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ON WEAK CONVERGENCE OF FILTRATIONS

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Abstract. *A sequence of filtrations $(\mathcal{F}_t^n)_{t\le T}$ converges weakly to a filtration $(\mathcal{F}_t)_{t\le T}$ iff, for all $B \in \mathcal{F}_T$, the sequence of processes $(E[1_B|\mathcal{F}_t^n])_{t\le T}$ converges in probability under the Skorokhod topology to the process $(E[1_B|\mathcal{F}_t])_{t\le T}$. We give some examples of this kind of convergence; then we study, under the weak convergence of filtrations, the convergence in probability of processes $(E[X_t|\mathcal{F}_t^n])_{t\le T}$ to X where X is an $\mathcal{F}_t$-adapted semimartingale.*

I. Introduction.

In what follows, we are given a probability space $(\Omega, \mathcal{G}, \mathbf{P})$. Every σ-algebra we deal with, is supposed to be included in $\mathcal{G}$. We also fix a positive integer T. Unless otherwise stated, every process or filtration will be indexed by $t \in [0,T]$; a filtration $(\mathcal{F}_t)_{t\le T}$ is denoted by $\mathcal{F}$; the relation $\mathcal{F}^n \subset \mathcal{F}$ means that for all $t \le T$, $\mathcal{F}_t^n \subset \mathcal{F}_t$. All filtrations considered are assumed to be right-continuous; in particular by natural filtration of a process X we denote the right-continuous filtration associated to the natural filtration of X. $\mathbb{D}$ (resp: $\mathbb{D}(S)$) denotes the space of càdlàg (right-continuous with left limits) functions from $\mathbb{R}^+$ into $\mathbb{R}$ (resp: S).

Following Hoover [12], we say that:

Definition 1. *A sequence of σ-algebras $\mathcal{A}^n$ converges weakly to a σ-algebra $\mathcal{A}$ (we write $\mathcal{A}^n \xrightarrow{w} \mathcal{A}$) iff, for all $B \in \mathcal{A}$, the sequence of random variables $(E[1_B|\mathcal{A}^n])$ converges in probability to 1_B, and we write $(E[1_B|\mathcal{A}^n]) \xrightarrow{\mathbf{P}} 1_B$.*

Following again [12], but with another topology, we introduce the following:

Definition 2. *A sequence of filtrations $\mathcal{F}^n$ converges weakly to the filtration $\mathcal{F}$ (and we write $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$) iff, for all $B \in \mathcal{F}_T$, the sequence of càdlàg martingales $(E[1_B|\mathcal{F}_{\cdot}^n])$ converges in probability under the Skorokhod J_1-topology on $\mathbb{D}$ to the martingale $(E[1_B|\mathcal{F}_{\cdot}])$.*

This notion does not present any interest when the limit filtration is trivial; for example, when $\mathcal{F}^n$ is generated by a càdlàg process Y^n such that $Y^n \xrightarrow{\mathbf{P}} y$ where y is deterministic, and $\mathcal{F}_{\cdot} = \mathcal{F}_{\cdot}^y$: then the weak convergence of filtrations is trivially satisfied. For such a case we should introduce a stronger notion of convergence.

Remark 1. Let us describe some easy consequences of the definition of weak convergence of filtrations.

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1) For σ-algebras, clearly enough, $\mathcal{A}^n \xrightarrow{w} \mathcal{A}$ iff $E[Z|\mathcal{A}^n] \xrightarrow{\mathbf{P}} Z$ for every integrable random variables Z measurable with respect to $\mathcal{A}$.

Similarly for filtrations, $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ iff $E[X|\mathcal{F}^n_\cdot] \xrightarrow{\mathbf{P}} E[X|\mathcal{F}_\cdot]$ under the Skorokhod J_1 topology for all integrable random variables X measurable with respect to $\mathcal{F}_T$. In this case, however, the equivalence is not immediate, because the Skorokhod J_1 topology is not linear. In view of Doob's inequality for martingales, for every $\varepsilon > 0$ and every integrable random variable X', we have

$$\mathbf{P}[\sup_{t\leq T} |\mathbf{E}[X|\mathcal{F}^n_t] - \mathbf{E}[X'|\mathcal{F}^n_t]| > \varepsilon] \leq \frac{1}{\varepsilon}\mathbf{E}[|X - X'|],$$

so it is sufficient to check that $E[X|\mathcal{F}^n_\cdot] \xrightarrow{\mathbf{P}} E[X|\mathcal{F}_\cdot]$ under the Skorokhod J_1 topology for every random variable of the form $X = \sum_{i=1}^k b_i 1_{B_i}$, where $B_i \cap B_j = \emptyset$ for $i \neq j$ and $b_i \in \mathbb{R}$, $B_i \in \mathcal{F}_T$, $i, j = 1,...k$. If $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ then $X^{ni}_\cdot = E[1_{B_i}|\mathcal{F}^n_\cdot] \xrightarrow{\mathbf{P}} X^i_\cdot = E[1_{B_i}|\mathcal{F}_\cdot]$, $i = 1,...k$, and $\sum_{i=1}^p X^{ni}_\cdot \xrightarrow{\mathbf{P}} \sum_{i=1}^p X^i_\cdot$ for $p \leq k$, since $\sum_{i=1}^p 1_{B_i} = 1_{\cup_{i=1}^p B_i}$, $p \leq k$. Hence $(X^{n1}_\cdot, ...X^{nk}_\cdot) \xrightarrow{\mathbf{P}} (X^1_\cdot, ..., X^k_\cdot)$ under the Skorokhod J_1 topology on $\mathbb{D}(\mathbb{R}^k)$, which implies that

$$E[X|\mathcal{F}^n_\cdot] = \sum_{i=1}^k b_i X^{ni}_\cdot \xrightarrow{\mathbf{P}} \sum_{i=1}^k b_i X^i_\cdot = E[X|\mathcal{F}_\cdot].$$

Actually, from a classical criterion of convergence for J_1, for example Proposition 29.2 of [1] or Proposition 3-6-5 in [11], we get and use the following immediate characterization:

Lemma 1. *Let us consider a sequence* $((x^1_n, ..., x^k_n))$ *in* $\mathbb{D}(\mathbb{R}^k)$; $(x^1_n, ..., x^k_n) \to (x^1, ..., x^k)$ *under* J_1 *in* $\mathbb{D}(\mathbb{R}^k)$ *if and only if:*

i) For every $i \leq k$, $x^i_n \to x^i$ *under* J_1 *in* $\mathbb{D}$

ii) For every $p \leq k$, $\sum_{i=1}^p x^i_n \to \sum_{i=1}^p x^i$ *under* J_1 *on* $\mathbb{D}$.

2) Moreover, suppose that we are given a sequence (X_n) of random variables converging in $L^1(\Omega, \mathcal{F}_T, \mathbf{P})$ to some X, and that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$; then, using again the Doob inequality for martingales we have the convergence: $\mathbf{E}[X_n|\mathcal{F}^n_\cdot] \xrightarrow{\mathbf{P}} \mathbf{E}[X|\mathcal{F}_\cdot]$ under the J_1 topology.

3) Let us consider a pair (X, X') of integrable random variables; then $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ implies the convergence $(E[X|\mathcal{F}^n_\cdot], E[X'|\mathcal{F}^n_\cdot]) \xrightarrow{\mathbf{P}} (E[X|\mathcal{F}_\cdot], E[X'|\mathcal{F}_\cdot])$. Actually, by linearity of conditioning, we have the convergence for J_1:

$$E[X|\mathcal{F}^n_\cdot] + E[X'|\mathcal{F}^n_\cdot] \xrightarrow{\mathbf{P}} E[X|\mathcal{F}_\cdot] + E[X'|\mathcal{F}_\cdot]$$

which, as above, implies convergence of pairs of processes under J_1.

4) Let $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$, and let Q be a probability on $(\Omega, \mathcal{G})$, absolutely continuous with respect to $\mathbf{P}$. Writing $\mathbf{E}_Q[X|\mathcal{F}_t] = \dfrac{\mathbf{E}_P[XZ|\mathcal{F}_t]}{\mathbf{E}_P[Z|\mathcal{F}_t]}$, Q a.s., where Z denotes the

Radon-Nikodym derivative $\frac{dQ}{d\mathbf{P}}$, and noticing that Q a.s. $\mathbf{E}[Z|F_.]$ is not vanishing, we immediately deduce the following interesting result:

the weak convergence $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ also holds on the probability space $(\Omega, \mathcal{F}_T, Q)$.

Remark 2. It is important to note that the weak convergence of a sequence of filtrations involves more than pointwise convergence of the corresponding σ-algebras, as the following example shows; it is taken from [7] (where all necessary details are given).

