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# OCCUPATION TIMES OF LÉVY PROCESSES AS QUADRATIC VARIATIONS

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**Abstract:** Bouleau and Yor [BY] have shown that the occupation time of a continuous martingale can be obtained as a quadratic variation. We extend this result to a large class of Lévy processes.

## 1 - Introduction

We first recall a result established by Bouleau and Yor [BY]. Let  $Y$  be a semimartingale such that  $\sum_{0 \leq s \leq t} |\Delta Y_s| < \infty$ .  $Y$  admits then the following decomposition

$$Y = Y_0 + M + A \tag{1}$$

where  $M$  is a continuous local martingale and  $A$  a process with bounded variation. Let  $(\pi_n)_{n \in \mathbf{N}}$  be a sequence of partitions of an interval  $[a, b]$  of  $\mathbb{R}$  such that  $|\pi_n|$  converges to 0 as  $n$  tends to  $\infty$ .

They show that  $(\sum_{x_i \in \pi_n} (\int_0^t 1_{(x_i, x_{i+1}]}(Y_{s-}) dY_s)^2; n \in \mathbf{N})$  converges in probability uniformly in  $t$  on any compact of  $\mathbb{R}^+$ , to

$$\int_0^t 1_{(a,b)}(Y_{s-}) d[Y]_s + 2 \int_0^t dA_s \int_{(s,t]} dA_u 1_{(a,b)}(Y_{s-}) 1_{(Y_u = Y_{s-})}. \tag{2}$$

In the particular case when  $Y$  is a continuous square-integrable martingale, they obtain the following convergence .

$$\sum_{x_i \in \pi_n} \left( \int_0^t 1_{(x_i, x_{i+1}]}(Y_s) dY_s \right)^2 \xrightarrow[n \rightarrow \infty]{L^2} \int_0^t 1_{(a,b)}(Y_{s-}) d[Y]_s \tag{3}$$

uniformly in  $t$  on compacts.

In the special case when  $Y$  is a Brownian motion, (3) has also been proved by Perkins [Per].

We establish here a similar result for a large class of Lévy processes.

**Theorem 1:** *Let  $(Y_t, t \geq 0)$  be a Lévy process starting from 0. Assume that  $Y$  is not a pure step process and that  $\sum_{0 \leq s \leq t} |\Delta Y_s| < \infty$ .*

*Let  $(\pi_n)_{n \in \mathbf{N}}$  be a sequence of partitions of an interval  $[a, b]$  of  $\mathbb{R}$  such that  $|\pi_n|$  converges to 0 as  $n$  tends to  $\infty$ . Then*

$$\lim_{n \rightarrow \infty} \sum_{x_i \in \pi_n} \left( \int_0^t 1_{(x_i, x_{i+1}]}(Y_{s-}) dY_s \right)^2 = \int_0^t 1_{(a,b)}(Y_{s-}) d[Y]_s$$

in probability, uniformly in  $t$  on any bounded interval of  $\mathbb{R}^+$ .

The assumption on the jumps of the Lévy process  $Y$  is equivalent to

$$\int (1 \wedge |x|) \nu(dx) < \infty$$

where  $\nu$  is the Lévy measure of  $Y$ .

**Remark :** If  $Y$  is a pure step process, we have immediately thanks to (2)

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{x_i \in \pi_n} \left( \int_0^t 1_{(x_i, x_{i+1}]}(Y_{s-}) dY_s \right)^2 \\ &= \int_0^t 1_{(a,b]}(Y_{s-}) d[Y]_s + \sum_{\substack{0 \leq u, v \leq t \\ u \neq v}} 1_{\{a \leq Y_{u-} = Y_{v-} \leq b\}} (\Delta Y_u)(\Delta Y_v) \end{aligned}$$

in probability. There are simple examples of compound Poisson processes such that the second term on the right hand side is different from 0.

