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Séminaire de probabilités (Strasbourg), tome 35 (2001), p. 421-424
<http://www.numdam.org/item?id=SPS_2001__35__421_0>

ARE SQUARED BESSEL BRIDGES INFINITELY DIVISIBLE ?

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Abstract: Consider a squared Bessel bridge between two positive values x and y. If x or y is equal to 0, then this process is infinitely divisible. In the case when both x and y are strictly positive, Pitman and Yor conjectured in [P-Y] that the process is not infinitely divisible. We show here that it is not infinitely decomposable in the sense of Shiga and Watanabe [S-W].

1 - Introduction

Let $\mathcal{C}$ be the canonical space $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ and $\mathcal{F}$ be the σ-field, $\sigma\{\omega \to \omega(s) = X_s(\omega); s \geq 0\}$. For $d \geq 0$ and $x \geq 0$, let $\mathbb{Q}_x^d$ be the distribution on $(\mathcal{C}, \mathcal{F})$ of the square of a Bessel process with dimension d starting from $\sqrt{x}$. In [S-W], Shiga and Watanabe have established the following important additivity property:

$$\mathbb{Q}_x^d \;\oplus\; \mathbb{Q}_{x'}^{d'} \;=\; \mathbb{Q}_{x+x'}^{d+d'} \tag{1}$$

where , for P and Q two probabilities on $(\mathcal{C}, \mathcal{F})$, $P \oplus Q$ denotes the distribution of $(X_t + Y_t, t \geq 0)$ with $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ two independent processes respectively P and Q distributed.
An immediate consequence of the above additivity property is that squared Bessel processes are infinitely divisible. Indeed, we have for any $n \in \mathbb{N}$:

$$\mathbb{Q}_x^d = \mathbb{Q}_{x/n}^{d/n} \oplus \mathbb{Q}_{x/n}^{d/n} \oplus \ldots \oplus \mathbb{Q}_{x/n}^{d/n}$$

But we also have the following stronger property :

$$\mathbb{Q}_x^d = \mathbb{Q}_{x_1}^{d/n} \oplus \mathbb{Q}_{x_2}^{d/n} \oplus \ldots \oplus \mathbb{Q}_{x_n}^{d/n}$$

for any sequence $(x_i)_{1 \leq i \leq n}$ such that : $\sum_{i=1}^n x_i = x$.
Shiga and Watanabe have introduced this last property as the property of infinite decomposability. More precisely, let $\mathbb{P} = \{\mathbb{P}_x, x \in \mathbb{R}_+\}$ be a system of probabilities on $(\mathcal{C}, \mathcal{F})$ such that :

- for any $B \in \mathcal{F}$, $\;x \longrightarrow \mathbb{P}_x(B)\;$ is measurable
- for every $x \in \mathbb{R}_+$, $\;\mathbb{P}_x(X(0) = x) = 1$.

We denote by $\mathcal{P}$ the set of such systems $\mathbb{P}$. Shiga and Watanabe set in [S-W] the definition below.

Definition 1.1 : *Let $\mathbb{P}$ be an element of $\mathcal{P}$. $\mathbb{P}$ is said to be infinitely decomposable if for any $n \in \mathbb{N}^*$, there exists $\mathbb{P}^{(n)} \in \mathcal{P}$ such that for any $x \in \mathbb{R}$:*

$$\mathbb{P}_x = \mathbb{P}^{(n)}_{x_1} \oplus \mathbb{P}^{(n)}_{x_2} \oplus ... \oplus \mathbb{P}^{(n)}_{x_n}$$

for any sequence $(x_i)_{1 \le i \le n}$ such that : $\sum_{i=1}^n x_i = x$.