Choose T greater than (say) 1. Let B be a standard Brownian motion on $[0, T]$, and let X be a random variable independent of B and such that $P(X = 1) = P(X = \frac{1}{2}) = \frac{1}{2}$. Put

$$Y_t = \begin{cases} 0 & \text{for } t < \frac{1}{2} \\ (B_t - B_{1/2})X & \text{for } t \geq \frac{1}{2}. \end{cases}$$

Let then $\mathcal{F}$ be the filtration generated by Y, and $(\mathcal{F}^n)$ be the sequence of filtrations generated by discrete-time approximations of Y along a sequence of subdivisions of $[0, T]$ with mesh going to 0. Then X is $\mathcal{F}_t$-measurable for every $t > \frac{1}{2}$, and moreover $\mathcal{F}_t^n \xrightarrow{w} \mathcal{F}_t$ for every t except $t = 1/2$. But $\mathbf{E}[X|\mathcal{F}_.^n]$ does not converge to $\mathbf{E}[X|\mathcal{F}_.]$ under the J_1 topology, hence the sequence of filtrations $(\mathcal{F}^n)$ does not converge weakly to $\mathcal{F}$.

However in Proposition 4, we shall see examples where the pointwise convergence of filtrations implies the weak convergence when all martingales with respect to the limiting filtration are continuous.

Remark 3. Convergence in probability of a sequence of random variables (X_n) to X implies convergence of the σ-algebras $\mathcal{F}^{X_n}$ to $\mathcal{F}^X$ (see [12]). But, we have no general analogous result for processes and the weak convergence of filtrations. The same example as above shows that if (X^n) is a sequence of processes converging in probability to a process X under J_1 (or uniform) topology, then, in general convergence : $\mathcal{F}^{X^n} \xrightarrow{w} \mathcal{F}^X$ does not hold.

We can find, however, some examples where convergence in probability of processes X^n to X implies the weak convergence of their associated natural filtrations (see Propositions 2-6).

Considering such convergence of filtrations was suggested by the paper of Antonelli and Kohatsu-Higa [4], where a general problem of convergence for solutions of backward SDE's was considered. Under suitable conditions, they introduced the backward SDE

$$V_t = \mathbf{E}\Big[\int_t^T g_s(V_s)\, dA_s + X|\mathcal{F}_t\Big], \tag{$*$}$$

and the $\mathcal{F}^n$-adapted solutions V^n of the "perturbed" equations

$$V_t^n = \mathbf{E}\Big[\int_t^T g_s^n(V_s^n)\, dA_s^n + X^n|\mathcal{F}_t^n\Big] \tag{$*^n$}$$

and they proved convergence of V^n to V in law under the Meyer-Zheng topology, when the filtration $\mathcal{F}^n$ is taken by discretization of a process Y generating $\mathcal{F}$ as in our example above, or in probability, when $\mathcal{F}^n$ is taken from $\mathcal{F}$ by a particular change of time.

In paper [7], corrected in [8], convergence in probability of V^n to V, in the discretized situation, was proved under the assumption that the process Y is Markov.

Below in section IV, we shall see (Theorem 4) that this convergence holds when, more generally, weak convergence of filtrations $\mathcal{F}^n$ is in force.

Another general problem is the following. Suppose that X is an $\mathcal{F}$-adapted càdlàg process, and let X^n be the càdlàg version of the processes $\mathbf{E}[X_.|\mathcal{F}^n_.]$: i.e. X^n is the $\mathcal{F}^n$-optional projection of X (see [9], VI-43 and VI-47). Under which assumption shall we get the convergence of X^n to X in probability under the J_1 topology ? This problem will be studied in section III; Theorems 1 and 2 will give a partial answer in the quasi-left-continuous case.

Let us consider now for this situation, the problem of pointwise convergence. Under the weak convergence of filtrations $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$, we cannot obtain the convergence in probability of $\mathbf{E}[X_t|\mathcal{F}^n_t]$ to $\mathbf{E}[X_t|\mathcal{F}_t]$ for every $t \leq T$, but we have the following:

Remark 4. Let X be a càdlàg $\mathcal{F}_t$-adapted process, such that $\sup_{t\leq T}|X_t| \in L^1$. Let us assume that s is a continuity point of X (i.e. $\mathbf{P}[\Delta X_s \neq 0] = 0$) and that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. Then:

$$\mathbf{E}[X_s|\mathcal{F}^n_s] \xrightarrow{\mathbf{P}} X_s.$$

Proof. Let us fix s; by the convergence $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ we have

$$\mathbf{E}[X_s|\mathcal{F}^n_.] \xrightarrow{\mathbf{P}} \mathbf{E}[X_s|\mathcal{F}_.].$$

Take $t < s$, we have $\mathbf{E}[X_s|\mathcal{F}_t] = X_t + \mathbf{E}[X_s - X_t|\mathcal{F}_t]$; then for $t \uparrow s$, $X_t \xrightarrow{\mathbf{P}} X_s$ and $\mathbf{E}[X_s - X_t|\mathcal{F}_t] \xrightarrow{\mathbf{P}} 0$; we deduce that s is a continuity point of the process $\mathbf{E}[X_s|\mathcal{F}_.]$, hence the convergence: $\mathbf{E}[X_s|\mathcal{F}^n_s] \xrightarrow{\mathbf{P}} X_s$. ∎

In the following section II, we shall give some examples of situations where the weak convergence of filtrations is in force.

For general notions or notation concerning general theory of processes and their limit theorems, we refer to [9] and [13].

II. Examples of weak convergences of filtrations

We begin with results on stability of weak convergence of filtrations under time changes: then we describe some examples, where convergence in probability of processes under the J_1 topology implies convergence of associated filtrations.

The following lemma is key for our stability results.

Lemma 2. *Assume that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. If τ^n, $n \in \mathbb{N}$, and τ are time changes (i.e. for all $t \in \mathbb{R}^+$, τ_t^n, τ_t are stopping times with respect to the filtrations $\mathcal{F}^n$, $n \in \mathbb{N}$, and $\mathcal{F}$, respectively and τ^n (respectively τ) are increasing càdlàg processes). Suppose that τ is continuous and $\tau^n \xrightarrow{P} \tau$ uniformly. Then $\mathcal{F}^n \circ \tau^n \xrightarrow{w} \mathcal{F} \circ \tau$.*

Proof. The result easily follows from the following simple remark, (which can be seen for example, as an immediate consequence of Proposition 3-6-5 of Ethier-Kurtz [11]):

$x^n \longrightarrow x$, and $a^n \longrightarrow a$, under $J_1 \Longrightarrow x^n \circ a^n \longrightarrow x \circ a$ under J_1, provided that a is continuous and a^n, a are nondecreasing and non negative.

As a corollary, we immediately get

Proposition 1. *Assume that $\mathcal{F}_t^n = \mathcal{F}_{\phi^n(t)}$, where ϕ^n denotes a sequence of non-decreasing càdlàg functions such that $\phi^n(t) \to t$. Then $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*

Lemma 3. *Let X be a càdlàg process, and let $\mathcal{F}^X$ denote its natural filtration.*

The following three conditions are equivalent:

(i) $\mathcal{F}^n \xrightarrow{w} \mathcal{F}^X$,

(ii) $E[f(X_{t_1}, ..., X_{t_k})|\mathcal{F}^n_\cdot] \xrightarrow{P} E[f(X_{t_1}, ..., X_{t_k})|\mathcal{F}^X_\cdot]$ *under J_1, for all bounded and continuous $f : \mathbb{R}^k \to \mathbb{R}$, and all $t_1, ...t_k$ points of continuity of X, $k \in \mathbb{N}$,*

(iii) $E[\sum_{j=1}^m c_j \exp\{i \sum_{l=1}^k \lambda_j^l X_{t_l}\}|\mathcal{F}^n_\cdot] \xrightarrow{P} E[\sum_{j=1}^m c_j \exp\{i \sum_{l=1}^k \lambda_j^l X_{t_l}\}|\mathcal{F}^X_\cdot]$

under J_1, for all $(c_1, ., c_m) \in \mathbb{R}^m$ and all $(\lambda_1^l, ...\lambda_m^l) \in \mathbb{R}^m$, every $t_1, ...t_k$ from the set of continuity points of X, $k \in \mathbb{N}$.

