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1 Introduction

Let S be a real-valued random walk starting at 0 and denote by $(T_n)_{n\in N}$ the ladder times of S, i.e. the successive times at which S reaches a new maximum: $T_0 = 0$ and $T_{n+1} = \inf\{k > T_n, S_k > S_{T_n}\}$. Denote by H the ladder heights defined as usual by $H_n = S_{T_n}$. One of the various formulations of the Wiener-Hopf factorization in this setting is the celebrated Spitzer-Baxter identity [3]:

$$\left(1 - E(r^{T_1}e^{-\mu H_1})\right)^{-1} = \exp\left(\sum_{m=1}^{\infty}\frac{r^m}{m}E(e^{\mu S_m},\ S_m > 0)\right) \qquad (1)$$

for $0 < r < 1$ and $\mu > 0$. An equivalent form is the following:

$$\frac{1}{n}P(S_n \in dx) = \sum_{k=1}^{\infty}\frac{1}{k}P(T_k = n,\ H_k \in dx), \quad x > 0$$

Alili and Doney [1] recently established the following refinement of this formula:

$$nP(T_k = n, S_n \in dx) = kP(H_{k-1} < S_n \leq H_k, S_n \in dx) \qquad (2)$$

for all integers $k, n > 0$ and every real $x > 0$.

This can also be stated in terms of transforms. Set $N_x = \min\{k,\ H_k \geq x\}$, then for all real $r, \alpha \in [0, 1)$ and $\theta > 0$,

$$\sum_{n=1}^{\infty}\frac{r^n}{n}E\left(e^{-\theta S_n}\alpha^{N_{S_n}},\ S_n > 0\right) = -\log(1 - \alpha E(r^{T_1}e^{-\theta H_1}))$$

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Taking $\alpha = 1$, we recover (1). Two proofs of this result are given in [1]. One is analytical, the other uses a path transform together with a well-known lemma by Feller ([2], p. 412) involving all the possible cyclical permutations of the steps of the random walk up to time n. Alili and Doney derive from (2) several results on the asymptotic behaviour of the ladder variables, in the case when $E(S_1) = 0$ and $\mathrm{var}(S_1) < \infty$.

The aim of this note is to prove (2) by means of another path transform, avoiding the use of Feller's lemma.

2 Proof

2.1 Two classes of paths

Fix $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Conditioning on the steps X_i, $(1 \leq i \leq n)$ of S up to time n, with $\sum_{i=1}^{n} X_i = x$, we consider all the possible n-step paths on $\mathbb{R}$ whose increments are the X_i's, possibly in a different order. We shall use for these deterministic paths the same notations T_i, H_i as above, and an n-step excursion of the reflected process will refer to a path such that $T_1 = n$.

To prove (2), it suffices to establish a one-to-one correspondance between two classes of such paths.

- The paths satisfying $T_k = n$, with a distinguished time $a \in \{0, 1, \dots n-1\}$.
- The paths such that $T_k \leq n$, $H_{k-1} < x \leq H_k$, with a distinguished ladder time T_b, $b \in \{0, 1, \dots k-1\}$

The one-to-one correspondance is obtained by cutting each path into 5 parts.

For the first type, put $i = \sup\{j,\ T_j \leq a\}$. Suppose $T_i \neq a$. Let a' be the first time at which the path reaches its overall maximum between a and $T_{i+1} - 1$. Then the five parts are:

- (1) from 0 to T_i
- (2) from T_i to a
- (3) from a to a'
- (4) from a' to T_{i+1}
- (5) from T_{i+1} to $T_k = n$

Particular cases If $T_i = a$ then parts (2) and (3) are suppressed. If $a = a'$ then part (4) is suppressed. If $T_{i+1} = T_k$ then part (5) is suppressed. If $k = 1$ then parts (1) and (5) are suppressed. It is a good exercise to check that no other particular case can occur.

For the second type, let b' be the first time of the overall supremum between 0 and n. The five parts are:

