x_2}$$

Since infinite decomposability is a stronger property than infinite divisibility, Theorem 1.2 does not prove Pitman and Yor's conjecture. But it confirms the gap between the cases $y = 0$ and $y > 0$.

In Section 2, we prove Theorem 1.2. The argument is based on the results of Pitman and Yor in [P-Y].

2 - Proof

Pitman and Yor have established that for any $\alpha > 0$ and $t \in [0, 1]$:

$$\mathbb{Q}_{x \rightarrow y}^d(e^{-\alpha X_t}) = A_0(t, \alpha)^x A_0(1-t, \alpha)^y B_0(t, \alpha)^2 I_\nu(\sqrt{xy} B_0(t, \alpha)^2) / I_\nu(\sqrt{xy}) \quad (2)$$

where $\nu = \frac{d}{2} - 1$, I_ν is the Bessel function of index ν , $A_0(t, \alpha)$ and $B_0(t, \alpha)$ are the constants determined by the equality :

$$\mathbb{Q}_{x \rightarrow 0}^d(e^{-\alpha X_t}) = A_0(t, \alpha)^x B_0(t, \alpha)^d$$

Our argument does not require the precise expression of these constants, but we note that they are computable.

Now $y > 0$ is fixed and we assume that there exists a couple $(P, \tilde{P})$ of elements of $\mathcal{P}$ such that for any $(x, x_1, x_2) \in \mathbb{R}_+^3$ verifying : $x_1 + x_2 = x$ we have :

$$\mathcal{Q}_{x \rightarrow y}^d = P_{x_1} \oplus \tilde{P}_{x_2} \quad (3)$$

This implies that :

$$\mathcal{Q}_{x \rightarrow y}^d(e^{-\alpha X_t}) = P_{x_1}(e^{-\alpha X_t}) \tilde{P}_{x_2}(e^{-\alpha X_t})$$

In particular, we have :

$$\begin{cases} P_{x_1+x_2}(e^{-\alpha X_t}) \tilde{P}_0(e^{-\alpha X_t}) = P_{x_1}(e^{-\alpha X_t}) \tilde{P}_{x_2}(e^{-\alpha X_t}) \\ P_0(e^{-\alpha X_t}) \tilde{P}_{x_2}(e^{-\alpha X_t}) = P_{x_2}(e^{-\alpha X_t}) \tilde{P}_0(e^{-\alpha X_t}) \end{cases} \quad (4)$$

which leads to : $P_{x_1+x_2}(e^{-\alpha X_t}) \tilde{P}_0(e^{-\alpha X_t}) = P_{x_1}(e^{-\alpha X_t}) \tilde{P}_{x_2}(e^{-\alpha X_t})$

Consequently :

$$P_x(e^{-\alpha X_t}) = P_0(e^{-\alpha X_t}) e^{bx}$$

and similarly :

$$\tilde{P}_x(e^{-\alpha X_t}) = \tilde{P}_0(e^{-\alpha X_t}) e^{\tilde{b}x}$$

The second equation of (4) gives : $b = \tilde{b}$, and we note that :

$$\mathcal{Q}_{0 \rightarrow y}^d(e^{-\alpha X_t}) = P_0(e^{-\alpha X_t}) \tilde{P}_0(e^{-\alpha X_t})$$

Hence, going back to our assumption (3), we obtain :

$$\mathcal{Q}_{x \rightarrow y}^d(e^{-\alpha X_t}) = \mathcal{Q}_{0 \rightarrow y}^d(e^{-\alpha X_t}) e^{bx}$$

Thanks to (2), this equation becomes :

$$\mathcal{Q}_{0 \rightarrow y}^d(e^{-\alpha X_t}) e^{bx} = A_0(t, \alpha)^x A_0(1-t, \alpha)^y B_0(t, \alpha)^2 I_\nu(\sqrt{xy} B_0(t, \alpha)^2) / I_\nu(\sqrt{xy})$$

By time reversal, we note that : $\mathcal{Q}_{0 \rightarrow y}^d(e^{-\alpha X_t}) = \mathcal{Q}_{y \rightarrow 0}^d(e^{-\alpha X_{1-t}})$.

We set then : $\beta = b - \text{Log} A_0(t, \alpha)$, to finally obtain :

$$e^{\beta x} = [B_0(t, \alpha)]^{2-d} I_\nu(\sqrt{xy} [B_0(t, \alpha)]^2) / I_\nu(\sqrt{xy}) \quad (5)$$

for any $x \in \mathbb{R}_+$.

Now , we use another result of Pitman and Yor [P-Y]. They define, for every $\nu > -1$ and $z > 0$, the Bessel (ν, z) distribution on $\mathbb{N}$, $b_{\nu, z}$, by :

$$b_{\nu, z}(n) = \left(\frac{z}{2}\right)^{2n+\nu} \frac{1}{n! \Gamma(n+\nu+1) I_\nu(z)}$$

They established that its generating function is :

$$\sum_{n=0}^{\infty} b_{\nu, z}(n) x^n = x^{-\nu/2} \frac{I_\nu(z\sqrt{x})}{I_\nu(z)}$$

and they noticed that this distribution is not infinitely divisible.

Let Y be a Bessel (ν, z) random variable and set : $B = [B_0(t, \alpha)]^4$. We write (5) under the following form :

$$\mathbb{E}[B^Y] = e^{\beta z^2}$$

for any $z > 0$. Hence for every $p \in \mathbb{N}^*$, we have :

$$\mathbb{E}[B^Y] = (e^{\beta \frac{z^2}{p}})^p = \mathbb{E}[B^{(Y_1+Y_2+\dots+Y_p)}]$$

where $Y_1, Y_2, \dots, Y_p$ are independent variables, Bessel $(\nu, \frac{z}{\sqrt{p}})$ distributed.

Note also that for a fixed t in $(0, 1)$, $B_0(t, \alpha)$ is a continuous decreasing function of α such that : $B_0(t, 0) = 1$ and $\lim_{\alpha \rightarrow \infty} B_0(t, \alpha) = 0$.

Consequently, if $\{\mathcal{Q}_{x \rightarrow y}^d, x \in \mathbb{R}_+\}$ were "2-decomposable" then the Bessel $(\nu, \sqrt{xy})$ distribution would be infinitely divisible, which is absurd . $\square$

The above proof of Theorem 1.2 does not allow to conclude that, for a fixed $t > 0$, the law of X_t under $\mathcal{Q}_{x \rightarrow y}^d$ is not infinitely divisible. This last simple question remains open.

References

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