Proof. For every $A \in \mathcal{F}^X, \varepsilon > 0$ one can find $k \in \mathbb{N}$, and $t_1, ...t_k$ points of continuity of X and f such that

$$E[|I_A - f(X_{t_1}, ..., X_{t_k})|] < \varepsilon$$

as well as one can find $m \in \mathbb{N}$, $(c_1, ...c_m) \in \mathbb{R}^m$, $(\lambda_1^l, ...\lambda_m^l) \in \mathbb{R}^m$ such that

$$E[|f(X_{t_1}, ..., X_{t_k}) - \sum_{j=1}^m c_j \exp\{i \sum_{l=1}^k \lambda_j^l X_{t_l}\}|] < \varepsilon.$$

Now the result follows from Doob's inequality for martingales. ∎

Proposition 2. *Let $\{X^n\}$ be a sequence of càdlàg processes with independent increments (initial values are considered as increments). If $X^n \xrightarrow{\mathbf{P}} X$ under J_1, then $\mathcal{F}^{X^n} \xrightarrow{w} \mathcal{F}^X$.*

Proof. We will check condition (iii) of Lemma 3. First observe that, for $\bar{\lambda}_j^l = \sum_{u=l}^k \lambda_j^u$,

$$\sum_{j=1}^m c_j \exp\{i \sum_{l=1}^k \lambda_j^l X_{t_l}\} = \sum_{j=1}^m c_j \exp\{i \sum_{l=1}^k \bar{\lambda}_j^l (X_{t_l} - X_{t_{l-1}})\}.$$

Since, except for a countable set of $[0,T]$, finite distributions of X^n converge in law to the corresponding finite distributions of X, it is clear that X also is a process with independent increments and, for $t \in [t_{l-1}, t_l)$,

$$\begin{aligned} F_t &= E[\sum_{j=1}^m c_j \exp\{i \sum_{l=1}^k \bar{\lambda}_j^l (X_{t_l} - X_{t_{l-1}})\} | \mathcal{F}_t^X] \\ &= \sum_{j=1}^m c_j \prod_{u>l} E \exp\{i\bar{\lambda}_j^u (X_{t_u} - X_{t_{u-1}})\} E \exp\{i\bar{\lambda}_j^l (X_{t_l} - X_t)\} \\ &\qquad \times \exp\{i\bar{\lambda}_j^l (X_t - X_{t_{l-1}})\} \prod_{u<l} \exp\{i\bar{\lambda}_j^u (X_{t_u} - X_{t_{u-1}})\}, \end{aligned}$$

as well as

$$\begin{aligned} F_t^n &= E[\sum_{j=1}^m c_j \exp\{i \sum_{l=1}^k \bar{\lambda}_j^l (X_{t_l}^n - X_{t_{l-1}}^n)\} | \mathcal{F}_t^{X^n}] \\ &= \sum_{j=1}^m c_j \prod_{u>l} E \exp\{i\bar{\lambda}_j^u (X_{t_u}^n - X_{t_{u-1}}^n)\} E \exp\{i\bar{\lambda}_j^l (X_{t_l}^n - X_t^n)\} \\ &\qquad \times \exp\{i\bar{\lambda}_j^l (X_t^n - X_{t_{l-1}}^n)\} \prod_{u<l} \exp\{i\bar{\lambda}_j^u (X_{t_u}^n - X_{t_{u-1}}^n)\}. \end{aligned}$$

Since $t_1, ..., t_k$ are points of continuity of X, from convergence $X^n \xrightarrow{\mathbf{P}} X$ it follows that $F^n \xrightarrow{\mathbf{P}} F$, which implies condition (iii). ∎

We recall below Theorem 1 of [7]; a partial extension to a sequence of filtrations generated by Feller processes is given in Proposition 4.

Proposition 3 ([7], Theorem 1). *Assume that $\mathcal{F}$ is generated by a càdlàg Markov process Y, and $\mathcal{F}^n$ by the discretization Y^n of Y along a sequence of subdivisions of $[0,T]$, with mesh going to 0 (as in the example of Remark 2). Then $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*

Proposition 4. *A) Assume that every $\mathcal{F}$-martingale is continuous, and:*
- either, for every $t \in [0,T]$, $\mathcal{F}_t^n$ increases (resp: decreases) with $n \to \infty$, to $\mathcal{F}_t$,
- or $\mathcal{F}$ is generated by a càdlàg process Y, $\mathcal{F}^n$ is generated by a càdlàg process Y^n, $\mathcal{F}^n \subset \mathcal{F}$, and $Y^n \xrightarrow{\mathbf{P}} Y$.

Then $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.

B) Let Y^n *be a càdlàg Feller process and* Y *be a continuous Feller process .*

We suppose that every time when $Y_0^n \xrightarrow{\mathbf{P}} Y_0$ *then the convergence* $Y^n \xrightarrow{\mathbf{P}} Y$ *under* J_1*holds. Then,* $\mathcal{F}^{Y^n} \xrightarrow{w} \mathcal{F}^Y$ *provided that* $Y_0^n \xrightarrow{\mathbf{P}} Y_0$.

Proof of A. In the first situation, for fixed t and B, the sequence $(\mathbf{E}[1_B|\mathcal{F}_t^n])_n$ defines a bounded discrete (resp: reverse) martingale, hence converges in probability to $\mathbf{E}[1_B|\mathcal{F}_t]$ as $n \to \infty$. Therefore, we have the pointwise convergence in probability of the sequence $(\mathbf{E}[1_B|\mathcal{F}^n_.])$ of $\mathcal{F}^n$-martingales towards the continuous martingale $\mathbf{E}[1_B|\mathcal{F}]$. It follows then from Aldous [2] that the convergence must be uniform (hence under J_1).

In the second situation, using the same argument (Aldous [2]) as in the first one, it is sufficient to check that for every continuity point t of Y, we have convergence in probability of the sequence $(\mathbf{E}[1_B|\mathcal{F}_t^n])_n$ to $\mathbf{E}[1_B|\mathcal{F}_t]$ as $n \to \infty$. Proceeding as in Lemma 3 above, for every ε, there exist points of continuity $t_1, ...t_k \leq t$ of Y and a bounded continuous function Φ from $\mathbb{R}^k$ in $\mathbb{R}$ such that

$$\|E[1_B|\mathcal{F}_t] - \Phi(Y_{t_1}, ...Y_{t_k})\|_{L^1} < \varepsilon.$$

Since $Y^n \xrightarrow{\mathbf{P}} Y$, $\Phi(Y_{t_1}^n, .., Y_{t_k}^n)$ converges to $\Phi(Y_{t_1}, ..., Y_{t_k})$ in L^1. Then

$$\begin{aligned}
&E[|E[1_B|\mathcal{F}_t^n] - \Phi(Y_{t_1}^n, ...Y_{t_k}^n)|] \\
&\leq E[E[|E[1_B|\mathcal{F}_t] - \Phi(Y_{t_1}^n, ..., Y_{t_k}^n)||\mathcal{F}_t^n]] \\
&\leq \|E[1_B|\mathcal{F}_t] - \Phi(Y_{t_1}^n, ..., Y_{t_k}^n)\|_{L^1} \\
&\leq \|E[1_B|\mathcal{F}_t] - \Phi(Y_{t_1}, ..., Y_{t_k})\|_{L^1} + \|\Phi(Y_{t_1}^n, ..., Y_{t_k}^n) - \Phi(Y_{t_1}, ..., Y_{t_k})\|_{L^1}.
\end{aligned}$$

This last expression is smaller than 2ε when n is large enough, hence

$$\|E[1_B|\mathcal{F}_t^n] - E[1_B|\mathcal{F}_t]\|_{L^1} \leq 3\varepsilon$$

and the assertion follows. ∎

Proof of B. Since Y is a continuous Feller process, every $\mathcal{F}^Y$-martingale is a.s. continuous. Then we can again apply Aldous's result and have only to prove the pointwise convergence of our $\mathcal{F}^{Y^n}$-martingales.

Using Lemma 3, it suffices to prove that, for every $k \in \mathbb{N}$, $t_1, ...t_k$ in $[0, T]$, and every $f : \mathbb{R}^k \to \mathbb{R}$, of the form: $f(x_1, ..., x_k) = f_1(x_1)...f_k(x_k)$, where f_i are bounded and continuous real valued functions, we have, for every $t \in [0, T]$,

$$E[f(Y_{t_1}, ..., Y_{t_k})|\mathcal{F}_t^{Y^n}] \xrightarrow{\mathbf{P}} E[f(Y_{t_1}, ..., Y_{t_k})|\mathcal{F}_t^Y]$$

Taking into account the convergence $Y^n \xrightarrow{\mathbf{P}} Y$ and continuity of f, it is sufficient to prove the convergence:

$$E[f(Y_{t_1}^n, ..., Y_{t_k}^n)|\mathcal{F}_t^{Y^n}] \xrightarrow{\mathbf{P}} E[f(Y_{t_1}, ..., Y_{t_k})|\mathcal{F}_t^Y].$$

But the Feller property for all Y^n and the convergence in probability of Y^n to Y, immediately yields the convergence of associated semigroups and the desired result. (See, for example, Theorem 17.25 of [16].) ∎

Comment. In many cases of approximation of diffusion processes by Markov chains, the hypotheses of part B of Proposition 4 are in force.

