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Séminaire de probabilités (Strasbourg), tome 35 (2001), p. 334-347
<http://www.numdam.org/item?id=SPS_2001__35__334_0>

Some consequences of the cyclic exchangeability property for exponential functionals of Lévy processes.

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Abstract : In this paper we derive some distributional properties of Lévy processes and bridges from their cyclic exchangeability property. We first describe the σ-field which is invariant under the cyclic transformations. Then, by conditioning on this σ-field, we obtain some information about the laws of many functionals of Lévy processes and bridges, such as exponential functionals, quantiles and local time.

Key words: Cyclic exchangeability, Lévy and Brownian bridges, exponential functionals.

A.M.S. Classification: 60 J 30, 60 J 20.

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1 Introduction

The original motivation of the present work was to give a simple explanation of the following property satisfied by the Brownian bridge $(b_u,\, 0 \le u \le 1)$:

$$\textit{for every } \alpha \in \mathbb{R}, \quad \mathbb{E}\left[\left(\int_0^1 du\,(\exp(\alpha b_u)\right)^{-1}\right] = 1\,. \tag{1.1}$$

Throughout the sequel, we will use the notation, $I_\alpha \stackrel{\text{def}}{=} \int_0^1 du \, \exp(\alpha b_u)$. The puzzling fact that the expectation of the variable I_α^{-1} does not depend on α has been noticed in [10], and the same authors explained this fact in [11] using an elementary decomposition of Brownian bridges.

One of our purposes is to show that the identity (1.1) is actually a direct consequence of the cyclic exchangeability property of the Brownian bridge. Therefore (1.1) is still valid for any process satisfying this property, including in particular, Lévy bridges.

For every $u \in [0,1]$, we denote by Θ_u the transformation which consists in splitting the path of b at time u, and then re-ordering the two parts so obtained:

$$\Theta_u(b) \stackrel{\text{def}}{=} (b_{\{t+u\}} - b_u, \, 0 \le t \le 1) \,, \tag{1.2}$$

where, here and in the sequel, $\{x\} \stackrel{\text{def}}{=} x \ (mod\ 1)$. It is well known and elementary to prove that:

$$\Theta_u(b) \stackrel{(d)}{=} b \,. \tag{1.3}$$

In section 4, we show that the identity in law (1.3) applied at any time u in [0,1], replacing b by any Lévy bridge $X^{(b)}$, leads straightforwardly to the identity (1.1). More precisely, by focussing on the description of the invariant σ-field associated with the transformations Θ_u, $u \in [0,1]$, we obtain some nice properties of functionals such as: $\int_0^1 du \, \exp(\alpha X_u^{(b)})$, the quantiles, (see (4.12) below for their definition), and the local time process of $X^{(b)}$. In section 3, we recall the necessary and sufficient conditions under which Lévy bridges may be defined and admit local times.

In section 4, we also establish some similar results for any process on [0,1], with cyclic exchangeable increments. Indeed, in [10] and [11], the following identity for Brownian bridge was also obtained,

$$\text{for every } \alpha, y \in \mathbb{R}^2, \quad \mathbb{E}\left[\left(\int_0^1 du \, (\exp(\alpha b_u + yu)\right)^{-1}\right] = \frac{y}{e^y - 1} \,. \tag{1.4}$$

If B is the standard Brownian motion, then by writing the left hand side of (1.4) as

$$\mathbb{E}\left[\left(\int_0^1 du \, (\exp(\alpha(B_u - uB_1 + u\frac{B_1}{\alpha}))\right)^{-1} \,|\, B_1 = y\right] \,,$$

we guessed that the arguments used for the Brownian bridge had an adequate extension for Brownian motion, and so, for any process with cyclic exchangeable increments.

Apart from giving a simple derivation of (1.1) and (1.4), one of the aims of this paper is to gather a number of already known consequences of the cyclic exchangeability property. These are found in the following section.

2 A characterization of processes with cyclic exchangeable increments.

Let $\mathcal{D}([0,1])$ be the space of càdlàg functions on $[0,1]$ endowed with the Skohorod topology and the associated Borel σ-field $\mathcal{F}$. Define the transformations Θ_u, $u \in [0,1]$, acting on any function f of $\mathcal{D}([0,1])$ by

$$\Theta_u(f)_t \stackrel{\text{def}}{=} \begin{cases} f(0) + f(t+u) - f(u), & \text{if } t < 1-u \\ f(t-(1-u)) + f(1) - f(u) & \text{if } 1-u \le t \le 1 \,. \end{cases} \tag{2.1}$$

This transformation consists in splitting the path f at time u and then in re-ordering the two parts so obtained, so that $\Theta_u(f)(0) = f(0)$ and $\Theta_u(f)(1) = f(1)$. A real valued stochastic process X defined on the space $\mathcal{D}([0,1])$, with law $\mathbb{P}$ is said to be a CEI process (i.e. a stochastic process with cyclic exchangeable increments) if

$$\textit{for every } u \in [0,1], \quad \Theta_u(X) \stackrel{(d)}{=} X\,, \quad \mathbb{P} - a.s. \tag{2.2}$$

In particular, any Lévy process on the time interval [0,1] is a CEI process (see the following section). But more generally, for any stochastic process Y defined on $\mathcal{D}([0,1])$, if $\mathcal{U}$ is a uniformly distributed random variable on [0,1], independent of Y, then the process $\Theta_{\mathcal{U}}(Y)$ is CEI.