**Applications :** Let  $(Y_t, t \geq 0)$  be a stable process of index  $\alpha \in (0, 2)$ . We have :

$$[Y]_t = \sum_{0 \leq s \leq t} (\Delta Y_s)^2$$

We note that for any  $\lambda > 0$

$$([Y]_{\lambda t}, t \geq 0) \stackrel{(loi)}{=} (\lambda^{2/\alpha} [Y]_t, t \geq 0)$$

Consequently  $[Y]$  is a stable subordinator of index  $\alpha/2$ . If  $\alpha$  is in  $(0, 1)$ ,  $Y$  satisfies the assumptions of Theorem 1. We see then that  $[Y]$  can be obtained as the limit in probability of  $(\sum_{x_i \in \pi_n} (\int_0^t 1_{(x_i, x_{i+1}]}(Y_{s-}) dY_s)^2, t \geq 0)$  as  $|\pi_n|$  tends to  $\infty$ , where  $(\pi_n, n \in \mathbb{N})$  is a sequence of subdivisions of  $\mathbb{R}$ .

## 2 - Proof of Theorem 1

From now on  $Y$  is a Lévy process satisfying the assumptions of Theorem 1. Consider the process  $W$  defined by :  $W_t = Y_t - \sum_{0 \leq s \leq t} \Delta Y_s$ . This process is a continuous Lévy process. Let  $\sigma$  be the constant such that :  $\mathbb{E}(W_t) = \sigma t$ . The process  $(W_t - \sigma t, t \geq 0)$  is hence a continuous martingale. Consequently, the process  $A$  associated to  $Y$ , in the decomposition (1), is equal to  $(\sum_{0 \leq s \leq t} (\Delta Y_s)^2 + \sigma t, t \geq 0)$ .

Similarly, for every  $\alpha > 0$ , there exists a constant  $b_\alpha$  such that :

$$Y_t = M_t^\alpha + b_\alpha t + V_t^\alpha$$

where :  $V_t^\alpha = \sum_{0 < s \leq t} \Delta Y_s 1_{|\Delta Y_s| \geq \alpha}$ ,

and  $M^\alpha$  is a martingale with bounded jumps such that :  $\langle M^\alpha \rangle_t = c_\alpha t$  (for more details about this general decomposition, see for example [Pro] p.32)

We have to prove that :  $\int_0^t dA_s \int_{(s,t]} dA_u 1_{(a,b]}(Y_{s-}) 1_{(Y_{u-} = Y_{s-})}$  is equal to 0.

We start by showing that

$$\int_0^t dV_s^\alpha \int_{(s,t]} dV_u^\alpha 1_{(a,b)}(Y_{s-}) 1_{(Y_{u-}=Y_{s-})} \quad (4)$$

is equal to 0. We write for a fixed  $t > 0$  :

$$\begin{aligned} & \int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dY_v \\ &= \int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dM_v^\alpha + \int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dV_v^\alpha + b_\alpha \int_0^t 1_{\{0\}}(Y_{v-}) dv \end{aligned} \quad (5)$$

Since  $M^\alpha$  is a square integrable martingale, we have :

$$\begin{aligned} \mathbb{E}[(\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dM_v^\alpha)^2] &= \mathbb{E}[\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) d[M^\alpha]_v] \\ &= c_\alpha \mathbb{E}[\int_0^t 1_{\{0\}}(Y_{v-}) dv] \end{aligned}$$

Since  $Y$  is not a pure step process, we have thanks to Theorem 1 of [BR] :  $\mathbb{P}(Y_{v-} = 0) = 0$ .

Hence we obtain :  $\int_0^t 1_{\{0\}}(Y_{v-}) dv = 0$  and  $\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dM_v^\alpha = 0$  a.s.

We note then that  $V^\alpha$  converges to 0 as  $\alpha$  tends to  $\infty$ . Consequently, thanks to (5) :

$$\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dY_v = 0$$

Making use once more of (5), we obtain :

$$\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dV_v^\alpha = 0 \quad \text{a.s.} \quad (6)$$

Thanks to the right continuity of  $V^\alpha$ , the process  $(\int_0^t 1_{(v>0)} 1_{\{0\}}(Y_{v-}) dV_v^\alpha, t \geq 0)$  is a.s. identically equal to 0. This result remains true if the function  $1_{\{0\}}$  is replaced by  $1_{\{X\}}$  with  $X$  any random variable, independent of  $Y$ .