Now, let us consider some more special cases.

Let us recall that a continuous local martingale M, such that $< M >_\infty = \infty$, is said to be pure, if $\mathcal{F}^M_\infty = \mathcal{F}^B_\infty$, where B is the Brownian motion defined by $B = M \circ < M >^{-1}$. Note that M is pure if and only if $\mathcal{F}^M_t = \mathcal{F}^B \circ < M >_t$, $t \in \mathbb{R}^+$ (see, e.g., Revuz and Yor [19], page 200, 1rst edition).

Proposition 5. *Assume that M^n and M are continuous pure local martingales such that $< M >$ is strictly increasing. If $M^n \xrightarrow{\mathbf{P}} M$ then $\mathcal{F}^{M^n} \xrightarrow{w} \mathcal{F}^M$.*

Proof. Since M^n and M are pure local martingales, there exist Brownian motions B^n and B such that

$$M^n = B^n \circ < M^n >, \quad \mathcal{F}^{M^n} = \mathcal{F}^{B^n} \circ < M^n >,$$
$$M = B \circ < M >, \quad \mathcal{F}^M = \mathcal{F}^B \circ < M > .$$

It is clear that

$$(M^n, < M^n >, < M^n >^{-1}) \xrightarrow{\mathbf{P}} (M, < M >, < M >^{-1}).$$

Hence, $B^n = M^n \circ < M^n >^{-1} \xrightarrow{\mathbf{P}} B = M \circ < M >^{-1}$ and, consequently, by Proposition 2, $\mathcal{F}^{B^n} \xrightarrow{w} \mathcal{F}^B$. In view of Lemma 1, this completes the proof. ∎

For the next proposition, let us recall some notions (see, for example, [10], Chap. XXI):

A normal martingale Y is a square integrable martingale such that its predictable quadratic variation is the identity (for every $t \leq T$, $< Y, Y >_t = t$).

Let us consider a normal martingale Y. For each integer k, we denote $S_k(t) = \{(t_1, ...t_k) : 0 < t_1 < ... < t_k \leq t\}$, and, for $f_k \in L^2(S_k(T), (dt)^{\otimes k})$ and $t \leq T$, $f_k \bullet I_k(Y)_t$ is the value at t of the k-iterated stochastic integral of f_k with respect to Y. It is well known that $E[f_k \bullet I_k(Y)_t)^2] = \|f_k\|^2_{L^2(S_k(t))}$, and that for $j \neq k$ random variables $f_j \bullet I_j(Y)_t$ and $f_k \bullet I_k(Y)_t$ are orthogonal.

A martingale Y has the chaotic representation if, for every $\mathcal{F}$-square integrable martingale M (denoting by $\mathcal{F}$ the filtration generated by Y), there exists a sequence $(f_p)_{p \in \mathbb{N}}$ with $f_p \in S_p(T)$ such that:

$$M_T = \sum_p f_p \bullet I_p(Y)_T.$$

Then, if Y is normal, equality $E[M_T^2] = \sum_p \|f_p\|^2_{L^2(S_p(T))}$ holds.

Proposition 6. *Let $\mathcal{F}$ be generated by Y, which is a $\mathcal{F}$-normal martingale having the chaotic representation property, and let $\mathcal{F}^n$ be generated by Y^n which are square integrable $\mathcal{F}^n$-martingales with $< Y^n, Y^n >_t = \int_0^t a_s^n ds$, where a_s^n are uniformly bounded in s, n, and ω. Let us assume that $Y^n \xrightarrow{\mathbf{P}} Y$. Then $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.*

Proof. Let $B \in \mathcal{F}_T$. Since Y has the chaotic representation property, we have, for some (f_p), $E[1_B|\mathcal{F}_.] = \sum_p f_p \bullet I_p(Y)_.$ Let us take an arbitrary $\varepsilon > 0$, there exists $K \in \mathbb{N}$, and, for $k \leq K$, there exist $g_k \in L^2(S_k(T))$ of the form

$$g_k(t_1, ..., t_k) = \sum_{i=1...q(k)} a_i \prod_{j=1}^{k} g_{k,i,j}(t_j)$$

where a_i are constants and $g_{k,i,j}$ are continuous functions, such that:

$$\sum_{k \leq K} \|f_k - g_k\|^2_{L^2(S_k(T))} + \sum_{k > K} \|f_k\|^2_{L^2(S_k(T))} < \varepsilon.$$

Then

$$E\big[(E[1_B|\mathcal{F}_T] - \sum_{k \leq K} g_k \bullet I_k(Y)_T)^2\big] < \varepsilon$$

and, by Doob's inequality,

$$E\big[\sup_{t \leq T}(E[1_B|\mathcal{F}_t] - \sum_{k \leq K} g_k \bullet I_k(Y)_t)^2\big] < 4\varepsilon.$$

Now, since g_k are continuous, by continuity of stochastic integration (see [15], Theorem 2-6, and [6] for the iterated case), the convergence $Y^n \xrightarrow{\mathbf{P}} Y$ for J_1 implies the convergence $\sum_{k \leq K} g_k \bullet I_k(Y^n) \xrightarrow{\mathbf{P}} \sum_{k \leq K} g_k \bullet I_k(Y)$ under J_1, and boundedness of the derivatives $\dfrac{d < Y^n, Y^n >_t}{dt}$ implies the convergence in L^2 of $\sum_{k \leq K} g_k \bullet I_k(Y^n)_T$ to $\sum_{k \leq K} g_k \bullet I_k(Y)_T$.

Then, to get the result, it suffices to check that, for n large enough

$$E[\sup_{t \leq T}(E[1_B|\mathcal{F}_t^n] - \sum_{k \leq K} g_k \bullet I_k(Y^n)_t)^2] < 10\varepsilon.$$

But, by Doob's inequality, the left-hand side is majorized by

$$4E[(E[1_B|\mathcal{F}_T^n] - \sum_{k \leq K} g_k \bullet I_k(Y^n)_T)^2],$$

then by

$$4E[E[(1_B - \sum_{k \leq K} g_k \bullet I_k(Y^n)_T)^2|\mathcal{F}_T^n]],$$

and, finally, by

$$8E[(1_B - \sum_{k\leq K} g_k \bullet I_k(Y)_T)^2] + 2E[(\sum_{k\leq K} g_k \bullet I_k(Y)_T - \sum_{k\leq K} g_k \bullet I_k(Y^n)_T)^2]$$

hence, by 10ε, when n is large enough. ■

In the previous Proposition 6 the conditions on boundedness of increasing process associated to Y^n are used only for getting convergence in L^2 of stochastic integrals and whenever this convergence in L^2 holds, we get the same result.

In the following subsection we examine the relation between the weak convergence $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ and the so-called "extended convergence" of $(X^n, \mathcal{F}^n)$ to $(X, \mathcal{F})$ when $\mathcal{F}^n$ and $\mathcal{F}$ are natural filtrations of càdlàg processes X^n and X.

Weak convergence of filtrations and extended convergence

In an important unpublished paper, Aldous [1] developped notion of extended weak convergence for a sequence of processes equipped with filtrations. Given a càdlàg process X adapted to a right continuous filtration $\mathcal{F}$, he considers the process $Z = (Z_t)_{t\geq 0}$ of regular conditional distributions of X given $\mathcal{F}_t$; then Z has almost all trajectories in the space $\mathbb{D}(\mathcal{P}(\mathbb{D}))$, where $\mathcal{P}(\mathbb{D})$ denotes the space of probability measures on $\mathbb{D}$, equipped with the topology of weak convergence. Z is called the "prediction process" of $(X, \mathcal{F})$.

Let us consider a sequence of càdlàg processes X^n adapted to filtrations $\mathcal{F}^n$ and the sequence Z^n of their prediction processes. The extended weak convergence of $(X^n, \mathcal{F}^n)$ to $(X, \mathcal{F})$ (denoted by $(X^n, \mathcal{F}^n) \to (X, \mathcal{F})$) means the weak convergence of (X^n, Z^n) to (X, Z). in $\mathbb{D}(\mathbb{R} \times \mathcal{P}(\mathbb{D}))$. If the sequence (X^n) is defined on a unique probability space, we can consider the notion of extended convergence in probability.

Extended convergence is also considered, for instance, in [14] and [17] where it is related to the study of necessary conditions in functional limit theorems for processes.