The family of transformations $(\Theta_u : u \in [0,1])$ is a group for the composition product and

$$\textit{for every } u, v \in [0,1], \quad \Theta_u \Theta_v = \Theta_{\{u+v\}} \,. \tag{2.3}$$

The cyclic exchangeability property is weaker than the exchangeability property discussed by Kallenberg [14] who also defines a group of transformations. In his paper, Kallenberg gives a generic decomposition of exchangeable processes on $[0,1]$, whereas Aldous [1], p.104 raises the following question (in discreteu time): *given a sub-group of transformations, characterize the processes whose law is invariant by this subgroup.* In this section we give an answer to the above question for cyclic exchange in continuous time. We point out that the following results can easily be adapted to discrete time.

Let $\mathcal{I}$ be the σ-field, invariant by the transformations $(\Theta_u : u \in [0,1])$, that is:

$$\textit{for every } u \in [0,1], \quad \Gamma \in \mathcal{I}, \quad \textit{if and only if} \quad \mathbb{I}_\Gamma = \mathbb{I}_\Gamma \circ \Theta_u, \quad \mathbb{P} - a.s. \tag{2.4}$$

The law $\mathbb{P}$ of a CEI process being given on the space $\mathcal{D}([0,1])$, we first state some crucial properties of the σ-field $\mathcal{I}$.

Proposition 1 *The following statements are equivalent:*
(i) $F \in L^1(\mathbb{P})$ *is* $\mathcal{I}$*-measurable.*
(ii)

$$F = \int_0^1 du\, F \circ \Theta_u\,, \quad \mathbb{P} - a.s. \tag{2.5}$$

(iii)

$$\mathbb{P} - a.s., \quad F = F \circ \Theta_u, \quad \textit{for every } u \in [0,1]. \tag{2.6}$$

(iv) There exists $u \in [0,1] \setminus \mathbb{Q}$ *such that*

$$\mathbb{P} - a.s., \quad F = F \circ \Theta_u\,. \tag{2.7}$$

Consequently, for any $F \in L^1(\mathbb{P})$

$$\mathbb{E}[F \,|\, \mathcal{I}] = \int_0^1 du F \circ \Theta_u\,, \qquad \mathbb{P}\text{-}a.s. \tag{2.8}$$

Proof The equivalences between (i), (ii) and (iii) follow from (2.3) and (2.4) and are not hard to check. It also is obvious that (i) implies (iv).

For $s \in [0,1]$, we denote by $\mathcal{I}_s$ the σ-field invariant by the transformation Θ_s, that is,

$$\Gamma \in \mathcal{I}_s, \quad \textit{if and only if} \quad \mathbb{I}_\Gamma = \mathbb{I}_\Gamma \circ \Theta_s, \quad \mathbb{P} - a.s. \tag{2.9}$$

To prove that (iv) implies (i), first observe that from (2.3), we have: $\mathcal{I}_u \subset \cap_{n \in \mathbb{N}} \mathcal{I}_{\{nu\}}$. Now, let F be a continuous functional on $\mathcal{C}([0,1])$, which is $\cap_{n\in\mathbb{N}}\mathcal{I}_{\{nu\}}$-measurable and let $v \in [0,1]$. Since the map $s \mapsto \Theta_s$ is continuous on $\mathcal{C}([0,1])$, and the set $(\{nu\} : n \in \mathbb{N})$ is dense in $[0,1]$ then $F \circ \Theta_v$ can be obtained as the limit of a sequence $F \circ \Theta_{\{n_i u\}}$, $i \in \mathbb{N}$, so $F \circ \Theta_v = F$, a.s. It follows that $\cap_{n\in\mathbb{N}}\mathcal{I}_{\{nu\}} \subset \mathcal{I}_v$, and then $\cap_{n\in\mathbb{N}}\mathcal{I}_{\{nu\}} \subset \mathcal{I}$. □