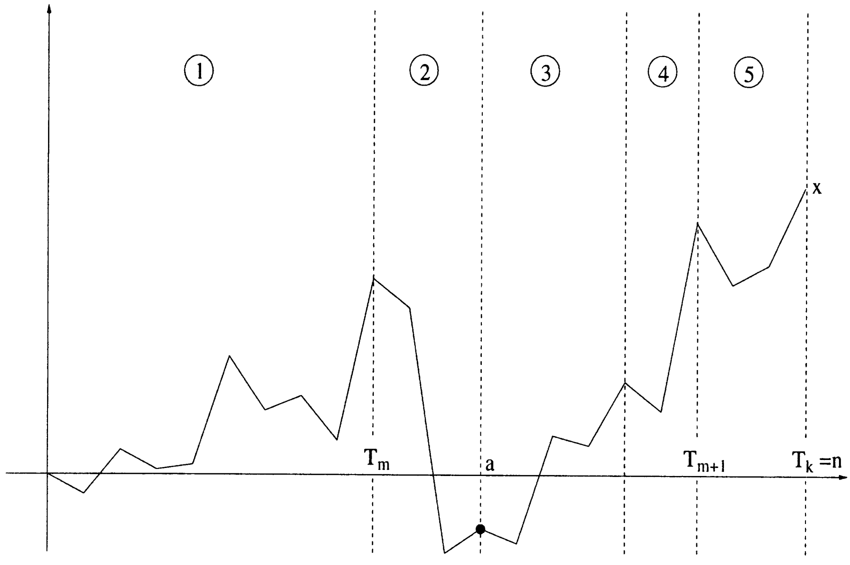

Figure 1: a path of the first type

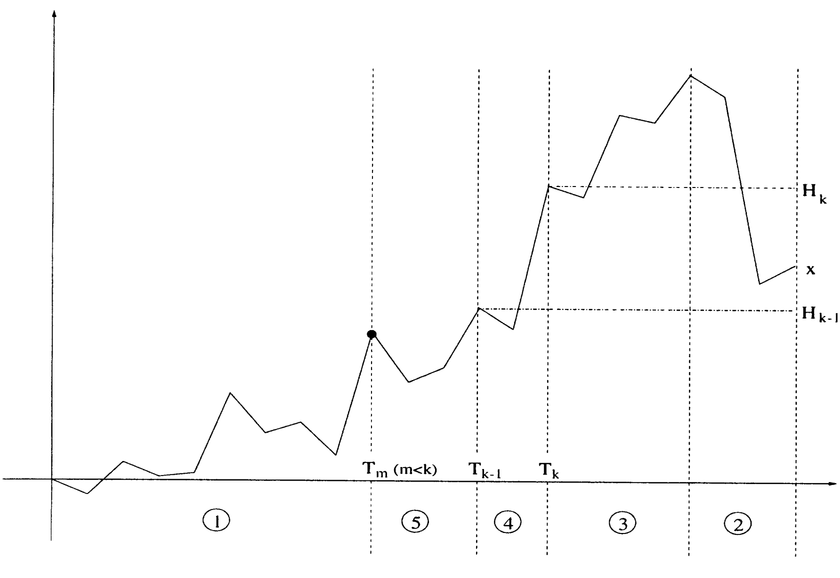

Figure 2: its transform

- (1') from 0 to T_b
- (2') from T_b to T_{k-1}
- (3') from T_{k-1} to T_k
- (4') from T_k to b'
- (5') from b' to n

Particular cases If $T_k = n$ then parts (4') and (5') are suppressed. If $T_b = T_{k-1}$ then part (2') is suppressed. If $b' = T_k$ then part (3') is suppressed. If $k = 1$ then parts (1') and (2') are suppressed.

2.2 The path transform

Let P_1 (resp P_2) be a the set of paths of the first (resp. second) type. Then (2) is a direct corollary of the following proposition:

Proposition 1 *The transformation obtained by cutting a path of P_1 into five parts as above, concatenating successively (1), (5), (4), (3) and (2), and distinguishing the time T_i, establishes a one-to-one correspondance between P_1 and P_2.*

The inverse transformation is obtained by dividing a path of P_2 into five parts, concatenating successively (1'), (5'), (4'), (3') and (2'), and distinguishing the time $T_b + (n - b')$.

Proof

We begin by proving that the first path transform effectively constructs a path of the second type. We check that the two paths have the same steps, although in a different order, and that the distinguished time T_i is a ladder time with $0 \leq i \leq k-1$. The new path has i ladder times in part (1) and $k - (i + 1)$ in part (5), so that the $(k-1)$-th ladder time is at the end of part (5). Furthermore part (4) is by construction an excursion of the reflected process, thus the k-th ladder time of the new path is at the end of part (4).

Put $h = S_a - S_{T_i}$, $h' = S'_a - S_a$, $h'' = S_{T_{i+1}} - S_{a'}$, S refering for the initial path. Then since the concatenation of (2), (3) and (4) is an excursion of the reflected process, the following inequalities hold: $h + h' + h'' > 0$ and $h + h' \leq 0$. Denote by y the value of the new path at the end of part (5), z the value of the new path at the end of part (3), and recall that the final value is x. Then the two inequalities above entail $y < x \leq z$, which completes this part of the proof.

Next we show that we get by the second transform a path of the first type. Indeed it suffices to show that the path E obtained by concatenating (5'), (4') and (3') is an excursion of the reflected process, because then the concatenation of (1'), E and (2') will have exactly k ladder times, the last being at time n.

Put $i = S_{T_k} - S_{T_{k-1}}$, $i' = S_b - S_{T_k}$, $i'' = S_n - S_b$, S refering for the initial path. Then by definition of a path of the second type the following inequalities hold : $i + i' + i'' \geq 0$, $i' + i'' < 0$. We deduce from the second inequality that the path obtained by concatenating (5') and (4') is always negative, and from the first inequality that E is a path whose last value is positive. But since part (3') is an excursion of the reflected process and E has a negative value at the end of (4'), it cannot reach a positive value before its endpoint. Hence E is an excursion of the reflected process.

The proof that the two path transforms are inverse of each other is left to the reader, as well as the study of the particular cases. See figures 1 and 2 for the first path transform.

Remark. Suppose that the random walk has integer increments and is right-continuous, that is, the only possible positive steps have size 1. Then $S_n = S_{T_k}$ for a path of the second type. Consequently the part of such a path between T_k and n (i.e. its restriction to (4') and (5')) is a bridge of the random walk. The exchange of the pre- and post-maximum parts, obtained by putting (5') before (4') and constructing by this manner a negative bridge, is exactly Vervaat's transform.

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References

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