This notion is closely related to the weak convergence of filtrations through the following result which is the Proposition (6.8) of preliminary (1978) version of [1] or Proposition (16.15) in the version of 1981.

Proposition A (Aldous). *Let us consider càdlàg processes X^n, X and their natural right continuous filtrations $\mathcal{F}^n, \mathcal{F}$. Then $(X^n, \mathcal{F}^n) \to (X, \mathcal{F})$ if and only if, for every integer k, and for all bounded continuous functions $\phi_1, ...\phi_k$ from $\mathbb{D}$ in $\mathbb{R}$, we have*

$$(X^n, \mathbf{E}[\phi_1(X^n)|\mathcal{F}^n_.], ..., \mathbf{E}[\phi_k(X^n)|\mathcal{F}^n_.]) \to (X, \mathbf{E}[\phi_1(X)|\mathcal{F}_.], ..., \mathbf{E}[\phi_k(X)|\mathcal{F}_.]).$$

weakly on $\mathbb{D}(\mathbb{R}^{k+1})$.

Now, let us consider real valued càdlàg processes X^n, X given on $(\Omega, \mathcal{G}, \mathbf{P})$, with their natural filtrations $\mathcal{F}^n, \mathcal{F}$, and let us assume that $X^n \xrightarrow{\mathbf{P}} X$ for J_1 and that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.

Remark 5. From the weak convergence of filtrations and convergence in probability of X^n to X, we get, for every h continuous bounded function from $\mathbb{D}$ in $\mathbb{R}$, the convergence: $\mathbf{E}[h(X^n)|\mathcal{F}^n_\cdot] \to \mathbf{E}[h(X)|\mathcal{F}_\cdot]$. Then, in some cases we can get convergence in probability of the sequence of pairs $(X^n, \mathbf{E}[h(X^n)|\mathcal{F}^n_\cdot])$ to $(X, \mathbf{E}[h(X)|\mathcal{F}_\cdot])$; then, taking into account point 1) of Remark 1 and Proposition A, the extended convergence $(X^n, \mathcal{F}^n) \to (X, \mathcal{F})$ will follow. The following proposition gives some examples of such a situation.

Proposition 7. *let us consider a sequence X^n of càdlàg processes converging in probability under J_1 to a process X. Let us assume that the sequence F^n of natural filtrations of X^n converges weakly to $\mathcal{F}$ the natural filtration of X. Let us suppose also that one of the following conditions is filled:*

(i) X is continuous.

(ii) Every $\mathcal{F}$-martingale is continuous.

(iii) For every n, X^n is a $\mathcal{F}^n$-martingale and X is a $\mathcal{F}$-martingale.

Then, we get the extended convergence $(X^n, \mathcal{F}^n) \to (X, \mathcal{F})$ in probability.

Comment. (i) and (ii) are immediate; in the case of (iii), we write that $X^n_\cdot = \mathbf{E}[X^n_T|\mathcal{F}^n_\cdot]$ and $X_\cdot = \mathbf{E}[X_T|\mathcal{F}_\cdot]$ and we apply points 2) and 3) of Remark 1.

We shall get another example of such a result in the following section as a corollary of Theorem 1.

III. Stability of processes under convergence of filtrations

Let X be a $\mathcal{F}$-semimartingale, with $X(0) = 0$ for simplification. From [9] (Theorem VII-59), we have that, if X is a quasimartingale and if $\mathcal{F}^n \subset \mathcal{F}$, then $\mathbf{E}[X|\mathcal{F}^n]$ is a $\mathcal{F}^n$-quasimartingale. For the following, let us recall some facts on the spaces $H^p(\mathcal{S})$ of semimartingales.

Definition and Theorem $H^p(\mathcal{S})$. *Let X be a $\mathcal{F}$-semimartingale, and let $p \geq 1$; X is said to belong to the space $H^p(\mathcal{S})$, if there exists a decomposition $X = M + A$, where M is a H^p-martingale and A is a predictable process with L^p-integrable variation (denoted $V(A)$). Such a decomposition is unique and will be called the "canonical decomposition". We have the following results:*

(i) A norm on $H^p(\mathcal{S})$ can be defined by:

$$\|X\|_{H^p(\mathcal{S})} = \|[M,M]_T^{1/2}\|_{L^p} + \|V(A)_T\|_{L^p},$$

where $[M,M]$ denotes the quadratic variation of M.

(ii) If $X \in H^p(\mathcal{S})$, the quadratic variation of X is $L^{\frac{p}{2}}$-integrable and:

$$\|[X,X]_T^{1/2}\|_{L^p} \leq \|X\|_{H^p(\mathcal{S})}.$$

(iii) If $X \in H^p(\mathcal{S})$ is a submartingale, there exist constants c^p and C^p depending only on p such that:

$$c^p\|X\|_{H^p(\mathcal{S})} \leq \|\sup_{t\leq T}|X_t|\|_{L^p} \leq C^p\|X\|_{H^p(\mathcal{S})}.$$

Comment. (i) and (ii) can be found in [9] chap. VII, and (iii) in [21].

Lemma 4. *If X is a $\mathcal{F}$-semimartingale in $H^p(\mathcal{S})$, for some $p > 1$, and if for a filtration $\mathcal{E}$ with $\mathcal{E} \subset \mathcal{F}$, then $\mathbf{E}[X|\mathcal{E}]$ is a $\mathcal{E}$-semimartingale in $H^p(\mathcal{S})$.*

Proof. Let us consider $X = M + A^+ - A^-$ the $\mathcal{F}$-canonical decomposition of X, where M is a H^p-martingale and A^+, A^- are predictable, L^p-integrable increasing processes; we can write:

$$\mathbf{E}[X|\mathcal{E}] = \mathbf{E}[\mathcal{M}|\mathcal{E}] + \mathbf{E}[\mathcal{A}^+|\mathcal{E}] - \mathbf{E}[\mathcal{A}^-|\mathcal{E}].$$

Then,

a) Since $\mathcal{E}_t \subset \mathcal{F}_t$, we have $\mathbf{E}[M_t|\mathcal{E}_t] = \mathbf{E}[M_T|\mathcal{E}_t]$ and it follows that the $\mathcal{E}$-martingale $\mathbf{E}[M|\mathcal{E}]$ is in H^p.

b) We easily see that $\mathbf{E}[A^+|\mathcal{E}]$ is a positive $\mathcal{E}$-submartingale, and we have:

$$\mathbf{E}[(\sup_t \mathbf{E}[A_t^+|\mathcal{E}_t])^p] \leq \mathbf{E}[(\sup_t \mathbf{E}[A_T^+|\mathcal{E}_t])^p] \leq c_p\mathbf{E}[(\mathbf{E}[A_T^+|\mathcal{E}_T])^p] \leq c_p\mathbf{E}[(A_T^+)^p] < \infty,$$

for some constant c_p. This proves that $\mathbf{E}[A^+|\mathcal{E}] \in H^p(\mathcal{S})$ ((iii) of Theorem $H^p(\mathcal{S})$). Of course the same result holds for A^-. ■

Here is our first limit theorem.

Theorem 1. *Let $(\mathcal{F}^n)$ be a sequence of filtrations and $\mathcal{F}$ be a filtration on $(\Omega, \mathcal{G}, \mathbf{P})$, such that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$, and $\mathcal{F}^n \subset \mathcal{F}$.*

Let X be a $\mathcal{F}$ adapted càdlàg process, such that $X_t = M_t + A_t$ with the following properties:

(i) A is a continuous process with $A_0 = 0$ and $\sup_{t \leq T} |A_t|$ is integrable.

(ii) M is a uniformly integrable $\mathcal{F}$−martingale.

Then $\mathbf{E}[X_.|\mathcal{F}^n_.] \xrightarrow{\mathbf{P}} X_.$ under J_1 topology.

Proof. From $\mathcal{F}^n \subset \mathcal{F}$ we have, for every t:

$$\mathbf{E}[M_t|\mathcal{F}^n_t] = \mathbf{E}[M_T|\mathcal{F}^n_t].$$

But, using the weak convergence of $\mathcal{F}^n$ to $\mathcal{F}$ we get:

$$(\mathbf{E}[M_T|\mathcal{F}^n_.]) \xrightarrow{\mathbf{P}} M$$

So, we are finished for the martingale part, and the following convergence holds in $\mathbb{D}$ under J_1

$$\mathbf{E}[M_.|\mathcal{F}^n_.] \xrightarrow{\mathbf{P}} M.$$

Now let

$$\pi_m = \{0 = t_0^m < t_1^m < \ldots < t_{k_m}^m = T\}, \qquad m \in \mathbb{N},$$

be a sequence of refining partitions of the interval $[0, T]$ such that $|\pi_m| := \max_i |t_i^m - t_{i-1}^m| \to 0$, $m \to \infty$.