Remark 1
1_ Note the different positions of '$\mathbb{P} - a.s.$*' in (2.4) and (2.6).*
2_ In Proposition 1 the equivalence between (iv) and the other statements can also be proved by using the well known fact that the transformation $x \mapsto \{x+u\}$, $(u \in [0,1]\setminus\mathbb{Q})$ *defined on the space* $[0,1]$, *endowed with the Lebesgue measure and the Borel* σ*-field, is ergodic.*
3_ The expression (2.8) is a particular case of (4.1) in [9], where the authors obtain a general expression for the conditional expectation given the invariant σ*-field by the action of a group on a group of transformations.*

We will now focus on a case for which the description of $\mathcal{I}$ is particularly explicit. Consider a CEI process X and construct from it the process

$$X_t^0 \stackrel{\text{def}}{=} X_t - X_0 - t(X_1 - X_0)\,, \quad t \geq 0\,, \tag{2.10}$$

which starts from 0 and return to it at time 1. One easily checks that X^0 is also a CEI process. We will denote by $\mathcal{I}^0$ the invariant σ-field defined in the sense of (2.4), relatively to X^0. In the sequel we restrict ourselves to the CEI processes such that X^0 almost surely reaches its minimum over [0,1] at a unique time m. That is

$$m \stackrel{\text{def}}{=} \inf\{t : X_t^0 \wedge X_{t-}^0 = \inf_{0\le s\le 1} X_s^0\} \tag{2.11}$$

is almost surely the only time such that $X_m^0 = \inf_{0\le s\le 1} X_s^0$.

The σ-fields $\mathcal{I}^0$ and $\mathcal{I}$ may be explicited as in the following theorem.

Theorem 1 *Let X be a CEI process and suppose that X^0, defined in* (2.10), *almost surely reaches its minimum over* $[0,1]$ *at a unique time m.*
(i) (M. Malric) *The invariant σ-field $\mathcal{I}^0$ associated to X^0 may be described as follows:*

$$\mathcal{I}^0 = \sigma\{\Theta_m(X^0)\}\,, \tag{2.12}$$

where $\sigma\{\Theta_m(X^0)\}$ is the σ-field generated by the process $\Theta_m(X^0)$. Moreover, any $\mathcal{I}^0$-invariant functional F is such that $F(X^0) = F(\Theta_m(X^0))$.
(ii) The invariant σ-field $\mathcal{I}$ is generated by $\mathcal{I}^0$, X_0 and X_1, that is:

$$\mathcal{I} = \mathcal{I}^0 \vee \sigma\{X_0, X_1\}\,, \tag{2.13}$$

where $\sigma\{X_0, X_1\}$ is the σ-field generated by the couple of variables (X_0, X_1).
(iii) The random time m is uniformly distributed and independent of $\mathcal{I}$.

Proof (i) Let F be $\mathcal{I}^0$-measurable. From (2.6) we have $F(X^0) = F\circ\Theta_T(X^0)$, $\mathbb{P}$-a.s., for every random time T distributed in [0,1]. This relation is true in particular for the time m, which proves that $F(X^0) = F(\Theta_m(X^0))$, and then $\mathcal{I}^0 \subset \sigma\{\Theta_m(X^0)\}$.

Now observe that (2.3) extends to any random time T distributed in [0,1], as follows,

$$\Theta_T\circ\Theta_u = \Theta_{\{T\circ\Theta_u+u\}}\,, \qquad \mathbb{P}-a.s., \text{ for every } u\in[0,1].$$

But a particular property of the time m, which is the crucial point of the proof, is that, for every $u\in[0,1]$,

$$\{m\circ\Theta_u + u\} = m\,, \qquad \mathbb{P}-a.s. \tag{2.14}$$

The above identity is easy to check, once we notice that $\Theta_u(X^0) = \Theta_u(X)^0$, for any $u\in[0,1]$. So, we obtain,

$$\Theta_m\circ\Theta_u(X^0) = \Theta_m(X^0)\,, \qquad \mathbb{P}-a.s., \text{ for every } u\in[0,1].$$

This proves that Θ_m is an $\mathcal{I}^0$-measurable functional and $\sigma\{\Theta_m(X^0)\}\subset\mathcal{I}^0$.
(ii) It suffices to observe that $\sigma\{X\} = \sigma\{X^0\}\vee\sigma\{X_0, X_1\}$ and $\sigma\{X_0, X_1\}\subset\mathcal{I}$.