The distribution under $\mathbb{Q}^d_x$ of $(X_s, 0 \le s \le t)$ given that $(X_t = y)$ has been clearly defined by Pitman and Yor in [P-Y] for $d, x, y, t \ge 0$. This distribution represents the law of the d-dimensional squared Bessel bridge from x to y over time t. Without loss of generality, we will choose $t = 1$ and write $\mathbb{Q}^d_{x \to y}$ for the law of the d-dimensional squared Bessel bridge from x to y over time 1.
Thanks to the additivity property (1) , we have :

$$\mathbb{Q}^d_{x \to 0} \oplus \mathbb{Q}^{d'}_{x' \to 0} = \mathbb{Q}^{d+d'}_{x+x' \to 0}$$

which gives immediately the infinite decomposability of $\{\mathbb{Q}^d_{x \to 0}, x \in \mathbb{R}_+\}$ for any $d \ge 0$.
We are going to prove that for $y > 0$, $\{\mathbb{Q}^d_{x \to y}, x \in \mathbb{R}_+\}$ is not infinitely decomposable. Actually, we will show that $\{\mathbb{Q}^d_{x \to y}, x \in \mathbb{R}_+\}$ is not even "2-decomposable". Namely, we have the following property :

Theorem 1.2 : *For $y > 0$ and $d \ge 0$, there is no couple $(\mathbb{P}, \tilde{\mathbb{P}})$ of $\mathcal{P} \times \mathcal{P}$ such that for any $(x, x_1, x_2) \in \mathbb{R}^3_+$ verifying : $x_1 + x_2 = x$, we have :*

$$\mathbb{Q}^d_{x \to y} = \mathbb{P}_{x_1} \oplus \tilde{\mathbb{P}}_{x_2}$$

Since infinite decomposability is a stronger property than infinite divisibility, Theorem 1.2 does not prove Pitman and Yor's conjecture. But it confirms the gap between the cases $y = 0$ and $y > 0$.
In Section 2, we prove Theorem 1.2. The argument is based on the results of Pitman and Yor in [P-Y].

2 - Proof

Pitman and Yor have established that for any $\alpha > 0$ and $t \in [0, 1]$:

$$\mathbb{Q}^d_{x \to y}(e^{-\alpha X_t}) = A_0(t,\alpha)^x A_0(1-t,\alpha)^y B_0(t,\alpha)^2 I_\nu(\sqrt{xy} B_0(t,\alpha)^2)/I_\nu(\sqrt{xy}) \quad (2)$$

where $\nu = \frac{d}{2} - 1$, I_ν is the Bessel function of index ν, $A_0(t,\alpha)$ and $B_0(t,\alpha)$ are the constants determined by the equality :

$$\mathbb{Q}^d_{x \to 0}(e^{-\alpha X_t}) = A_0(t,\alpha)^x B_0(t,\alpha)^d$$

Our argument does not require the precise expression of these constants, but we note that they are computable.

Now $y > 0$ is fixed and we assume that there exists a couple $(\mathbb{P}, \tilde{\mathbb{P}})$ of elements of $\mathcal{P}$ such that for any $(x, x_1, x_2) \in \mathbb{R}_+^3$ verifying : $x_1 + x_2 = x$ we have :

$$\mathbb{Q}^d_{x\to y} = \mathbb{P}_{x_1} \oplus \tilde{\mathbb{P}}_{x_2} \tag{3}$$

This implies that :

$$\mathbb{Q}^d_{x\to y}(e^{-\alpha X_t}) = \mathbb{P}_{x_1}(e^{-\alpha X_t})\, \tilde{\mathbb{P}}_{x_2}(e^{-\alpha X_t})$$

In particular, we have :

$$\begin{cases} \mathbb{P}_{x_1+x_2}(e^{-\alpha X_t})\, \tilde{\mathbb{P}}_0(e^{-\alpha X_t}) = \mathbb{P}_{x_1}(e^{-\alpha X_t})\, \tilde{\mathbb{P}}_{x_2}(e^{-\alpha X_t}) \\ \mathbb{P}_0(e^{-\alpha X_t})\, \tilde{\mathbb{P}}_{x_2}(e^{-\alpha X_t}) = \mathbb{P}_{x_2}(e^{-\alpha X_t})\, \tilde{\mathbb{P}}_0(e^{-\alpha X_t}) \end{cases} \tag{4}$$

which leads to : $\mathbb{P}_{x_1+x_2}(e^{-\alpha X_t})\mathbb{P}_0(e^{-\alpha X_t}) = \mathbb{P}_{x_1}(e^{-\alpha X_t})\mathbb{P}_{x_2}(e^{-\alpha X_t})$
Consequently :

$$\mathbb{P}_x(e^{-\alpha X_t}) = \mathbb{P}_0(e^{-\alpha X_t})e^{bx}$$

and similarly :

$$\tilde{\mathbb{P}}_x(e^{-\alpha X_t}) = \tilde{\mathbb{P}}_0(e^{-\alpha X_t})e^{\tilde{b}x}$$