We define a sequence of processes A^m by $A_t^m = A_{t_k^m}, t \in [t_k^m, t_{k+1}^m[$.

It is clear, by (i), that

$$E[\sup_{t \leq T} |A_t^m - A_t|] \to 0, n \to \infty. \tag{1}$$

By the weak convergence $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$, we have that, for every m, k in $\mathbb{N}$:

$$\mathbf{E}[A_{t_k^m} - A_{t_{k-1}^m}|\mathcal{F}^n_.] \xrightarrow{\mathbf{P}} \mathbf{E}[A_{t_k^m} - A_{t_{k-1}^m}|\mathcal{F}_.]$$

under the J_1 topology.

Each t_k^m is a continuity point of the process $\mathbf{E}[A_{t_k^m} - A_{t_{k-1}^m}|\mathcal{F}_.]$. To see that property, let us take $s > t_{k-1}^m, s \uparrow t_k^m$ and note that, by hypothesis (i), we have:

$$\begin{aligned}
\mathbf{E}[A_{t_k^m} - A_{t_{k-1}^m}|\mathcal{F}_s] &= \mathbf{E}[A_{t_k^m} - A_s|\mathcal{F}_s] + \mathbf{E}[A_s - A_{t_{k-1}^m}|\mathcal{F}_s] \\
&= \mathbf{E}[A_{t_k^m} - A_s|\mathcal{F}_s] + (A_s - A_{t_{k-1}^m}) \\
&\xrightarrow{\mathbf{P}} A_{t_k^m} - A_{t_{k-1}^m}.
\end{aligned}$$

Hence, for every $m, k \in \mathbb{N}$ we have

$$
(2)\quad \begin{aligned} \mathbf{E}[(A_{t_k^m} - A_{t_{k-1}^m})1_{\{.\geq t_k^m\}}|\mathcal{F}_.^n] &= 1_{\{.\geq t_k^m\}}\mathbf{E}[A_{t_k^m} - A_{t_{k-1}^m}|\mathcal{F}_.^n] \\ &\xrightarrow{\mathbf{P}} 1_{\{.\geq t_k^m\}}(A_{t_k^m} - A_{t_{k-1}^m}). \end{aligned}
$$

Since

$$
A_t^m = \sum_{t_k^m \leq t}(A_{t_k^m} - A_{t_{k-1}^m}) = \sum_{k=1}^{k_m}(A_{t_k^m} - A_{t_{k-1}^m})1_{\{t\geq t_k^m\}},
$$

from (2) we deduce:

$$
(3)\quad \begin{aligned} \mathbf{E}[A_.^m|\mathcal{F}_.^n] &= \sum_{k=1}^{k_m}\mathbf{E}[(A_{t_k^m} - A_{t_{k-1}^m})1_{\{.\geq t_k^m\}}|\mathcal{F}_.^n] \\ &\xrightarrow{\mathbf{P}} \sum_{k=1}^{k_m}(A_{t_k^m} - A_{t_{k-1}^m})1_{\{.\geq t_k^m\}} = A^m. \end{aligned}
$$

(The summands have disjoint points of discontinuity).

On the other hand, by (1) and by Doob's maximal inequality for martingales we get, for every ε,

$$
(4)\quad \begin{aligned} &\limsup_{m\to\infty}\sup_n \mathbf{P}[\sup_{t\leq T}|\mathbf{E}[A_t|\mathcal{F}_t^n] - \mathbf{E}[A_t^m|\mathcal{F}_t^n]| \geq \varepsilon] \\ &\leq \limsup_{m\to\infty}\sup_n \mathbf{P}[\sup_{t\leq T}\mathbf{E}[\sup_{u\leq T}|A_u - A_u^m||\mathcal{F}_t^n] \geq \varepsilon] \\ &\leq \limsup_{m\to\infty}\sup_n \frac{\mathbf{E}[\sup_{t\leq T}|A_t - A_t^m|]}{\varepsilon} \\ &= \lim_{m\to\infty}\frac{1}{\varepsilon}\mathbf{E}[\sup_{t\leq T}|A_t - A_t^m|] = 0. \end{aligned}
$$

Finally, by (3), (4), and (1) we get the convergences under J_1 in $\mathbb{D}$:

$\mathbf{E}[A_.|\mathcal{F}_.^n] \xrightarrow{\mathbf{P}} A$ and $\mathbf{E}[X_.|\mathcal{F}_.^n] \xrightarrow{\mathbf{P}} X$,

which finishes the proof. ■

Comment. When X is continuous, the hypothesis $\mathcal{F}^n \subset \mathcal{F}$ is not needed. Actually we take $M = 0$ and $X = A$.

Corollary. *Let $(\mathcal{F}^n)$ be the sequence of natural filtrations of càdlàg processes X^n and $\mathcal{F}$ be the natural filtration of a càdlàg process X given on $(\Omega, \mathcal{G}, \mathbf{P})$ such that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ and $\mathcal{F}^n \subset \mathcal{F}$.*

Let X be such that $X_t = M_t + A_t$ with the following properties

(i) A is a continuous process with $A_0 = 0$ and $\sup_{t\leq T}|A_t|$ is integrable.

(ii) M is a uniformly integrable $\mathcal{F}$-martingale.

We also suppose that convergence $X^n \xrightarrow{\mathbf{P}} X$ holds.

Then extended convergence $(X^n, \mathcal{F}^n) \to (X, \mathcal{F})$ in probability holds.

Proof. In the beginning of the proof of Theorem 1, we can replace $\mathbf{E}[M_t|\mathcal{F}_t^n] = \mathbf{E}[M_T|\mathcal{F}_T^n]$ by $\mathbf{E}[M_t + h(X)|\mathcal{F}_t^n] = \mathbf{E}[M_T + h(X)|\mathcal{F}_t^n]$, where h is a continuous bounded function from $\mathbb{D}$ to $\mathbb{R}$. Using the weak convergence of filtrations we deduce the convergence under J_1:

$$(\mathbf{E}[M_.|\mathcal{F}_.^n], \mathbf{E}[h(X)|\mathcal{F}_.^n]) \xrightarrow{\mathbf{P}} (\mathbf{E}[M_.|\mathcal{F}_.], E[h(X)|\mathcal{F}_.]).$$

Since A is continuous the rest of the proof of Theorem 1 can be unchanged and we get the convergence under J_1:

$$(\mathbf{E}[M_.|\mathcal{F}_.^n], \mathbf{E}[A|\mathcal{F}_.^n], \mathbf{E}[h(X)|\mathcal{F}_.^n]) \xrightarrow{\mathbf{P}} (\mathbf{E}[M_.|\mathcal{F}_.], \mathbf{E}[A|\mathcal{F}_.], E[h(X)|\mathcal{F}_.]).$$

On the other hand, from J_1-convergence of X^n to X, we get by Doob's inequality for martingales convergence of uniform distance: $d_U(\mathbf{E}[h(X_.^n)|\mathcal{F}_.^n], \mathbf{E}[h(X)|\mathcal{F}_.^n]) \xrightarrow{\mathbf{P}} 0$ and convergence of the Skorokhod distance $d_S(X^n, \mathbf{E}[X|\mathcal{F}_.^n]) \xrightarrow{\mathbf{P}} 0$. Hence we get:

$$d_S((X^n, \mathbf{E}[h(X^n)|\mathcal{F}_.^n]), (\mathbf{E}[X|\mathcal{F}_.^n], \mathbf{E}[h(X)|\mathcal{F}_.^n])) \xrightarrow{\mathbf{P}} 0,$$

we deduce $(X^n, \mathbf{E}[h(X^n)|\mathcal{F}_.^n]) \xrightarrow{\mathbf{P}} (X, \mathbf{E}[h(X)|\mathcal{F}_.])$, which gives the desired result of extended convergence. ■

The following theorem shows that we can extend the conclusion of Theorem 1 to quasi-left-continuous processes X.

Theorem 2. *Let X be a càdlàg quasi-left-continuous process such that $\sup_{t\leq T}|X_t|$ is integrable. If $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$ and $\mathcal{F}^n \subset \mathcal{F}$, then convergence*

$$\mathbf{E}[X_.|\mathcal{F}_.^n] \xrightarrow{\mathbf{P}} X$$

holds in $\mathbb{D}$ under J_1.