(iii) Applying Proposition 1 and using (2.14), we get that for every bounded measurable functional F,

$$\begin{aligned} \mathbb{E}[F(m)\,|\,\mathcal{I}] &= \int_0^1 du\, F(m\circ\Theta_u) \\ &= \int_0^m du\, F(m-u) + \int_m^1 du\, F(1+m-u) \\ &= \int_0^1 ds\, F(s)\,. \qquad \square \end{aligned}$$

Remark 2 *When X^0 is the Brownian bridge, Θ_m applied to X^0 is nothing but the well known Vervaat's transformation [18]. The process $\Theta_m(X^0)$ is then a normalized Brownian excursion. In that case, it was proved by Biane [5] that the latter is independent of time m.*

We end this section with an application of Proposition 1 to the time A^0 that a CEI process X such that $X_0 = X_1 = 0$ spends below level 0, that is

$$A^0 \overset{\text{def}}{=} \int_0^1 du\, \mathbb{I}_{\{X_u \le 0\}}\,. \tag{2.15}$$

It has been proved by Knight [15] that for a process X with exchangeable increments, the time A^0 is uniformly distributed over [0,1] if and only if the sojourn function of X is continuous. We also know from [7], that moreover A^0 is independent of the invariant σ-field $\mathcal{I}$, (see also [5] for the brownian case). Here, we show that these results are also properties of CEI processes and are straightforward consequences of Proposition 1.

Proposition 2 *Let X be a CEI process such that $X_0 = X_1 = 0$ and introduce its sojourn function $F(x) = \int_0^1 du \mathbb{I}_{\{X_u \le x\}}$, $x \in \mathbb{R}$.*

The random time A^0 is uniformly distributed over $[0,1]$ if and only if F is continuous. In that case, A^0 is independent of $\mathcal{I}$.

Proof First recall the following crucial argument which we pick up in the proof of Theorem 5 in [7]: Let $\mathcal{U}$ be a uniformly distributed random variable over [0,1], independent of X, then it is easy to check that that F is the distribution function of the r.v. $X_{\mathcal{U}}$. It is well known that $F(X_{\mathcal{U}})$ is uniformly distributed if and only if F is continuous. Therefore, under this condition, $F(X_{\mathcal{U}})$ is independent of the whole process X and moreover, applying Proposition 1, we obtain that for every bounded measurable function G,

$$\begin{aligned} \mathbb{E}[G(A^0)\,|\,\mathcal{I}] &= \int_0^1 du\, G(A^0\circ\Theta_u) \\ &= \int_0^1 du\, G\left(\int_0^1 ds\, \mathbb{I}_{\{X_s\le X_u\}}\right) \\ &= \mathbb{E}[G\circ F(X_{\mathcal{U}})] \\ &= \int_0^1 dh\, G(h)\,, \end{aligned}$$

where the last identity holds if and only if F is continuous. $\square$

Remark 3 *If X is the Brownian bridge and more generally, any Lévy bridge which admits local times (see the next section for conditions under which it holds), then the identity*

$$\int_0^1 du\, G\left(\int_0^1 ds\, \mathbb{I}_{\{X_s \le X_u\}}\right) = \int_0^1 dh\, G(h)$$

can be proved by applying the occupation time density formula. We refer to ([17], Exercice 1.31 p. 237), and Nasyrov [16].

3 Lévy processes and bridges are CEI.

Any Lévy process (i.e. a process with stationary and independent increments) is CEI and the same is true for Lévy bridges whenever they can be defined. Therefore, all the results of the previous section are still valid for Lévy processes and bridges.

In this section, without any further assumption, $(X_t, 0 \le t \le 1)$ is a real Lévy process starting at 0. As we already noticed, the process $X^0 \stackrel{\text{def}}{=} (X_t - tX_1, 0 \le t \le 1)$ defined in (2.10) is CEI but its law is not equal in general, to the law of the Lévy bridge (see the definition (3.2) below). Actually, Brownian motion with drift times a constant is the only Lévy process such that the laws of X^0 and the bridge coincide, see [15], Theorem 2.2.

The definition of the Lévy bridge requires the following additional assumption:

(H1) *The probability transitions $\mathbb{P}(X_t \in dy \mid X_0 = x)$ are absolutely continuous with respect to the Lebesgue measure dy.*

We denote the above densities by $p_t(y-x)$, that is

$$\mathbb{P}(X_t \in dy \mid X_0 = x) \stackrel{\text{def}}{=} p_t(y-x)\, dy\,, \quad x, y \in \mathbb{R}. \tag{3.1}$$

Condition (H1) is equivalent to the fact that the semigroup of X fulfills the strong Feller property, (see for instance [3], Proposition 3). Suppose moreover that X is not a subordinator. Then Lemma 2.10 in [12] asserts that $p_t(0) > 0$ for all t and the Lévy bridge associated with X is the process $X^{(b)}$ with law defined by:

$$\mathbb{E}[F(X_u^{(b)}, 0 \le u \le t)] \stackrel{\text{def}}{=} \mathbb{E}\left[F(X_u, 0 \le u \le t)\frac{p_{1-t}(-X_t)}{p_1(0)}\right], \tag{3.2}$$

for every $t < 1$, and every bounded measurable functional F. One will find a general discussion about the definition of bridges for Markov processes in [13]. Intuitively, $X^{(b)}$ corresponds to the process X conditioned to return to 0 at time 1. There are many instances of Lévy processes where this last definition has a rigorous meaning. The case of stable Lévy processes is described in Bertoin [2], ch. VIII.3.