The second equation of (4) gives : $b = \tilde{b}$, and we note that :

$$\mathbb{Q}^d_{0\to y}(e^{-\alpha X_t}) = \mathbb{P}_0(e^{-\alpha X_t})\, \tilde{\mathbb{P}}_0(e^{-\alpha X_t})$$

Hence, going back to our assumption (3), we obtain :

$$\mathbb{Q}^d_{x\to y}(e^{-\alpha X_t}) = \mathbb{Q}^d_{0\to y}(e^{-\alpha X_t})e^{bx}$$

Thanks to (2), this equation becomes :

$$\mathbb{Q}^d_{0\to y}(e^{-\alpha X_t})e^{bx} = A_0(t,\alpha)^x A_0(1-t,\alpha)^y B_0(t,\alpha)^2 I_\nu(\sqrt{xy}B_0(t,\alpha)^2)/I_\nu(\sqrt{xy})$$

By time reversal, we note that : $\mathbb{Q}^d_{0\to y}(e^{-\alpha X_t}) = \mathbb{Q}^d_{y\to 0}(e^{-\alpha X_{1-t}})$.
We set then : $\beta = b - \mathrm{Log} A_0(t,\alpha)$, to finally obtain :

$$e^{\beta x} = [B_0(t,\alpha)]^{2-d} I_\nu(\sqrt{xy}[B_0(t,\alpha)]^2)/I_\nu(\sqrt{xy}) \tag{5}$$

for any $x \in \mathbb{R}_+$.
Now , we use another result of Pitman and Yor [P-Y]. They define, for every $\nu > -1$ and $z > 0$, the Bessel (ν, z) distribution on $\mathbb{N}$, $b_{\nu,z}$, by :

$$b_{\nu,z}(n) = (\frac{z}{2})^{2n+\nu}\frac{1}{n!\Gamma(n+\nu+1)I_\nu(z)}$$

They established that its generating function is :

$$\sum_{n=0}^{\infty} b_{\nu,z}(n)x^n = x^{-\nu/2}\frac{I_\nu(z\sqrt{x})}{I_\nu(z)}$$

and they noticed that this distribution is not infinitely divisible.

Let Y be a Bessel (ν, z) random variable and set : $B = [B_0(t,\alpha)]^4$. We write (5) under the following form :

$$\mathbb{E}[B^Y] = e^{\beta z^2}$$

for any $z > 0$. Hence for every $p \in \mathbb{N}^*$, we have :

$$\mathbb{E}[B^Y] = (e^{\beta \frac{z^2}{p}})^p = \mathbb{E}[B^{(Y_1+Y_2+\ldots+Y_p)}]$$

where $Y_1, Y_2, \ldots, Y_p$ are independent variables, Bessel $(\nu, \frac{z}{\sqrt{p}})$ distributed.
Note also that for a fixed t in $(0,1)$, $B_0(t,\alpha)$ is a continuous decreasing function of α such that : $B_0(t,0) = 1$ and $\lim_{\alpha\to\infty} B_0(t,\alpha) = 0$.
Consequently, if $\{\mathbb{Q}^d_{x\to y}, x \in \mathbb{R}_+\}$ were "2-decomposable" then the Bessel $(\nu, \sqrt{xy})$ distribution would be infinitely divisible, which is absurd . □

The above proof of Theorem 1.2 does not allow to conclude that, for a fixed $t > 0$, the law of X_t under $\mathbb{Q}^d_{x\to y}$ is not infinitely divisible. This last simple question remains open.

References

[P-Y] **Pitman J. and Yor M.** : A decomposition of Bessel bridges. *Z. Wahrscheinlichkeitstheorie verw. Gebiete 59,425-457 (1982).*

[S-W] **Shiga T. and Watanabe S.** : Bessel diffusions as a one-parameter family of diffusion processes.*Z. Wahrscheinlichkeitstheorie verw. Gebiete 27,37-46 (1973).*