Proof. For every $m \in \mathbb{N}$, we define a sequence of stopping times by

$$\sigma_0^m = 0$$

and, for $k > 0$,

$$\sigma_{k+1}^m = \min\{\sigma_k^m + \delta_k^m, \inf\{t > \sigma_k^m : |\Delta X_t| > \delta^m\}\},$$

where $(\delta^m)_m$ and for all k, $(\delta_k^m)_m$ are sequences of constants such that $\delta^m \downarrow 0$, $\frac{\delta^m}{2} \leq \delta_k^m \leq \delta^m$, $\mathbf{P}[\exists t \in [0,T] : |\Delta X_t| = \delta^m] = 0$, and $\mathbf{P}[|\Delta X_{\sigma_k^m + \delta_k^m}| = 0] = 1$, for all k and m.

Let us consider, for every m, the step process X^m defined by

$$X_t^m = X_{\sigma_k^m} \text{ for } \sigma_k^m \leq t < \sigma_{k+1}^m.$$

Then (see, for example, [20], Proposition 2), we have the convergence

$$\sup_{t \leq T} |X_t - X_t^m| \xrightarrow{\mathbf{P}} 0, \quad m \to \infty. \tag{5}$$

The process X^m can be written under the form

$$X_t^m = \sum_{k=1}^{\infty} (X_{\sigma_k^m} - X_{\sigma_{k-1}^m}) 1_{\{t \geq \sigma_k^m\}} + X_0$$

, it is an $\mathcal{F}_t$-adapted process of finite variation and $\sup_{t \leq T} |X_t^m|$ is integrable, since, for every m, $\sup_{t \leq T} |X_t^m| \leq \sup_{t \leq T} |X_t|$ (X^m is a discretization of X). Since, for every m, X^m has only a finite number of jumps greater than δ^m, one can find an increasing sequence of constants $k_m \uparrow \infty$ such that

$$\sup_{t \leq T} |X_t^{m,m} - X_t^m| \xrightarrow{\mathbf{P}} 0, \quad m \to \infty, \tag{6}$$

where $X_t^{m,m} = \sum_{k=1}^{k_m} (X_{\sigma_k^m} - X_{\sigma_{k-1}^m}) 1_{\{t \geq \sigma_k^m\}}$.

For every m, $\mathrm{X}^{m,m}$ is an $\mathcal{F}_t$-adapted process with integrable variation, therefore we have a decomposition of the form:

$$X^{m,m} = M^m + A^m,$$

where M^m is a uniformly integrable martingale, and A^m is a predictable (hence continuous) process of integrable variation. By the proof of Theorem 1, we get that, for each m,

$$\mathbf{E}[M_\cdot^m | \mathcal{F}_\cdot^n] \xrightarrow{\mathbf{P}} M^m, \quad n \to \infty, \tag{7}$$

as well as

$$\mathbf{E}[A_\cdot^m | \mathcal{F}_\cdot^n] \xrightarrow{\mathbf{P}} A^m, \quad n \to \infty. \tag{8}$$

In view of (7) and (8) we get:

$$\mathbf{E}[X_\cdot^{m,m} | \mathcal{F}_\cdot^n] \xrightarrow{\mathbf{P}} X^{m,m}, \quad n \to \infty$$

which, together with (5) and (6), implies the result, using the same argument as in (4) for the proof of Theorem 1. ∎

Let us return to the situation of Lemma 4, where $X \in H^p(\mathcal{S})$ for some $p > 1$ with the canonical decomposition $X = M + A$. We suppose that A is continuous. Let us denote $X^n = \mathbf{E}[X|\mathcal{F}^n]$ and write the canonical decomposition $X^n = M^n + A^n$. We shall sharpen the convergence statement $X^n \xrightarrow{\mathbf{P}} X$ of Theorem 1.

Theorem 3. *Suppose that $\mathcal{F}^n \subset \mathcal{F}$ and $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. Then, under the J_1 topology, the following convergences hold:*

a)
$$(X^n, [X^n, X^n]) \xrightarrow{\mathbf{P}} (X, [X, X])$$

b)
$$(M^n, [M^n, M^n]) \xrightarrow{\mathbf{P}} (M, [M, M])$$

c)
$$A^n \xrightarrow{\mathbf{P}} A.$$

Proof. a) Let us write $X^n = N^n + B^n$, where $N^n = \mathbf{E}[M|\mathcal{F}^n]$ and $B^n = \mathbf{E}[A|\mathcal{F}^n]$.

$$\|X^n\|^p_{H^p(\mathcal{S})} \leq c_p(\mathbf{E}[(\sup_t \mathbf{E}[M_t|\mathcal{F}^n_t]^p)] + \mathbf{E}[(\sup_t \mathbf{E}[V(A)_t|\mathcal{F}^n_t]^p)])$$

where c_p is a constant which can change from line to line and $V(A)$ denotes the "variation-process" of A,

$$\leq c_p(\mathbf{E}[|M_T|^p] + \mathbf{E}[(V(A)_T)^p]) < \infty.$$

Then, $\sup_n \|X^n\|_{H^p(\mathcal{S})} < \infty$, and the sequence (X^n) has the U.T. property (see [15]). Applying Theorem 1, and using ([15], corollary 2.8), we get the first assertion.

b) and c). Of course, under the weak convergence of $\mathcal{F}^n$ to $\mathcal{F}$, we have $N^n \xrightarrow{\mathbf{P}} M$ and the sequence (N^n) also has the U.T. property, since in part a) we incidentally proved that $\sup_n \|N^n\|_{H^p} < \infty$: therefore

$$(N^n, [N^n, N^n]) \xrightarrow{\mathbf{P}} (M, [M, M]).$$

Let us denote $B^n(+) = \mathbf{E}[A^+|\mathcal{F}^n]$; $B^n(+)$ is $\mathcal{F}^n$-submartingale in $H^p(\mathcal{S})$ with the canonical decomposition $B^n(+) = \bar{M}^n(+) + \bar{A}^n(+)$; since, by a) $\sup_n \|B^n(+)\|_{H^p}$ is finite, $B^n(+)$ has the U.T. property and we get:

$$(B^n(+), [B^n(+), B^n(+)]) \xrightarrow{\mathbf{P}} (A^+, [A^+, A^+]).$$

Since A^+ is continuous and increasing , $[A^+, A^+] = 0$, and $[B^n(+), B^n(+)] \xrightarrow{\mathbf{P}} 0$; using again the $H^p(\mathcal{S})$ boundedness of sequence $(B^n(+))$, $([B^n(+), B^n(+)]^{1/2})$ is a sequence L^p-bounded (ii of Theorem $H^p(\mathcal{S})$), and $[B^n(+), B^n(+)]^{1/2}_T$ converges to 0 in L^1.

Hence, using the inequality

$$\|[\bar{M}^n(+), \bar{M}^n(+)]_T\|_{L^{p/2}} \leq \|[B^n(+), B^n(+)]_T\|_{L^{p/2}}$$

([9] Theorem VII 95), we get: $[\bar{M}^n(+), \bar{M}^n(+)] \xrightarrow{\mathbf{P}} 0$, whence finally: $\bar{M}^n(+) \xrightarrow{\mathbf{P}} 0$ and $\bar{A}^n(+) \xrightarrow{\mathbf{P}} A^+$.

Denoting by $\bar{M}^n(-) + \bar{A}^n(-)$ the canonical decomposition of $\mathbf{E}[A^-|\mathcal{F}^n_.]$, we get similarly the convergences: $\bar{M}^n(-) \xrightarrow{\mathbf{P}} 0$ and $\bar{A}^n(-) \xrightarrow{\mathbf{P}} 0$. Now, since $M^n = N^n + \bar{M}^n(+) - \bar{M}^n(-)$ and $A^n = \bar{A}^n(+) - \bar{A}^n(-)$, we are finished with assertions b) and c). ∎

Remark 6. As a continuation of Theorem 3, with the same assumptions and notation, we get stability results for stochastic differential equations under the weak convergence of filtrations.

Precisely, let us consider $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that the equations

$$(**) \qquad Y_t = y_0 + \int_0^t f(s, Y_s) dX_s$$

and

$$(**^n) \qquad Y_t^n = y_0 + \int_0^t f(s, Y_s^n) dX_s^n$$

have $\mathcal{F}$ (resp: $(\mathcal{F}^n)$)- adapted solutions defined on $(\Omega, \mathcal{G}, \mathbf{P})$. We also assume that equation $(**)$ has the property of pathwise uniqueness.

Then, under J_1 we have the convergence

$$(X^n, Y^n) \xrightarrow{\mathbf{P}} (X, Y).$$

Comment. Actually, in the proof of Theorem 3, we got the U.T. property for the sequence (X^n), then, applying Theorem 1 and [18] (Theorem 3-6), we have the desired result. ∎

IV Stability of backward equations under convergence of filtrations

Let us consider the situation of paper [7] with V^n and V unique solutions of backward equations $(*^n)$ and $(*)$ respectively and the hypotheses assumed in [4] and [7].