Some applications in the next section concern the local times of $X^{(b)}$. Its existence requires the following further assumption: When

(H2) 0 *is regular for itself for the Lévy process* X,

it makes sense to define the occupation local time of X. We refer to Bertoin [2], ch. II.5 for necessary and sufficient conditions under which (H2) holds, and to ch. V of the same book for the definition of the occupation local time. Suppose moreover that (H1) holds, then in a natural way, one may define the occupation local time of the bridge $X^{(b)}$. In the following, this local time will be considered only at time 1. We will denote this process by l^x, $x \in \mathbb{R}$, when no confusion is possible. It is formally defined by the occupation time density formula:

$$\int_0^1 f(X_s^{(b)})\, ds = \int_{\mathbb{R}} f(x) l^x\, dx\,, \tag{3.3}$$

where f is any bounded measurable function, see [2], ch. V. Note also that from [2], Proposition V.2, one has for every $x \in \mathbb{R}$,

$$l^x = \lim_{a\to 0} (2a)^{-1} \int_0^1 du\, \mathbb{I}_{\{|X_u^{(b)} - x| \le a\}}\,, \qquad \mathbb{P}\text{-a.s.} \tag{3.4}$$

4 Applications.

This section is mainly devoted to the study of the law of the functional

$$A_\alpha \stackrel{\text{def}}{=} \int_0^1 du\, \exp(\alpha X_u)\,, \tag{4.1}$$

conditionally on $\mathcal{I}$, for some CEI processes. With no loss of generallity and for convenience reason in the satement of the next results, we may suppose that $X_0 = 0$, a.s. First, note that without any particular assumption on the process X, we have for every $u \in [0, 1]$,

$$A_\alpha \circ \Theta_u = e^{-\alpha X_u} \left(A_\alpha + (e^{\alpha X_1} - 1) \int_0^u ds\, e^{\alpha X_s} \right)\,, \tag{4.2}$$

thus, from Proposition 1,

$$\begin{aligned} \mathbb{E}\left[A_\alpha^{-1} \,|\, \mathcal{I}\right] &= \int_0^1 du \frac{e^{\alpha X_u}}{A_\alpha + (e^{\alpha X_1} - 1) \int_0^u dh e^{\alpha X_h}} \\ &= \frac{\alpha X_1}{e^{\alpha X_1} - 1} \mathbb{I}_{(X_1 \neq 0)} + \mathbb{I}_{(X_1 = 0)}\,. \end{aligned} \tag{4.3}$$

This direct application of Proposition 1 yields a reinforcement of identities (1.1) and (1.4) mentioned in the introduction. In the next proposition, we extend formula (4.3) to obtain a recurrence formula bearing upon the conditional Mellin transform of A_α.

Theorem 2 *For every real ν,*

$$\mathbb{E}\left[A_{\nu\alpha}\,A_\alpha^{-(\nu+1)}\,|\,\mathcal{I}\right] = \left(\frac{e^{\nu\alpha X_1}-1}{\nu(e^{\alpha X_1}-1)}\mathbb{I}_{(X_1\neq 0)} + \mathbb{I}_{(X_1=0)}\right)\mathbb{E}[A_\alpha^{-\nu}\,|\,\mathcal{I}]\,. \tag{4.4}$$

More generally, for every bounded, measurable function φ,

$$\begin{aligned}&\mathbb{E}\left[A_\alpha^{-1}\int_0^1 du\,\varphi\left(\frac{e^{\alpha X_u}}{A_\alpha}\right)\,|\,\mathcal{I}\right]\\ = &\ \frac{\alpha}{(e^{\alpha X_1}-1)}\int_0^{X_1} dy\,\mathbb{E}\left[\varphi\left(\frac{e^{\alpha y}}{A_\alpha}\right)\,|\,\mathcal{I}\right]\mathbb{I}_{(X_1\neq 0)} + \mathbb{E}\left[\varphi\left(\frac{1}{A_\alpha}\right)\,|\,\mathcal{I}\right]\mathbb{I}_{(X_1=0)}\,.\end{aligned} \tag{4.5}$$

Proof. From (4.2), we have,

$$e^{-\nu\alpha X_u}A_\alpha^{-(\nu+1)}\circ\Theta_u = e^{\alpha X_u}\left(A_\alpha + (e^{\alpha X_1}-1)\int_0^u ds\,e^{\alpha X_s}\right)^{-(\nu+1)}.$$