We define now the sequence of stopping times  $(T_n)_{n \geq 1}$  by :

$$T_1 = \inf\{s \geq 0 : |\Delta Y_s| \geq \alpha\}$$

$$T_{n+1} = \inf\{s \geq T_n : |\Delta Y_s| \geq \alpha\}$$

Let  $n$  be a fixed integer. Conditionally on  $\{T_n < +\infty\}$ , the process  $\tilde{Y} = (Y_{T_n+t} - Y_{T_n}, t \geq 0)$  is independent of  $\mathcal{F}_{T_n}$ , and has the law of  $Y$ . Similarly, we can write the following decomposition of  $\tilde{Y}$

$$\tilde{Y}_t = \tilde{M}_t^\alpha + b_\alpha t + \tilde{V}_t^\alpha$$

Note that :  $\tilde{V}_t^\alpha = V_{T_n+t}^\alpha - V_{T_n}^\alpha$ .

Since the variable  $\Delta Y_{T_n}$  is independent of  $\tilde{Y}$ , we have, thanks to (6)

$$\int_0^t 1_{(v>0)} 1_{\{-\Delta Y_{T_n}\}}(\tilde{Y}_{v-}) d\tilde{V}_v^\alpha = 0$$

which means that a.s.

$$\int_{T_n}^{\cdot} 1_{(v>T_n)} 1_{\{Y_{(T_n)-}\}}(Y_{v-}) dV_v^\alpha = 0$$

We come back now to the expression (4) :

$$\begin{aligned} & \int_0^t 1_{(a,b)}(Y_{u-}) dV_u^\alpha \int_0^t 1_{(v>u)} 1_{(Y_{u-}=Y_{v-})} dV_v^\alpha \\ &= \sum_{T_n \leq t} 1_{(a,b)}(Y_{T_n-}) \Delta Y_{T_n} \int_{T_n}^t 1_{(v>T_n)} 1_{(Y_{T_n-}=Y_{v-})} dV_v^\alpha \end{aligned}$$

Consequently, we have obtained

$$\int_0^t 1_{(a,b)}(Y_s) dV_s^\alpha \int_{(s,t]} 1_{(Y_{s-}=Y_{v-})} dV_v^\alpha = 0 \quad (7)$$

Similarly, we have for every  $n > 0$

$$\int_{T_n}^{\cdot} 1_{\{Y_{(T_n)-}\}}(Y_{v-}) dv = 0 \quad \text{a.s.}$$

which leads to

$$\sum_{T_n \leq t} 1_{(a,b)}(Y_{T_n-}) \Delta Y_{T_n} \int_{T_n}^t 1_{(Y_{T_n-}=Y_{v-})} dv = 0.$$

We have actually obtained

$$\int_0^t 1_{(a,b)}(Y_{s-}) dV_s^\alpha \int_{(s,t]} 1_{(Y_{s-}=Y_{v-})} dv = 0 \quad (8)$$

The previous argument made at the stopping time  $T_n$ , can similarly be written for a fixed time  $s$  such that  $0 < s < t$ . We hence obtain

$$\int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dV_v^\alpha = 0 \quad \text{and} \quad \int_s^t 1_{(Y_{v-}=Y_{s-})} dv = 0 \quad \text{a.s.}$$

which lead to

$$\int_0^t 1_{(a,b)}(Y_s) ds \int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dV_v^\alpha = 0 \quad \text{and} \quad \int_0^t 1_{(a,b)}(Y_s) ds \int_s^t 1_{(Y_{v-}=Y_{s-})} dv \quad (9)$$

We set then

$$A_t^\alpha = V_t^\alpha + \sigma t$$

Thanks to (7),(8) and (9), we can write

$$\int_0^t 1_{(a,b)}(Y_s) dA_s^\alpha \int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dA_v^\alpha = 0$$

Letting  $\alpha$  tend to 0, we finally obtain, by dominated convergence

$$\int_0^t 1_{(a,b)}(Y_s) dA_s \int_s^t 1_{(v>s)} 1_{(Y_{v-}=Y_{s-})} dA_v = 0 \quad \square$$

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