(HX) $\exists\delta > 0 : X^n \to X$ in $L^{1+\delta}(\mathbf{P})$;

(HA) All A^n are increasing $(\mathcal{F}_t^n)$-adapted processes, and $A^n \xrightarrow{\mathbf{P}} A$;
$A_T^n \leq \beta_n$; $\sup_n \beta_n = \beta_\infty < \infty$;

(Hg) $g, g^n: \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are respectively $(\mathcal{F}_t)$ and $\mathcal{F}_t^n$-adapted, Lipschitz with constants c, c_n; g is continuous in s, g^n are càdlàg in s; $\sup_s |g_s(0)|$ is bounded;

$$\sup_n e^{c_n\beta_n(1+\delta)}\Big\{\mathbf{E}\Big[\int_0^T |g_s^n(0)|\, dA_s^n\Big]^{1+\delta} + \mathbf{E}|X^n|^{1+\delta}\Big\} < \infty;$$

$\mathbf{P}$-a.s. $g^n \to g$ uniformly in s and x on compact sets.

(Hco) Either A is a continuous process, or all $(\mathcal{F}_t)$-martingales are continuous.

The following theorem will be proved using exactly the same arguments as in [8].

Theorem 4. *Suppose that hypotheses* (HX), (HA), (Hg), *and* (Hco) *are satisfied. Suppose that* $(\mathcal{F}^n) \xrightarrow{w} (\mathcal{F})$. *Then the sequence* (V^n) *of the solutions of equations* $(*^n)$ *converges to the solution* V *of* $(*)$ *in probability under the Skorokhod* J_1 *topology.*

Proof. Our method for proving this result is considering, for each n, the iterates $U^{n,k}$ given by the Picard approximation converging, as $k \to \infty$, to the solution V^n of $(*^n)$ uniformly in n and proving that, for each k, $U^{n,k}$ converges in probability, as $n \to \infty$, to U^k, the k-iterated process of the Picard approximation of V, the solution of $(*)$.

To be precise, we put:
for equation $(*^n)$,

$$U_t^{n,0} = 0,$$

and, for $k \geq 1$, by induction,

$$U_t^{n,k} = \mathbf{E}\Big[\int_t^T g_s^n(U_s^{n,k-1})\, dA_s^n + X^n | \mathcal{F}_t^n\Big];$$

for equation $(*)$,

$$U_t^0 = 0,$$

and, for $k \geq 1$, by induction,

$$U_t^k = \mathbf{E}\Big[\int_t^T g_s(U_s^{k-1})\, dA_s + X | \mathcal{F}_t\Big].$$

Step 1: In Theorem 2-4 of [3], Antonelli proved the inequality

$$\|U^{n,k+1} - U^{n,k}\|_{L^1(\mu_n)} \leq \frac{(c_n\beta_n)^{k+1}}{(k+1)!}\|U^{n,1}\|_{L^1(\mu_n)},$$

where $\mu_n(dt, d\omega) = dA_t^n(\omega)P(d\omega)$.

Therefrom we deduce:

$$\|V^n - U^{n,k}\|_{L^1(\mu_n)} \leq \sum_{p=k+1}^{\infty} \frac{(c_n\beta_n)^p}{p!} M_n,$$

where

$$M_n = \beta_n \mathbf{E}\Big(\int_0^T |g_s^n(0)| dA_s^n + |X^n|\Big).$$

Using the Doob maximal inequality, we easily get that

$$\forall \varepsilon > 0,\ \mathbf{P}\big[\sup_{t\leq T}|V_t^n - U_t^{n,k+1}| \geq \varepsilon\big] \leq \mathbf{P}\Big[\sup_{t\leq T} c_n \mathbf{E}\Big[\int_0^T |V_s^n - U_s^{n,k}|\, dA_s^n \big| \mathcal{F}_t^n\Big] > \varepsilon\Big]$$
$$\leq \frac{1}{\varepsilon} c_n \|V^n - U^{n,k}\|_{L^1(\mu_n)},$$

whence

$$\forall \varepsilon > 0,\ \mathbf{P}\big[\sup_{t\leq T}|V_t^n - U_t^{n,k+1}| \geq \varepsilon\big] \leq \frac{1}{\varepsilon} \sum_{p=k+1}^{\infty} \frac{(c_n\beta_n)^p}{p!} c_n M_n.$$

Finally, the assumptions of Theorem 4 give uniform (in n)convergence:

$$\forall \varepsilon > 0,\ \sup_n \mathbf{P}\big[\sup_{t\leq T}|V_t^n - U_t^{n,k}| \geq \varepsilon\big] \to 0, \qquad k \to \infty. \tag{9}$$

Moreover, from hypothesis (Hg), we get by induction that, for every k,

$$\sup_n \mathbf{E}\big[\sup_{t\leq T}|U_t^{n,k}|^{1+\delta}\big] < \infty \tag{10}$$

and

$$\sup_n \mathbf{E}\Big[\Big(\int_0^T |g_s^n(U_s^{n,k})|\, dA_s^n\Big)^{1+\delta} + |X^n|^{1+\delta}\Big] < \infty. \tag{11}$$

Step 2: All convergences below are under the Skorokhod J_1 topology. From (HA) and (Hg) we get:

$$\Big(A^n, \int_0^\cdot g_s^n(0)\, dA_s^n\Big) \xrightarrow{\mathbf{P}} \Big(A, \int_0^\cdot g_s(0)\, dA_s\Big).$$

Then, with the weak convergence of $(\mathcal{F}^n)$ to $\mathcal{F}$, (HX), (Hco), and above inequalities (10) and (11), we get

$$\Big(A^n, \int_0^{\cdot} g_s^n(0)\, dA_s^n, \mathbf{E}\Big[\int_0^T g_s^n(0)\, dA_s^n + X^n|\mathcal{F}_{\cdot}^n\Big]\Big) \xrightarrow{\mathbf{P}} \Big(A, \int_0^{\cdot} g_s(0)\, dA_s, \mathbf{E}\Big[\int_0^T g_s(0)\, dA_s + X|\mathcal{F}_{\cdot}\Big]\Big).$$

Hence

$$(A^n, U^{n,1}) \xrightarrow{\mathbf{P}} (A, U^1).$$

We can iterate the procedure: from

$$(A^n, U^{n,k}) \xrightarrow{\mathbf{P}} (A, U^k),$$

we deduce, using the continuity of g and convergence of Stieltjes integrals (see, for example, [15]):

$$\Big(A^n, \int_0^{\cdot} g_s^n(U_s^{n,k})\, dA_s^n\Big) \xrightarrow{\mathbf{P}} \Big(A, \int_0^{\cdot} g_s(U_s^k)\, dA_s\Big).$$

Using again inequalities (10), (11), and the hypotheses (HA), (H$\mathcal{F}$), (Hco), (Hg) we get

$$(A^n, U^{n,k+1}) \xrightarrow{\mathbf{P}} (A, U^{k+1}).$$

Finally, for every k, we have convergence for the Skorokhod topology of processes:

$$U^{n,k} \xrightarrow{\mathbf{P}} U^k.$$

Inequality (9) of Step 1 finally gives the desired result $V^n \xrightarrow{\mathbf{P}} V$. ∎

Example. Let us consider the situation of Theorem 3. We would like to approximate the solution V of the backward equation $(*)$ by some sequence of $(\mathcal{F}^n)$-adapted processes; a simple way is to consider the sequence $(\mathbf{E}[V_{\cdot}|\mathcal{F}_{\cdot}^n])$. Then, by Theorem 1, this sequence converges to V in probability for the J_1 topology. But we can get such an approximation by another way: we consider the perturbed equation $(*^n)$, with $g^n = g$ (g not depending on ω), $X^n = X$, and $A^n = \mathbf{E}[A_{\cdot}|\mathcal{F}_{\cdot}^n]$. Using the canonical decomposition $A^n = \bar{M}^n + \bar{A}^n$, by the martingale property of $\bar{M}^n$, we get

$$V_t^n = \mathbf{E}[\int_t^T g_s(V_s^n)\, d\bar{M}_s^n|\mathcal{F}_t^n] = 0,$$

where V^n is the unique solution of the backward equation:

$$V_t^n = \mathbf{E}[\int_t^T g_s(V_s^n)\, d\bar{A}_s^n + X|\mathcal{F}_t^n].$$

Thus, $(*^n)$ has a unique solution V^n; then the convergence $\bar{A}^n \xrightarrow{\mathbf{P}} A$ and Theorem 4 yield the convergence $V^n \xrightarrow{\mathbf{P}} V$. ∎

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