Using the fact that $X_u = (X_1 - X_{1-u})\circ\Theta_u$, we obtain that for every bounded $\mathcal{I}$-measurable functional F,

$$\begin{aligned}&\mathbb{E}\left[F\,e^{-\nu\alpha(X_1-X_{1-u})}\,A_\alpha^{-(\nu+1)}\right]\\ = &\ \mathbb{E}\left[F\,e^{\alpha X_u}\left(A_\alpha + (e^{\alpha X_1}-1)\int_0^u ds\,e^{\alpha X_s}\right)^{-(\nu+1)}\right].\end{aligned}$$

Integrating with respect to u over [0,1], gives

$$\begin{aligned}&\mathbb{E}\left[F\,e^{-\nu\alpha X_1}A_{\nu\alpha}A_\alpha^{-(\nu+1)}\right]\\ = &\ \mathbb{E}\left[F\int_0^1 du\,e^{\alpha X_u}\left(A_\alpha + (e^{\alpha X_1}-1)\int_0^u ds\,e^{\alpha X_s}\right)^{-(\nu+1)}\right].\end{aligned}$$

But noticing that,

$$\begin{aligned}&\frac{d}{du}\left(A_\alpha + (e^{\alpha X_1}-1)\int_0^u ds\,e^{\alpha X_s}\right)^{-\nu}\\ = &\ \nu(e^{\alpha X_1}-1)e^{\alpha X_u}\left(A_\alpha + (e^{\alpha X_1}-1)\int_0^u ds\,e^{\alpha X_s}\right)^{-(\nu+1)},\end{aligned}$$

allows us to conclude,

$$\mathbb{E}\left[F\,e^{-\nu\alpha X_1}A_{\nu\alpha}\,A_\alpha^{-(\nu+1)}\right] = \mathbb{E}\left[F\left(\frac{1-e^{-\nu\alpha X_1}}{\nu(e^{\alpha X_1}-1)}\mathbb{I}_{(X_1\neq 0)} + \mathbb{I}_{(X_1=0)}\right)A_\alpha^{-\nu}\right].$$

The second statement follows from (4.4) by uniqueness of Mellin transforms. □

Throughout the rest of the paper we will deal with the Lévy bridge. So, suppose that the Lévy process X is not a subordinator and verifies hypotheses (H1) and (H2) of the previous section. From X, construct a bridge $X^{(b)}$ on the canonical space $\mathcal{D}([0,1])$, whose law is defined by (3.2). We will refer to $\mathcal{I}$ as the invariant σ-field defined in (2.4) with respect to $X^{(b)}$. Let also $\mathcal{L} = \sigma\,(l^x\,, x \in \mathbb{R})$ be the σ-field generated by the local time process defined in (3.3) and (3.4). This σ-field is equivalently generated by the variables $F = \int_0^1 dh f(X_h^{(b)})$, where the functions f are in $L^1(\mathbb{R}, dx)$; then, for every $u \in [0,1]$, $F \circ \Theta_u = \int_0^1 dh\, f(X_h^{(b)} - X_u^{(b)})$, so the variables,

$$\int_0^1 du\, F \circ \Theta_u = \int_0^1 du \int_0^1 dh\, f(X_h^{(b)} - X_u^{(b)}) \tag{4.6}$$

are $\mathcal{I}$-measurable. These variables are actually $\mathcal{L} \cap \mathcal{I}$-measurable. Indeed, by the occupation time density formula,

$$\int_0^1 du \int_0^1 dh\, f(X_h^{(b)} - X_u^{(b)}) = \int_{-\infty}^{\infty} dx\, f(x) \int_{-\infty}^{\infty} dy\; l^y\, l^{x+y}\,.$$

This last equality actually defines the intersection local time $(\gamma^x\,, x \in \mathbb{R})$, and can be expressed as :

$$\int_0^1 du \int_0^1 dh\, f(X_h^{(b)} - X_u^{(b)}) = \int_{-\infty}^{\infty} dx\, f(x)\, \gamma^x\,,$$

so that we can state

Proposition 3 *For every $x \in \mathbb{R}$,*

$$\mathbb{E}[\, l^x \,|\, \mathcal{I}\,] = \gamma^x\,.$$

Among the $\mathcal{L}$-measurable functionals, we consider some of particular interest, e.g: those whose conditional law upon $\mathcal{I}$ have a nice expression.

(a) The first example is closely connected with I_α. Taking $f(x) = e^{\alpha x}$ in (4.6), we get that the variables $I_\alpha I_{-\alpha}$ are $\mathcal{I}$-measurable and, $\mathbb{E}[\, I_\alpha \,|\, \mathcal{I}\,] = I_\alpha I_{-\alpha}$. More generally, for every $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$, we have

$$\mathbb{E}[(I_{\alpha_1})^{\beta_1} \dots (I_{\alpha_n})^{\beta_n} \,|\, \mathcal{I}\,] = I_{(-\sum_{i=1}^n \alpha_i \beta_i)} (I_{\alpha_1})^{\beta_1} \dots (I_{\alpha_n})^{\beta_n}\,. \tag{4.7}$$

Now, set

$$\begin{aligned} L_\alpha &\stackrel{\text{def}}{=} \alpha^{-1} \log I_\alpha\,, \quad \text{if } \alpha \neq 0 \\ &\stackrel{\text{def}}{=} \int_0^1 du\, X_u^{(b)}\,, \quad \text{if } \alpha = 0. \end{aligned} \tag{4.8}$$

(Note that $L_\alpha \to L_0$, as $\alpha \to 0$, $\mathbb{P} - a.s.$) By taking $n = 2$, $\alpha_1 = \alpha$, $\beta_1 = 1/\alpha$, $\alpha_2 = \beta$, $\beta_2 = -1/\beta$ in (4.7), we obtain that for every $\alpha, \beta \in \mathbb{R}$, such that $\alpha\beta \neq 0$, the variables,

$$L_\alpha - L_\beta\,, \tag{4.9}$$

are $\mathcal{I}$-measurable. This property can be justified more directly by noticing that,

$$\begin{aligned} \mathbb{E}[\, L_\alpha \,|\, \mathcal{I}\,] &= L_\alpha - \int_0^1 du\, X_u^{(b)} \\ &= L_\alpha - L_0\,. \end{aligned} \tag{4.10}$$

We emphasize that the functionals L_α play some natural role in the study of the asymptotic windings of planar Brownian motion. We refer to [17], Exercice (1.18), chap. I, p.23 and [6] where there is a discussion concerning L_α for Brownian motion.

The above results on L_α can be reinforced by the following proposition:

Proposition 4 *The conditional law of L_α given $\mathcal{I}$ satisfies,*

$$\mathbb{P}(L_\alpha \in dy \,|\, \mathcal{I}) = dy\, l^{L_\alpha - y}, \quad y \in \mathbb{R}. \tag{4.11}$$

Proof This is a straightforward consequence of the relation

$$\mathbb{E}[f(\alpha^{-1}\log I_\alpha) \,|\, \mathcal{I}] = \int_0^1 du\, f(\alpha^{-1}\log I_\alpha - X_u^{(b)}),$$

for every bounded measurable functional f, which follows from (2.8) and the occupation time density formula. □

(b) The second example concerns the quantiles of the bridge which present some intriguing analogies with L_α. For $\alpha \in [0,1]$, the α-quantile of $X^{(b)}$ is the level under which the process $X^{(b)}$ spends a time equal to α; it formally can be defined as follows:

$$M_\alpha \stackrel{\text{def}}{=} \inf\left\{x : \int_0^1 dh\, \mathbb{I}_{(X_h^{(b)} \le x)} = \alpha\right\}. \tag{4.12}$$

Since $\alpha \mapsto M_\alpha$ is the inverse of the function $x \mapsto \int_0^1 dh\, \mathbb{I}_{(X_h^{(b)} \le x)}$, then it is $\mathcal{L}$-measurable. Moreover, we easily check from (2.8) in Proposition 1, the analogue for M_α, of formula (4.10),

$$\mathbb{E}[\, M_\alpha \,|\, \mathcal{I}\,] = M_\alpha - \int_0^1 du\, X_u^{(b)}. \tag{4.13}$$

By the same token as in (4.9), for every $\alpha, \beta \in [0,1]$, the variables

$$M_\alpha - M_\beta \tag{4.14}$$

are $\mathcal{I}$-measurable and we prove the following proposition by the same arguments as those used in the proof of Proposition 4.

Proposition 5 *The conditional law of M_α given $\mathcal{I}$ satisfies,*

$$\mathbb{P}(M_\alpha \in dy \,|\, \mathcal{I}) = dy\, l^{M_\alpha - y}, \quad y \in \mathbb{R}. \tag{4.15}$$

A simple consequence of (4.15) is that:

$$\mathbb{E}[\alpha^{-1}\mathbb{I}_{\{M_\alpha > 0\}} \,|\, \mathcal{I}] = 1, \tag{4.16}$$

which may be seen as the counterpart of (4.3) applied to $X^{(b)}$. But (4.16) is also explained by the identity: $\{M_\alpha > 0\} = \{A^0 < \alpha\}$, and the fact that A^0 is uniform and independent of $\mathcal{I}$, see Proposition 2.

We note that the quantiles of a Lévy bridge have already been studied under this aspect in [7]. We also refer to [19], [8] and [4] for results about their laws.

The functionals L_α and M_α allow us to obtain several descriptions of the σ-field $\mathcal{I}\cap\mathcal{L}$, as the next proposition shows. First recall that under hypothesis (H1), the Lévy bridge $X^{(b)}$ reaches its minimum at almost surely one time, (defined in (2.11). For convenience reasons, set $\varepsilon = \Theta_m(X^{(b)})$, then we have

Proposition 6 *Let $\mathcal{L}(\varepsilon)$ be the σ-field generated by the local time process of ε, then the σ-fields $\mathcal{I}\cap\mathcal{L}$, $\mathcal{L}(\varepsilon)$, $\sigma(l^{L_\alpha}, \alpha\in\mathbb{R})$ and $\sigma(l^{M_\alpha}, \alpha\in\mathbb{R})$ are the same.*

Proof From Proposition 1 and Theorem 1 (i), it is easy to see the equality between $\mathcal{I}\cap\mathcal{L}$ and $\mathcal{L}(\varepsilon)$. Indeed, we have for every measurable bounded functional F,

$$\begin{aligned}\mathbb{E}(F(l^x,\, x\in\mathbb{R})\,|\,\mathcal{I}) &= \int_0^1 du\, F(l^{x-X_u^{(b)}},\, x\in\mathbb{R})\\ &= \int_0^1 du\, F(l^{x-\varepsilon_u}(\varepsilon),\, x\in\mathbb{R}),\end{aligned}$$

where $(l^x(\varepsilon),\, x\in\mathbb{R})$ is the local time process of ε.

On the other hand, it is clear that the processes $(l^{L_\alpha}, \alpha\in\mathbb{R})$ and $(l^{M_\alpha}, \alpha\in\mathbb{R})$ are $\mathcal{I}$-measurable. Moreover, if $L_\alpha(\varepsilon)$ and $M_\alpha(\varepsilon)$, are defined by

$$\begin{aligned}L_\alpha(\varepsilon) &\stackrel{\text{def}}{=} \alpha^{-1}\log\int_0^1 dh\, \exp(\alpha\varepsilon_h)\\ M_\alpha(\varepsilon) &\stackrel{\text{def}}{=} \inf\left\{x : \int_0^1 dh\, \mathbb{I}_{(\varepsilon_h\le x)} = \alpha\right\}\end{aligned}$$

then again, from Theorem 1 (i), we have

$$\begin{aligned}(l^{L_\alpha}, \alpha\in\mathbb{R}) &= (l^{L_\alpha(\varepsilon)}(\varepsilon), \alpha\in\mathbb{R})\\ (l^{M_\alpha}, \alpha\in\mathbb{R}) &= (l^{M_\alpha(\varepsilon)}(\varepsilon), \alpha\in\mathbb{R}).\end{aligned}$$

Finally, we easily check that the σ-fields generated by $(l^{L_\alpha(\varepsilon)}(\varepsilon), \alpha\in\mathbb{R})$, $(l^{M_\alpha(\varepsilon)}(\varepsilon), \alpha\in\mathbb{R})$ and $(l^x_\varepsilon,\, x\in\mathbb{R})$ are the same, which proves the equality between $\mathcal{L}(\varepsilon)$, $\sigma(l^{L_\alpha}, \alpha\in\mathbb{R})$ and $\sigma(l^{M_\alpha}, \alpha\in\mathbb{R})$. □

The analogy between $(L_\alpha,\, \alpha\in\mathbb{R})$ and $(M_\alpha,\, \alpha\in[0,1])$ is reinforced by the following remark: denote by S the supremum over [0,1] of the Lévy bridge $X^{(b)}$, then S is an extremal value of both the processes $(L_\alpha,\, \alpha\in\mathbb{R})$ and $(M_\alpha,\, \alpha\in[0,1])$, in the sense that almost surely,

$$\begin{aligned}S &= \lim_{\alpha\to+\infty} L_\alpha &(4.17)\\ &= \lim_{\alpha\to 1} M_\alpha \quad (= M_1). &(4.18)\end{aligned}$$

The same equalities hold for the infimum over [0,1] of $X^{(b)}$ by letting α respectively go to $-\infty$ and 0. In particular, from either Proposition 4 or 5, we get

$$\mathbb{P}(S \in dy \,|\, \mathcal{I}) = dy\, l^{S-y}\,, \quad y \in \mathbb{R}\,.$$

And from this last equality, we recover the following result which, at least in the Brownian case, was already a consequence of Vervaat's transformation:

$$S \stackrel{(d)}{=} \varepsilon_{\mathcal{U}}\,,$$

where $\mathcal{U}$ is an uniformly distributed random variable independent of ε.

In [7], the process $(M_\alpha,\ \alpha \in [0,1])$ is involved in an extension of Vervaat's transformation. The above analogies between L_α and M_α raise the question of the existence of such an extension involving L_α.

Acknowledgement:
We are grateful to M. Emery, F. Knight and J. Pitman for providing us with some constructive critisism and references. Thanks also to M. Malric to whom Theorem 1 (i) is due.

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